Appl. Math. J. Chinese Univ.
2025, 40(2): 480-500

Almost global existence for d-dimensional fractional nonlinear Schrödinger equation on flat torus

DING Wan-ran[1] LI Si-ming[1] XUE Hui-ning[2]

Abstract. In this paper, we will discuss the almost global existence result for d-dimensional fractional nonlinear Schrödinger equation on flat torus, which is based on BNF technique, the tame property and the analysis of the spectrum of $(-\Delta)^s$.

§1 Introduction

Understanding the qualitative aspects of the long time behavior of solutions to Hamiltonian partial differential equations (PDEs) on compact manifolds has been a fundamental problem over the past decades. Pioneering work on Nekhoroshev-like theorems for Hamiltonian PDEs was carried out by Bourgain and Bambusi, see [14,15,2] and references therein. However, as Kuksin noted in [35] which is "still the main challenge here is to prove or disprove the Nekhoroshev theorem".

The fractional Schrödinger equation is a key development in quantum physics, generalizing the Schrödinger equation in the context of fractional quantum mechanics. As noted in [34], it is an emerging area related to nonlocal quantum phenomena, offering new insights into quantum behavior beyond traditional locality principles. For related work, refer to [36,41,26,12]. In this paper, we will study the long time stability for the d-dimensional the fractional nonlinear Schrödinger equation (FNLS)

$$\mathrm{i}u_t + (-\Delta)^s u + V(x) * u + |u|^2 u = 0, \quad x \in \mathbb{T}^d_{\mathscr{L}} := \mathbb{R}^d/\mathscr{L} \tag{1}$$

on flat tori, where $(-\Delta)^s$ is the Riesz fractional differentiation with $s \in (0, 1/2)$, $V*$ is a convolution potential and $\mathscr{L}$ is a lattice of $\mathbb{R}^d$.

Significant progress towards understanding the long time stability for Hamiltonian PDEs was made by Bambusi-Grébert in [7]. They proved a Birkhoff normal form (BNF) theorem that is adapted to a large class of Hamiltonian PDEs, which exhibit the **tame** property in Sobolev

Received: 2024-11-20. Revised: 2024-12-10.
MR Subject Classification: 37K55, 37J40, 35B35, 35Q35.
Keywords: tame property, Birkhoff normal form theory, flat torus, nonresonant conditions.
Digital Object Identifier(DOI): https://doi.org/10.1007/s11766-025-5328-0.
Supported by the National Natural Science Foundation of China(12101542, 12371189, 12371241).

space

$$H^{\sigma}\left(\mathbb{T}^d\right) := \left\{ q = (q_{\mathbf{j}})_{\mathbf{j}\in\mathbb{Z}^d} : \sum_{\mathbf{j}\in\mathbb{Z}^d} |q_{\mathbf{j}}|^2 \langle \mathbf{j}\rangle^{2s} < \infty \right\}, \quad \sigma > d/2, \tag{2}$$

where $\langle \mathbf{j}\rangle := \max\{|\mathbf{j}|, 1\}$ and $|\mathbf{j}| := \sqrt{j_1^2 + \cdots + j_d^2}$ with $\mathbf{j} = (j_1, \cdots, j_d)$. At the same time, they discussed dynamical consequences on the **polynomial** long time behavior of the solutions with small initial data. In 2013, Faou-Grébert [30] proved a **subexponential** long stability time interval for d-dimensional NLS in the analytic space

$$\mathcal{A}_{\rho}(\mathbb{T}^d) := \left\{ q = (q_{\mathbf{j}})_{\mathbf{j}\in\mathbb{Z}^d} : \sum_{\mathbf{j}\in\mathbb{Z}^d} |q_{\mathbf{j}}| e^{\rho/2|\mathbf{j}|} < \infty \right\}, \quad \rho > 0. \tag{3}$$

After that Cong-Liu-Wang [20] and Chen-Cong-Meng-Wu [19] proved respectively a similar result for 1-dimensional nonlinear wave equation and 1-dimensional NLS in Gevrey space

$$G_{\sigma}(\mathbb{T}) := \left\{ q = (q_j)_{j\in\mathbb{Z}} : \sum_{j\in\mathbb{Z}} |q_j|^2 \psi(j,\sigma)^2 < \infty \right\}, \quad \sigma > 0$$

with $\psi(j,\sigma) = \exp\left\{\sigma\sqrt{|j|}\right\}$ in [20] and $\psi(j,\sigma) = \exp\left\{\sigma \ln^2 [j]\right\}$ in [19], where $[j] := \max\{|j|, e^3\}$ with $j \in \mathbb{Z}$.

Finally, let us mention some recent results. Berti-Delort [3] proved the almost global solution existence for quasi-linear water wave equations with periodic boundaries. Feola-Giuliani-Pasquali [31] developed a formal BNF for the Degasperis-Procesi equation, addressing resonances using conserved quantities. Bernier-Faou-Grébert [5] introduced a normal form for the 1-dimensional NLS without external parameters. Recently, Bernier-Grébert [8] studied the long time dynamics of the solutions of the generalized Korteweg-De Vries and Benjamin-Ono equations without external parameters. See more related work [4,21,22,23,32,33,37,38] for examples.

In 2022, Procesi raised an open question in his ICM report (see Q3 within the document): "Can one extend the stability results to general manifolds in higher dimension?" Delort-Szeftel [27] provide a lower bound estimate for the existence time of small smooth solutions to the Cauchy problem for a nonlinear Klein-Gordon equation (NLKG) on the sphere S^{d-1}. Subsequently, Bambusi-Delort-Grébert-Szeftel [4] utilized the BNF method and the characteristic distribution of eigenvalues of the Laplacian operator on Zoll manifolds to prove the existence of almost global solutions for the d-dimensional NLKG on these manifolds. For more work related to this question, see [11,25,28]. For other research on general manifolds, refer to [9,10,18,24,39].

Recently, a significant advancement was achieved by Bambusi-Feola-Montalto in [6], proving a result on the almost global existence for d-dimensional NLS, Beam and QHD on flat tori, with the phase space being the Sobolev space $H^{\sigma}(\mathbb{T}^d)$. In this work, in order to tackle the resonance issue related to the eigenvalues of the Laplace operator on flat tori, they introduced a key technique—a Lemma by Bourgain on the "localization of resonant sites" in $\mathbb{T}^d$—to construct a N-block normal form. Specifically, when two indexes $\mathbf{i}, \mathbf{j} \in \mathbb{Z}^d$ belong respectively to two different blocks $\Omega_{\alpha}, \Omega_{\beta} \subset \mathbb{Z}^d$, their frequencies $\omega_{\mathbf{i}}$ and $\omega_{\mathbf{j}}$ are well-separated, i.e.,

$$|\mathbf{i} - \mathbf{j}| + |\omega_{\mathbf{i}} - \omega_{\mathbf{j}}| \geq C(\delta)\left(|\mathbf{i}|^{\delta} + |\mathbf{j}|^{\delta}\right), \quad \exists\, \delta > 0. \tag{4}$$

Consequently, it becomes feasible to eliminate the monomials involving two high-frequency variables $q_{\mathbf{i}}, q_{\mathbf{j}}$, where $\mathbf{i} \in \Omega_\alpha$ and $\mathbf{j} \in \Omega_\beta$ with $\alpha \neq \beta$, by applying suitable nonresonance conditions. For the remaining monomials, which involve two high-frequency variables $q_{\mathbf{i}}, q_{\mathbf{j}}$ within the same block, the Duhamel formula is used for estimation.

In analogy with [19], we also expect the almost global existence for d-dimensional NLS on flat tori in Gevrey space $G_\sigma(\mathbb{T}^d)$. However, note that in [6], when considering the Sobolev space $H^\sigma(\mathbb{T}^d)$, the norm $\|\cdot\|_{H^\sigma(\mathbb{T}^d)}$ is equivalent to the ℓ^2_s-norm $\|\cdot\|_s$, where $\| z \|^2_{H^\sigma(\mathbb{T}^d)} := \sum_\alpha n(\alpha)^{2s} \|z_\alpha\|_0$ with $n(\alpha) := \min_{j \in \Omega_\alpha} |\mathbf{j}|$ and $\mathbb{Z}^d = \bigcup_\alpha \Omega_\alpha$. In contrast, when we consider the Gevrey space $G_\sigma(\mathbb{T}^d)$, this equivalence does not hold.

Therefore, the focus of our paper is to investigate the almost global existence for the FNLS given by (1) on flat tori, with initial data belonging to the Gevrey space. Furthermore, to facilitate a more comprehensive comparison with the results in [6], we have also taken into account the Sobolev space and the modified Sobolev space in our paper. Precisely, in this paper, we will study the long time stability of solutions for (1) in the following four phase spaces:

$$\ell^2_\sigma(\mathbb{C}) = \left\{ q = (q_{\mathbf{j}})_{\mathbf{j} \in \mathbb{Z}^d} \in \mathbb{C}^{\mathbb{Z}^d} : \sum_{\mathbf{j} \in \mathbb{Z}^d} |q_{\mathbf{j}}|^2 \psi(\mathbf{j}, \sigma)^2 < \infty \right\} \tag{5}$$

with

$$\psi(\mathbf{j}, \sigma) = \exp\left\{\sigma \sqrt{|\mathbf{j}|}\right\}, \exp\left\{\sigma \ln^2 \lfloor \mathbf{j} \rfloor\right\}, \exp\left\{\sigma \ln \langle \mathbf{j} \rangle\right\}, \exp\left\{\sigma \ln \lfloor \mathbf{j} \rfloor\right\},$$

where $\lfloor \mathbf{j} \rfloor := \max\{|\mathbf{j}|, 2\}$. (The first two functions $\psi(\mathbf{j}, \sigma)$ correspond to Gevrey spaces, the third corresponds to the Sobolev space, and the last one corresponds to the modified Sobolev space.) And denote

$$\|q\|^2_\sigma := \sum_{\mathbf{j} \in \mathbb{Z}^d} |q_{\mathbf{j}}|^2 \psi(\mathbf{j}, \sigma)^2.$$

For the Gevrey spaces and Sobolev spaces, we will investigate the almost global existence of solutions for (1) by using the BNF theorem given in [30]. In order to apply the BNF theorem, it is essential to handle the issue of small divisors (see Section 3 for more details) and discuss the **tame** property

$$\|q * q'\|_\sigma \leq C(\sigma, \sigma') \left(\|q\|_\sigma \|q'\|_{\sigma'} + \|q'\|_\sigma \|q\|_{\sigma'}\right),$$

where $0 < \sigma' < \sigma$, in various phase spaces (see Lemma 5.3 for more details). As for the modified Sobolev space, we will explore it through the application of a **tame**-like inequality given in [38] (see (70) for more details).

As previously mentioned, applying the BNF theorem depends on properly handling the issue of small divisors. However, since the frequency ω (see (14)) for the FNLS exhibits only sublinear growth, it poses significant challenges to prove that ω meets certain appropriate nonresonance conditions. At the same time, the difficulty caused by the sublinear increasing frequency is also a problem of the stability of linear models. To be specific, for the linear Schrödinger-type equation $\mathrm{i}u_t = (\mathrm{H} + \mathrm{V}(t))u$, it is widely believed that the spectral gap of the unperturbed Hamiltonian H is closely related to the long time behavior of solutions. If the spectral gap increases, there are enormous results to explore the stability of solutions of such equations, including polynomial (logarithmic) bounds on the growth of Sobolev norms [16,17,24,40], as well as globally bounded Sobolev norms [1,29]. However, if the eigenvalues of unperturbed Hamiltonian H grow only sublinearly, i.e., spectral gap shrinks to zero, the situation has been changed significantly. It is much harder to obtain stability results for the linear models.

Recall the FNLS given by (1), where $\mathscr{L}$ is a lattice of $\mathbb{R}^d$ with linearly independent generators $\mathbf{v}_1, \cdots, \mathbf{v}_d \in \mathbb{R}^d$, i.e.,

$$\mathscr{L} := \left\{ \sum_{i=1}^{d} m_i \mathbf{v}_i : m_i \in \mathbb{Z} \right\}$$

and the potential $V(x)$ is smooth, real valued, periodic on the lattice $\mathscr{L}$, namely

$$V\left(x + \sum_{i=1}^{d} m_i \mathbf{v}_i\right) = V(x), \quad m_i \in \mathbb{Z}, 1 \leq i \leq d.$$

Then equation (1) is equivalent to the following FNLS on the standard torus

$$\mathrm{i}u_t + (-\Delta_{\mathscr{L}})^s u + V(x) * u + |u|^2 u = 0, \quad x \in \mathbb{T}^d, \tag{6}$$

where the anisotropic Laplacian

$$\Delta_{\mathscr{L}} := \sum_{i,j=1}^{d} \partial_{x_i} \left[W^{\mathrm{T}} W\right]_i^j \partial_{x_j}$$

with

$$W := \mathbf{V}^{-1}, \quad \mathbf{V} := (\mathbf{v}_1 | \cdots | \mathbf{v}_d).$$

And the eigenvalues of $-\Delta_{\mathscr{L}}$ are $\mu_{\mathbf{j}} = |W\mathbf{j}|^2$ for any $\mathbf{j} \in \mathbb{Z}^d$. Since W is invertible, it follows that $\mu_{\mathbf{j}} \sim |\mathbf{j}|^2$. Let u be a solution of (6), which can be expanded into a Fourier series

$$u = \sum_{\mathbf{j} \in \mathbb{Z}^d} q_{\mathbf{j}} \cdot \phi_{\mathbf{j}}(x) \quad \text{with} \quad \phi_{\mathbf{j}}(x) = \frac{1}{(2\pi)^{d/2}} e^{\mathrm{i}\langle \mathbf{j}, x\rangle}.$$

Moreover, $V(x)*$ is the convolution potential defined by

$$V(x) * u = \sum_{\mathbf{j} \in \mathbb{Z}^d} V_{\mathbf{j}} \cdot q_{\mathbf{j}} \cdot \phi_{\mathbf{j}}(x).$$

We assume that for $m > d/2$, V belongs to the space

$$\mathcal{W}_m = \left\{ V(x) = \sum_{\mathbf{j} \in \mathbb{Z}^d} V_{\mathbf{j}} \phi_{\mathbf{j}} \,\middle|\, V_{\mathbf{j}} \in \left[-\frac{1}{2\langle \mathbf{j} \rangle^m}, \frac{1}{2\langle \mathbf{j} \rangle^m}\right] \text{for any } \mathbf{j} \in \mathbb{Z}^d \right\}, \tag{7}$$

that we endow with the product probability measure. For any $\sigma > 0$, define by $\|u\|_\sigma := \|q\|_\sigma$.

Then we have the following results:

Theorem 1.1. Sobolev space and modified Sobolev space
Let

$$\psi(\mathbf{j}, \sigma) = \exp\left\{\sigma \ln \langle \mathbf{j} \rangle\right\},$$

which corresponds to the Sobolev space. Given any large $r > 0$, there exists $\sigma_*(r) > 0$ depending on r and a subset $\mathcal{V} \subset \mathcal{W}_m$ of full measure. Then for any $V \in \mathcal{V}$ and any $\sigma > \sigma_*(r)$ there is a small $\epsilon_0 = \epsilon_0(\sigma, m, d, s, W, r) > 0$ depending only on σ, m, d, s, W and r, such that for any $0 < \epsilon < \epsilon_0$, if

$$\|u(0, x)\|_\sigma \leq \epsilon,$$

then the solution $u(t,x)$ of (1) with the initial datum $u(0,x)$ satisfies

$$\|u(t,x)\|_\sigma \leq 2\epsilon, \quad \forall\ |t| \leq T_\epsilon,$$

where

$$T_\epsilon \geq \exp\left\{\frac{4}{5} r\left|\ln\epsilon\right|\right\}. \tag{8}$$

Furthermore, let

$$\psi(\mathbf{j},\sigma) = \exp\left\{\sigma \ln \lfloor \mathbf{j} \rfloor\right\},$$

which corresponds to the modified Sobolev space. For any $V \in \mathcal{W}_m$, there is a small $\epsilon_0(d) > 0$ depending only on d such that for any $0 < \epsilon < \epsilon_0(d)$, one has

$$T_\epsilon \geq \epsilon^{-1}\left(\frac{4}{3}\right)^\sigma. \tag{9}$$

In particular, if $\sigma \geq 1/\epsilon$, the stability time $T_\epsilon \geq \epsilon^{-1}\exp\left\{C\epsilon^{-1}\right\}$ is exponential about ϵ, where $C = \ln 4/3$.

Theorem 1.2. Gevrey space
There exists a subset $\mathcal{V} \subset \mathcal{W}_m$ of full measure and a small $\epsilon_0 = \epsilon_0(\sigma, m, d, s, W) > 0$ depending only on σ, m, d, s and W, such that for any $0 < \epsilon < \epsilon_0$, if $\|u(0,x)\|_\sigma \leq \epsilon$, then the solution $u(t,x)$ of (1) with the initial datum $u(0,x)$ satisfies

$$\|u(t,x)\|_\sigma \leq 2\epsilon, \quad \forall\ |t| \leq T_\epsilon,$$

where
(1) if $\psi(\mathbf{j},\sigma) = \exp\left\{\sigma \ln^2 [\mathbf{j}]\right\}$, the stability time T_ϵ is given by

$$T_\epsilon \geq \exp\left\{\frac{1}{2}\left|\ln\epsilon\right|^{1+\beta}\right\}; \tag{10}$$

(2) if $\psi(\mathbf{j},\sigma) = \exp\left\{\sigma\sqrt{|\mathbf{j}|}\right\}$, the stability time T_ϵ is given by

$$T_\epsilon \geq \exp\left\{-\frac{C_*\left|\ln\epsilon\right|^{4/3}}{\ln|\ln\epsilon|}\right\}, \tag{11}$$

where $0 < \beta < 1/7$, $0 < C_* \leq 1/200$.

Sketch of the proof and comments

The proof of (8) in Theorem 1.1, (10) and (11) in Theorem 1.2 follows from the idea of the **tame** property in various phase spaces (see Lemma 5.3) and an abstract BNF theorem which are developed by [30]. The proof of (9) follows from a **tame**-like inequality (70), which is given by [19].

Precisely, in the first step, the equation (6) is rewritten as an infinite dimensional Hamiltonian systems, where the Hamiltonian function given by

$$H(u,\bar{u}) = \int_{\mathbb{T}^d}\left(\left|\nabla_{\mathscr{L}} u\right|^{2s} + (V(x)*u)\bar{u} + \frac{1}{2}|u|^4\right)\mathrm{d}x \tag{12}$$

with the complex symplectic $\mathrm{i}\mathrm{d}u \wedge \mathrm{d}\bar{u}$. By Fourier series, the Hamiltonian (12) turns into

$$H(q,\bar{q}) = H_2(q,\bar{q}) + P(q,\bar{q}), \tag{13}$$

where

$$H_2 = \sum_{\mathbf{j}\in\mathbb{Z}^d} \omega_{\mathbf{j}} q_{\mathbf{j}} \bar{q}_{\mathbf{j}} \qquad \text{with} \qquad \omega_{\mathbf{j}} = |W\mathbf{j}|^{2s} + V_{\mathbf{j}} \tag{14}$$

and

$$P(q,\bar{q}) = \sum_{l,l'\in\mathbb{N}^{\mathbb{Z}^d}} P_{l,l'} q^l \bar{q}^{l'} \qquad \text{with} \qquad P_{l,l'} = \frac{1}{2(2\pi)^d} \tag{15}$$

satisfying

$$\sum_{\mathbf{j}\in\mathbb{Z}^d} (l_{\mathbf{j}} - l'_{\mathbf{j}}) = 0 \quad \text{and} \quad \sum_{\mathbf{j}\in\mathbb{Z}^d} \mathbf{j}(l_{\mathbf{j}} - l'_{\mathbf{j}}) = 0, \tag{16}$$

which are respectively called the **mass** conservation and the **momentum** conservation. Otherwise $P_{l,l'} = 0$.

Secondly, we will show that the Hamiltonian P satisfies **tame** property in various phase spaces when the weight function $\psi(\mathbf{j},\sigma)$ is chosen as

$$\exp\{\sigma \ln\langle\mathbf{j}\rangle\}, \quad \exp\{\sigma \ln^2 [\mathbf{j}]\}, \quad \exp\left\{\sigma\sqrt{|\mathbf{j}|}\right\},$$

which are based on a **tame** inequality (see Lemma 5.3 for the details).

Thirdly, using BNF technique under some suitable nonresonance conditions (see (27)), we obtain a partial normal form around the origin. Then combining with **tame** property, one obtains the long time stability result by optimizing the stability time T_ϵ.

§2 The tame norm of Hamiltonian vector field

2.1 Some notations

In this section, we first introduce some notations. Define the scale of the phase space

$$(q,\bar{q}) \in \mathcal{P}_\sigma(\mathbb{C}) := \ell^2_\sigma(\mathbb{C}) \oplus \ell^2_\sigma(\mathbb{C}).$$

where $\ell^2_\sigma(\mathbb{C})$ is given by (5). We identify a pair $z = (q,\bar{q}) \in \mathbb{C}^{\mathbb{Z}^d} \times \mathbb{C}^{\mathbb{Z}^d}$ with $(z_{\mathbf{a}})_{\mathbf{a}\in\mathfrak{Z}} \in \mathbb{C}^{\mathfrak{Z}}$ via the formula

$$\mathbf{a} = (\mathbf{j},\delta) \in \mathfrak{Z} \Rightarrow \begin{cases} z_{\mathbf{a}} = q_{\mathbf{j}}, & \text{if } \delta = 1, \\ z_{\mathbf{a}} = \bar{q}_{\mathbf{j}}, & \text{if } \delta = -1, \end{cases} \tag{17}$$

where $\mathfrak{Z} = \mathbb{Z}^d \times \{\pm 1\}$ and

$$\|z\|^2_\sigma := \|q\|^2_\sigma + \|\bar{q}\|^2_\sigma .$$

Denote by $B_{\mathbb{C},\sigma}(R)$ the open ball centered at the origin and of radius R in $\mathcal{P}_\sigma(\mathbb{C})$. For simplicity, we often write

$$\mathcal{P}_\sigma \equiv \mathcal{P}_\sigma(\mathbb{C}), \quad B_\sigma(R) \equiv B_{\mathbb{C},\sigma}(R).$$

Let $H : \mathcal{P}_\sigma \to \mathbb{C}$ be a homogeneous polynomial of degree n given by

$$H(z) = \sum_{\substack{l,l'\in\mathbb{N}^{\mathbb{Z}^d} \\ |l+l'|=n}} H_{l,l'} q^l \bar{q}^{l'},$$

where $H_{l,l'}$ is the coefficient of the monomial

$$q^l \bar{q}^{l'} := \prod_{\mathbf{j}\in\mathbb{Z}^d} q_{\mathbf{j}}^{l_{\mathbf{j}}} \bar{q}_{\mathbf{j}}^{l'_{\mathbf{j}}} \qquad \text{and} \qquad |l+l'| := \sum_{\mathbf{j}\in\mathbb{Z}^d} |l_{\mathbf{j}} + l'_{\mathbf{j}}|.$$

Then we define its modulus $\lfloor H \rceil$ by

$$\lfloor H \rceil(z) := \sum_{\substack{l,l' \in \mathbb{N}^{\mathbb{Z}^d} \\ |l+l'|=n}} |H_{l,l'}| q^l \bar{q}^{l'}.$$

Furthermore, it is naturally associated with a symmetric n-linear form $\mathcal{H}\left(z^{(1)}, \ldots, z^{(n)}\right)$ such that

$$\mathcal{H}(z, \ldots, z) = H(z),$$

and a symmetric $(n-1)$-linear form $\mathcal{X}_H\left(z^{(1)}, \ldots, z^{(n-1)}\right)$ such that

$$\mathcal{X}_H(z, \ldots, z) = X_H(z),$$

where $X_H : \mathcal{P}_\sigma \to \mathcal{P}_\sigma$ is the Hamiltonian vector field of homogeneous polynomial H and is defined by

$$X_H(z) := \mathrm{i}\left(\frac{\partial H}{\partial \bar{q}}, -\frac{\partial H}{\partial q}\right).$$

2.2 The tame norm of Hamiltonian vector field

Definition 1. *Let H be a homogeneous polynomial of degree $n+1$, and assume that H satisfies the following two conditions:*

1. ***Tame property**, i.e.*

$$A = \sup \frac{\|\lfloor \mathcal{X}_H \rceil(\omega)\|_\sigma}{\|\omega\|_\sigma^T} < \infty,$$

where

$$\|\omega\|_\sigma^T = \frac{1}{n}\sum_{l=1}^{n}\left(\left\|z^{(1)}\right\|_{\sigma'} \cdots \left\|z^{(l-1)}\right\|_{\sigma'} \left\|z^{(l)}\right\|_{\sigma} \left\|z^{(l+1)}\right\|_{\sigma'} \cdots \left\|z^{(n)}\right\|_{\sigma'}\right),$$

$0 < \sigma' < \sigma$ *and the sup is taken over all the multivectors* $\omega = \left(z^{(1)}, \ldots, z^{(n)}\right) \neq 0$;

2. ***Bounded property**, i.e.*

$$B = \sup \frac{\|\lfloor \mathcal{X}_H \rceil(\omega)\|_{\sigma'}}{\|\omega\|_{\sigma'}} < \infty,$$

where

$$\|\omega\|_{\sigma'} = \left\|z^{(1)}\right\|_{\sigma'} \cdots \left\|z^{(n)}\right\|_{\sigma'}$$

and the sup is taken over all the multivectors $\omega = \left(z^{(1)}, \ldots, z^{(n)}\right) \neq 0$.

Then the tame norm of the Hamiltonian vector field X_H is defined by

$$|H|_\sigma^T := \max\{A, B\}.$$

Furthermore, let H be a nonhomogeneous polynomial. Consider its Taylor expansion

$$H = \sum_{n \geq 3} H_n,$$

where H_n is a homogeneous polynomial of degree n. For any $R > 0$, we define

$$|H|_{\sigma,R}^T := \sum_{n \geq 3} |H_n|_\sigma^T R^{n-1}. \tag{18}$$

2.3 The Poisson bracket of two Hamiltonians

For two Hamiltonians H and F, we define the Poisson bracket by

$$\{H, F\} = \mathrm{i} \sum_{\mathbf{j} \in \mathbb{Z}^d} \left(\frac{\partial H}{\partial \bar{q}_{\mathbf{j}}} \frac{\partial F}{\partial q_{\mathbf{j}}} - \frac{\partial H}{\partial q_{\mathbf{j}}} \frac{\partial F}{\partial \bar{q}_{\mathbf{j}}} \right). \tag{19}$$

Lemma 2.1. Let H and F be homogeneous polynomials of degree $n+1$ and $m+1$, respectively. One has

$$|\{H, F\}|_{\sigma}^{T} \leq (n+m) |H|_{\sigma}^{T} |F|_{\sigma}^{T}. \tag{20}$$

Proof. The proof can be found in Lemma 2.4 in [19]. □

Definition 2. *Define the norm of the Hamiltonian vector field X_H by*

$$\|X_H\|_{\sigma}^{R} := \sup_{\|z\|_{\sigma} \leq R} \|X_H(z)\|_{\sigma}. \tag{21}$$

In fact, the tame norm (18) is stronger than the Hamiltonian vector field norm (21).

Lemma 2.2. For a given Hamiltonian H, one has

$$\|X_H\|_{\sigma}^{R} \leq |H|_{\sigma,R}^{T}. \tag{22}$$

Proof. The proof can be found in Lemma 2.7 in [19]. □

Remark 2.1. Let χ be an analytic Hamiltonian function with Hamiltonian vector field which is analytic as a map from $B_{\sigma}(R)$ to $\mathcal{P}_{\sigma}$. For any $0 < d < R$, assume that $|\chi|_{\sigma,R}^{T} < d$, and consider the time t flow Φ^t of X_{χ}. Then for $|t| < 1$, we have

$$\sup_{\|z\|_{\sigma} \leq R-d} \left\| \Phi_{\chi}^{t}(z) - z \right\|_{\sigma} \leq \|X_{\chi}\|_{\sigma}^{R-d} \leq |\chi|_{\sigma,R}^{T}. \tag{23}$$

Furthermore, for a vector $z = (z_{\mathbf{j}})_{\mathbf{j} \in \mathbb{Z}^d}$ and a given large $N > 0$, we define the projection

$$\widetilde{\Pi}_N z := \widetilde{z} = (\widetilde{z}_{\mathbf{j}})_{\mathbf{j} \in \mathbb{Z}^d}$$

by $\widetilde{z}_{\mathbf{j}} = z_{\mathbf{j}}$ when $|\mathbf{j}| \leq N$ and otherwise $\widetilde{z}_{\mathbf{j}} = 0$. Let $\widehat{z} := z - \widetilde{z}$. Then using the **tame** property (80), one has

Lemma 2.3. Assume that H has a zero of order three in the variable $\widehat{z}$. Then one has

$$\|X_H\|_{\sigma}^{R} \leq C(N) |H|_{\sigma,2R}^{T}, \tag{24}$$

where for $0 < \sigma' < \sigma$

$$C(N) = \frac{4}{\psi(\mathbf{j}^*, \sigma - \sigma')} \qquad \text{with} \qquad |\mathbf{j}^*| = N. \tag{25}$$

Proof. For the detailed proof process, see the proof of Lemma 3.8 in [38]. □

§3 Measure estimate

Lemma 3.1. For any sufficiently small γ, let N and r be large enough depending on W and m, d, s respectively, there exists a set $\mathcal{R} \subset \mathcal{W}_m$ satisfying

$$\text{mes } \mathcal{R} = O(\gamma), \tag{26}$$

such that for any $V \in (\mathcal{W}_m \setminus \mathcal{R})$, one has

$$|\langle k, \omega\rangle| \geq \frac{\gamma^{2/s_*+1}}{N^{r^3}}, \tag{27}$$

for $0 \neq k \in \mathbb{Z}^{\mathbb{Z}^d}$ with $|k| \leq r+2$ and $|\widehat{k}| \leq 2$, where $s_* = \min\{2s, 1-2s\}$ with $0 < s_* \leq 1/2$.

Proof. Define the resonant set $\mathcal{R}_k$ and $\mathcal{R}_k^*$ by

$$\mathcal{R}_{\widetilde{k}} := \left\{ V \in \mathcal{W}_m : \left|\langle \widetilde{k}, \omega\rangle\right| < \frac{\gamma}{N^{r^2}} \right\} \tag{28}$$

and

$$\mathcal{R}_k^* := \left\{ V \in \mathcal{W}_m : |\langle k, \omega\rangle| < \frac{\gamma^{2/s_*+1}}{N^{r^3}} \right\}.$$

Let $\mathcal{R} = \widetilde{\mathcal{R}} \bigcup \mathcal{R}^*$, where

$$\widetilde{\mathcal{R}} = \bigcup_{\substack{0 \neq \widetilde{k} \in \mathbb{Z}^{\mathbb{Z}^d} \\ |\widetilde{k}| \leq r+2}} \mathcal{R}_{\widetilde{k}} \quad \text{and} \quad \mathcal{R}^* = \bigcup_{\substack{0 \neq k \in \mathbb{Z}^{\mathbb{Z}^d} \\ |k| \leq r+2, 0 < |\widehat{k}| \leq 2}} \mathcal{R}_k^*.$$

Then it is easy to see that for any $V \in (\mathcal{W}_m \setminus \mathcal{R})$, the inequality (27) holds.

Now it suffices to prove the estimate (26) holds, which will be given in the following two steps.

Step 1: We will prove that mes $\widetilde{\mathcal{R}} = O(\gamma)$.

Note that $0 \neq \widetilde{k} \in \mathbb{Z}^{\mathbb{Z}^d}$ in $\mathcal{R}_{\widetilde{k}}$, then in view of (7) and (28), one has

$$\text{mes } \mathcal{R}_{\widetilde{k}} \leq \frac{\gamma}{N^{r^2-m}},$$

and then

$$\text{mes } \widetilde{\mathcal{R}} \leq \sum_{0 < |\widetilde{k}| \leq r+2} \frac{\gamma}{N^{r^2-m}} \leq \frac{\gamma(2N+1)^{d(r+2)}}{N^{r^2-m}} \leq \gamma,$$

where using the fact that r is large enough depending on m and d. Thus, one finishes the proof of Step 1.

Step 2: We will prove that mes $\mathcal{R}^* = O(\gamma)$.

Write

$$\mathcal{R}^* = \bigcup_{i=1}^{2} \mathcal{R}_i^* \qquad \text{with} \qquad \mathcal{R}_i^* = \bigcup_{\substack{0 \neq k \in \mathbb{Z}^{\mathbb{Z}^d} \\ |k| \leq r+2, |\widehat{k}| = i}} \mathcal{R}_k^*.$$

Case 1. $|\widehat{k}| = 1$. Then one has

$$\langle k, \omega\rangle = \langle \widetilde{k}, \omega\rangle \pm \omega_{\mathbf{l}},$$

where $|\mathbf{l}| > N$. According to the **momentum** conservation (16), we have $|\mathbf{l}| \leq rN$. Then, following the proof of Step 1, we get mes $\mathcal{R}_1^* = O(\gamma)$.

Case 2. $|\widehat{k}| = 2$. Then one has

$$\langle k, \omega\rangle = \langle \widetilde{k}, \omega\rangle + k_{\mathbf{l}_1}\omega_{\mathbf{l}_1} + k_{\mathbf{l}_2}\omega_{\mathbf{l}_2}, \tag{29}$$

where $k_{\mathbf{l}_1}, k_{\mathbf{l}_2} \in \{-1, 1\}$ and $|\mathbf{l}_1|, |\mathbf{l}_2| > N$.

Subcase 2.1. $k_{\mathbf{l}_1}k_{\mathbf{l}_2} = 1$. Without loss of generality, we assume that

$$k_{\mathbf{l}_1} = 1, \quad k_{\mathbf{l}_2} = 1 \quad \text{and} \quad |\mathbf{l}_1| \leq |\mathbf{l}_2|.$$

Note that by (74)

$$\left|\langle\widetilde{k},\omega\rangle\right| \le |\widetilde{k}| \max_{|\mathbf{j}|\le N}\left(|W\mathbf{j}|^{2s}+1\right) \le r(CN+1)^{2s} \le r^2N, \tag{30}$$

and

$$|\omega_{\mathbf{l}_1}| = |W\mathbf{l}|^{2s} + V_{\mathbf{l}_1} \ge \left(C^{-1}|\mathbf{l}_1|\right)^{2s} - 1,$$

where $C = C(W,d) > 1$ and $0 < s < 1/2$. Then if $|\mathbf{l}_1|^{2s} > r^3N$, one has by (29)

$$|\langle k,\omega\rangle| \ge 2|\omega_{\mathbf{l}_1}| - \left|\langle\widetilde{k},\omega\rangle\right| \ge 2r^2N - 2 - r^2N \ge 1,$$

which is not small. Hence we always assume that $|\mathbf{l}|^{2s} \le r^3N$, following the proof of Step 1, we obtain mes $\mathcal{R}_2^* = O(\gamma)$.

Subcase 2.2. $k_{\mathbf{l}_1}k_{\mathbf{l}_2} = -1$. Without loss of generality, we assume that

$$k_{\mathbf{l}_1} = 1, \quad k_{\mathbf{l}_2} = -1 \quad \text{and} \quad |\mathbf{l}_1| \le |\mathbf{l}_2|.$$

Then we have by (29)

$$|\langle k,\omega\rangle| \ge \left|\langle\widetilde{k},\omega\rangle\right| - \left||W\mathbf{l}_2|^{2s} - |W\mathbf{l}_1|^{2s}\right| - |V_{\mathbf{l}_1} - V_{\mathbf{l}_2}|. \tag{31}$$

Firstly, for any $V \in \left(\mathcal{W}_m \setminus \widetilde{\mathcal{R}}\right)$, according to (28) we have

$$\left|\langle\widetilde{k},\omega\rangle\right| \ge \frac{\gamma}{N^{r^2}}. \tag{32}$$

Secondly, by (75) in Lemma 5.2, one has

$$\left||W\mathbf{l}_2|^{2s} - |W\mathbf{l}_1|^{2s}\right| \le \frac{2\left||W\mathbf{l}_2| - |W\mathbf{l}_1|\right|}{|W\mathbf{l}_1|^{s_*}}, \tag{33}$$

where $s_* = \min\{2s, 1-2s\}$ wit momentum conservation (16), we get

$$2\left||W\mathbf{l}_2| - |W\mathbf{l}_1|\right| \le 2|W\mathbf{l}_2 - W\mathbf{l}_1| \le 2rCN, \tag{34}$$

where $C = C(W,d) > 1$. On the other hand, by (74) again and when $|\mathbf{l}_1|^{s_*} > \gamma^{-1}N^{2r^2}$, we have

$$|W\mathbf{l}_1|^{s_*} \ge \left(C^{-1}|\mathbf{l}_1|\right)^{s_*} \ge (C\gamma)^{-1}N^{2r^2}. \tag{35}$$

Therefore, in view of (33)-(35), we obtain

$$\left||W\mathbf{l}_2|^{2s} - |W\mathbf{l}_1|^{2s}\right| \le \frac{\gamma}{4N^{r^2}}. \tag{36}$$

Finally, one has by (7)

$$|V_{\mathbf{l}_1} - V_{\mathbf{l}_2}| \le \frac{R}{\langle\mathbf{l}_1\rangle^m} \le \frac{\gamma}{4N^{r^2}}, \tag{37}$$

where using $m > d/2$ and N large enough depending on R. Thus, for any $V \in \left(\mathcal{W}_m \setminus \widetilde{\mathcal{R}}\right)$, in view of (31), (32), (36) and (37), one has

$$|\langle k,\omega\rangle| \ge \frac{\gamma}{2N^{r^2}},$$

which is not small.

Now we only consider $|\mathbf{l}_1|^{s_*} \le \gamma^{-1}N^{2r^2}$, which implies

$$|\mathbf{l}_1| \le \left(\gamma^{-1}N^{2r^2}\right)^{1/s_*}.$$

Based on the momentum conservation (16), one has

$$|\mathbf{l}_2| \le \left(\gamma^{-1}N^{2r^2}\right)^{1/s_*} + rN.$$

Thus, following the proof of Step 1, we also obtain mes $\mathcal{R}_2^* = O(\gamma)$.

To sum up, according to Case 1 and Case 2, one finishes the proof of Step 2. In view of Step 1 and Step 2, one finishes the proof of (26). $\square$

§4 Normal form theorem

4.1 Iterative Lemma

In this subsection, we will construct a partial BNF of high order, which mostly follows from the BNF iteration given in [30] under the following nonresonant conditions. We say the frequency ω defined by (14) is nonresonance if ω satisfies for $0 \neq k \in \mathbb{Z}^{\mathbb{Z}^d}$ with $|k| \leq r+2$ and $|\widehat{k}| \leq 2$,

$$|\langle k, \omega\rangle| \geq \frac{\gamma_s}{N^{r^3}}, \tag{38}$$

where $0 < \gamma_s \ll 1$ is given by (27) in Lemma 3.1.

Definition 3. *(r, N)-normal form: We say that a polynomial W is in (r, N)-normal form if it can be written*

$$W(q, \bar{q}) = \sum_{n=3}^{r} \sum_{\substack{|l+k|=n,\ l,k \in \mathbb{N}^{\mathbb{Z}^d} \\ l=k \ \text{or}\ |\widehat{l}+\widehat{k}| \geq 3}} W_{l,k} q^l \bar{q}^k.$$

Namely, $W := Z + \boldsymbol{W}$, where Z depends only on the action variables I (in the case $l = k$) and $\boldsymbol{W}$ has a zero of order three in the variables $\widehat{z}$ (in the case $|\widehat{l}+\widehat{k}| \geq 3$).

Now we introduce the recursive equation. The solution of recursive equation can generate a canonical transformation Φ such that in the new variables, the Hamiltonian $H_2 + P$ given by (13) is in normal form modulo a small remainder term. To obtain the recursive equation, we consider the following problem.

Seek polynomials χ and W in normal form and a smooth Hamiltonian $\mathbf{R}$ satisfying $\partial^\alpha \mathbf{R}(0) = 0$ for all $\alpha \in \mathbb{N}^{\mathbb{Z}^d}$ with $|\alpha| \leq r$, namely a polynomial of degree at most $r+1$, such that

$$(H_2 + P) \circ \Phi_\chi^1 = H_2 + W + \mathbf{R}, \tag{39}$$

where Φ_χ^1 is the time-1 map of the Hamiltonian vector field Φ_χ,

$$\chi(q, \bar{q}) = \sum_{n=3}^{r} \chi_n(q, \bar{q}), \qquad \chi_n(q, \bar{q}) = \sum_{\substack{|l+l'|=n \\ l,l' \in \mathbb{N}^{\mathbb{Z}^d}}} \chi_{l,l'} q^l \bar{q}^{l'},$$

$$W(q, \bar{q}) = \sum_{n=3}^{r} W_n(q, \bar{q}), \qquad W_n(q, \bar{q}) = \sum_{\substack{|l+l'|=n \\ l,l' \in \mathbb{N}^{\mathbb{Z}^d}}} W_{l,l'} q^l \bar{q}^{l'}$$

and

$$\mathbf{R}(q, \bar{q}) = \sum_{n \geq r+1} \mathbf{R}_n(q, \bar{q}), \qquad \mathbf{R}_n(q, \bar{q}) = \sum_{\substack{|l+l'|=n \\ l,l' \in \mathbb{N}^{\mathbb{Z}^d}}} \mathbf{R}_{l,l'} q^l \bar{q}^{l'}.$$

For two Hamiltonians χ and K, we have for all $k \geq 0$,

$$\frac{\mathrm{d}^k}{\mathrm{d}t^k}(K \circ \Phi_\chi^t) = \underbrace{\{\chi, \{\cdots\{\chi, K\}\cdot\}\}}_{k-\text{fold}}(\Phi_\chi^t) = (\mathrm{ad}_\chi^k K)(\Phi_\chi^t),$$

where $\mathrm{ad}_\chi K = \{\chi, K\}$.

After a straightforward calculation, we obtain the recursive equations

$$\{\chi_n, H_2\} - W_n = Q_n, \quad n = 3, \cdots, r, \tag{40}$$

where

$$\begin{aligned} Q_n = &- P_n + \sum_{k=3}^{n-1} \{P_{n+2-k}, \chi_k\} \\ &+ \sum_{k=1}^{n-3} \frac{B_k}{k!} \sum_{\substack{\ell_1+\cdots+\ell_{k+1}=n+2k \\ 3\le \ell_i \le n-k}} \mathrm{ad}_{\chi_{\ell_1}} \cdots \mathrm{ad}_{\chi_{\ell_k}} (W_{\ell_{k+1}} - P_{\ell_{k+1}}), \end{aligned} \tag{41}$$

and B_k with $1 \le k \le n-3$ are the Bernoulli numbers.

Once these recursive equations are solved, we define the remainder term as

$$\mathbf{R} = (H_2 + P) \circ \Phi_\chi^1 - H_2 - W.$$

By construction, $\mathbf{R}$ is analytic in a neighborhood of the origin in $\mathcal{P}_\sigma$. Thus, by the Taylor's formula,

$$\begin{aligned} \mathbf{R} = &\sum_{n\ge r+1} \sum_{k=2}^{n-2} \frac{1}{k!} \sum_{\substack{\ell_1+\cdots+\ell_k=n+2k-2 \\ 3\le \ell_i \le r}} \mathrm{ad}_{\chi_{\ell_1}} \cdots \mathrm{ad}_{\chi_{\ell_k}} H_2 \\ &+ \sum_{n\ge r+1} \sum_{k=0}^{n-3} \frac{1}{k!} \sum_{\substack{\ell_1+\cdots+\ell_{k+1}=n+2k \\ 3\le \ell_1,\cdots,\ell_k\le r, \ell_{k+1}\ge 3}} \mathrm{ad}_{\chi_{\ell_1}} \cdots \mathrm{ad}_{\chi_{\ell_k}} P_{\ell_{k+1}}. \end{aligned} \tag{42}$$

Lemma 4.1. The homological equation
Consider the Hamiltonian H_2 given by (13). Suppose that the nonresonance conditions (38) are satisfied, and

$$Q_n(q,\bar{q}) = \sum_{\substack{|l+l'|=n \\ l,l'\in \mathbb{N}^{\mathbb{Z}^d}}} Q_{l,l'} q^l \bar{q}^{l'}$$

is a homogeneous polynomial of degree n. Then the homological equation

$$\{\chi_n, H_2\} - W_n = Q_n \tag{43}$$

admits a polynomial solution (χ_n, W_n) homogeneous of degree n satisfying the following estimates

$$|W_n|_\sigma^T \le |Q_n|_\sigma^T \qquad \text{and} \qquad |\chi_n|_\sigma^T \le \gamma_s^{-1} N^{n^3} |Q_n|_\sigma^T. \tag{44}$$

Proof. Let

$$W_n(q,\bar{q}) = \sum_{\substack{|l+l'|=n \\ l,l'\in \mathbb{N}^{\mathbb{Z}^d}}} W_{l,l'} q^l \bar{q}^{l'} \qquad \text{and} \qquad \chi_n(q,\bar{q}) = \sum_{\substack{|l+l'|=n \\ l,l'\in \mathbb{N}^{\mathbb{Z}^d}}} \chi_{l,l'} q^l \bar{q}^{l'}.$$

Then (43) can be written in terms of polynomial coefficients

$$\mathrm{i}\,\langle l'-l,\ \omega\rangle\, \chi_{l,l'} - W_{l,l'} = Q_{l,l'}.$$

We then define

- $|\widehat{l}+\widehat{l'}| \ge 3$ or $l = l'$, $W_{l,l'} = -Q_{l,l'}$, $\chi_{l,l'} = 0$;
- $|\widehat{l}+\widehat{l'}| \le 2$ and $l \ne l'$, $W_{l,l'} = 0$, $\chi_{l,l'} = \dfrac{Q_{l,l'}}{\mathrm{i}\,\langle l'-l,\omega\rangle}$.

Since the frequency ω satisfies the nonresonance conditions (38), then we finish the proof of (44) in view of $|l'-l| \leq |l+l'| = n$. $\square$

Lemma 4.2. The estimate of the solution and the remainder term
Consider the Hamiltonian $H = H_2 + P$ given by (13). Suppose that the frequency ω satisfies the nonresonance conditions (27). Then there exists positive constants $C = C(\gamma, s)$ depending on γ, s such that for any r and N, and for $n = 3, \cdots r$, there exists two homogeneous polynomials χ_n and W_n of degree n, which are solutions of the recursive equation (40) and satisfy

$$|\chi_n|_\sigma^T + |W_n|_\sigma^T \leq (CNn)^{n^4}. \tag{45}$$

Moreover, the remainder term $\mathbf{R}$ given in (42) can be rewritten as

$$\mathbf{R}(z) = \sum_{n \geq r+1} \mathbf{R}_n(z).$$

Then $\mathbf{R}_n$ satisfies

$$|\mathbf{R}_n|_\sigma^T \leq (CNn)^{10nr^3}. \tag{46}$$

Proof. The proofs of (45) and (46) follow from the proofs of Lemma 4.5 and Lemma 4.6 in [38], respectively. $\square$

4.2 The Birkhoff normal form theorem

Based on the Iterative Lemma in subsection 4.1, we will construct the BNF theorem by choosing suitable N and r in phase spaces $\mathcal{P}_\sigma(\mathbb{C})$.

According to Lemma 5.3 and Remark 5.1, by choosing respectively

$$\sigma' = 4\sigma/5, \qquad \text{for} \quad \psi(j,\sigma) = \exp\left\{\sigma \ln^2 [\mathbf{j}]\right\}, \tag{47}$$

$$\sigma' = \sigma/2, \qquad \text{for} \quad \psi(j,\sigma) = \exp\left\{\sigma \sqrt{|\mathbf{j}|}\right\} \tag{48}$$

in Lemma 2.3,we have the following theorem:

Theorem 4.1. Assume that the frequency ω satisfies the nonresonance condition (38) and P is analytic on a ball $B_\sigma(R_0)$ for some $0 < R_0 < 1$.

Then there exists a constant $\epsilon_0 = \epsilon_0(\sigma, m, d, s, W) > 0$ depending only on σ, m, d, s, W such that for any $0 < \epsilon < \epsilon_0$ one can find polynomials $\chi, Z, \mathbf{W}$ and a Hamiltonian $\mathbf{R}$ analytic on $B_\sigma(4\epsilon)$ such that

$$(H_2 + P) \circ \Phi_\chi^1 = H_2 + Z + \mathbf{W} + \mathbf{R}, \tag{49}$$

where Z depends on the action variable I only, $\mathbf{W}$ and $\mathbf{R}$ satisfy
(1) for $\psi(\mathbf{j},\sigma) = \exp\left\{\sigma \ln^2 [\mathbf{j}]\right\}$,

$$\|X_{\mathbf{W}}\|_\sigma^{2\epsilon} + \|X_{\mathbf{R}}\|_\sigma^{2\epsilon} \leq \epsilon^{3/2} \exp\left\{-\frac{1}{2}|\ln \epsilon|^{1+\beta}\right\}; \tag{50}$$

(2) for $\psi(\mathbf{j},\sigma) = \exp\left\{\sigma \sqrt{|\mathbf{j}|}\right\}$,

$$\|X_{\mathbf{W}}\|_\sigma^{2\epsilon} + \|X_{\mathbf{R}}\|_\sigma^{2\epsilon} \leq \epsilon^{3/2} \exp\left\{-\frac{C_* |\ln \epsilon|^{4/3}}{\ln |\ln \epsilon|}\right\}, \tag{51}$$

where $0 < \beta < 1/7$, $0 < C_* \leq 1/200$.

Proof. Without loss of generality, we will focus solely on proving (50). Let

$$N = \exp\left\{ \frac{2}{\sqrt{\sigma}} |\ln \epsilon|^{\frac{1+\beta}{2}} \right\} \qquad \text{and} \qquad r = |\ln \epsilon|^{\beta}, \tag{52}$$

where $0 < \beta < 1/7$.

On one hand, by (45), we get for $3 \leq k \leq r$

$$|\chi_k|_\sigma^T \leq \exp\left\{k^4 \ln(CkN)\right\} \leq \exp\left\{3kr^3 \ln N\right\} \leq \epsilon^{-k/8},$$

where the last inequality is based on the fact

$$3r^3 \ln N = \frac{6}{\sqrt{\sigma}} |\ln \epsilon|^{\frac{1+7\beta}{2}} \leq \frac{1}{8} |\ln \epsilon|, \tag{53}$$

which follows from (52) and ϵ sufficiently small depending on σ. Then according to (18), we have for any $R \leq \epsilon$,

$$|\chi|_{\sigma,4R}^T \leq \sum_{k=3}^{r} |\chi_k|_\sigma^T (4R)^{k-1} \leq \sum_{k=3}^{r} \epsilon^{-k/8} (4\epsilon)^{k-1} \leq \epsilon^{3/2}. \tag{54}$$

Similarly, one also has

$$|W|_{\sigma,4R}^T \leq \epsilon^{3/2} \qquad \text{and} \qquad |\mathbf{W}|_{\sigma,4R}^T \leq \epsilon^{3/2}.$$

where the last inequality follows from Definition (3).

Thus, according to (24), (47) and (52), we have

$$\|X_{\mathbf{W}}\|_\sigma^{2R} \leq \frac{4|\mathbf{W}|_{\sigma,4R}^T}{e^{(\sigma-\sigma')\ln^2[N]}} \leq 4\epsilon^{3/2} \exp\left\{ -\frac{4}{5} |\ln \epsilon|^{1+\beta} \right\}. \tag{55}$$

On the other hand, we obtain from (46) and (53) that

$$|\mathbf{R}_k|_\sigma^T \leq \exp\left\{10kr^3 \ln(CkN)\right\} \leq \exp\left\{30kr^3 \ln N\right\} \leq \epsilon^{-k/8}.$$

Then one has for $R \leq \epsilon$,

$$|\mathbf{R}|_{\sigma,2R}^T = \sum_{k \geq r+1} |\mathbf{R}_k|_\sigma^T (2R)^{k-1} \leq \sum_{k \geq r+1} \epsilon^{-k/8} (2\epsilon)^{k-1} \leq \epsilon^{\frac{4}{5}r}.$$

Combining (22) and (52), we have

$$\|X_{\mathbf{R}}\|_\sigma^{2R} \leq \exp\left\{ -\frac{4}{5} |\ln \epsilon|^{1+\beta} \right\} \leq \epsilon^2 \exp\left\{ -\frac{3}{5} |\ln \epsilon|^{1+\beta} \right\}. \tag{56}$$

To sum up, in view of (55) and (56), one obtains (50).

Similarly, for $\psi(\mathbf{j}, \sigma) = e^{\sigma\sqrt{|\mathbf{j}|}}$, let

$$N = \left(\frac{C|\ln \epsilon|^{4/3}}{\sigma \ln|\ln \epsilon|} \right)^2 \qquad \text{and} \qquad r = \frac{C|\ln \epsilon|^{1/3}}{\ln|\ln \epsilon|}, \tag{57}$$

where $C = 2C_*$ with $0 < C \leq 1/100$. Then following the proof of (50) and combining (48), we also obtain (51). $\square$

Similarly, according to Lemma 5.3 and Remark 5.1, by choosing

$$\sigma' = \sigma - r^5 \qquad \text{for} \quad \psi(j, \sigma) = \exp\{\sigma \ln \langle \mathbf{j} \rangle\} \tag{58}$$

in Lemma 2.3, we have the following result:

Theorem 4.2. Assume that the frequency ω satisfies the nonresonance condition (38) and P is analytic on a ball $B_\sigma(R_0)$ for some $0 < R_0 < 1$. Given any large $r > 0$, then there exists a constant $\epsilon_0 = \epsilon_0(\sigma, m, d, s, W, r) > 0$ depending only on σ, m, d, s, W, r such that for any $0 < \epsilon < \epsilon_0$ one can find polynomials $\chi, Z, \mathbf{W}$ and a Hamiltonian $\mathbf{R}$ analytic on $B_\sigma(4\epsilon)$ such that

$$(H_2 + P) \circ \Phi_\chi^1 = H_2 + Z + \mathbf{W} + \mathbf{R},$$

where Z depends on the action variable I only, $\mathbf{W}$ and $\mathbf{R}$ satisfy

$$\|X_{\mathbf{W}}\|_\sigma^{2\epsilon} + \|X_{\mathbf{R}}\|_\sigma^{2\epsilon} \le \epsilon^{3/2} \exp\left\{-\frac{4}{5} r \left|\ln \epsilon\right|\right\}. \tag{59}$$

Proof. Let

$$N = \exp\left\{r^{-4} |\ln \epsilon|\right\} \qquad \text{for any} \quad r \ge 1. \tag{60}$$

Then following the proof of (55) and (56), combining with (58), we can deduce that

$$\|X_{\mathbf{W}}\|_\sigma^{2R} \le 4\epsilon^{3/2} \exp\left\{-r \left|\ln \epsilon\right|\right\} \qquad \text{and} \qquad \|X_{\mathbf{R}}\|_\sigma^{2R} \le \epsilon^2 \exp\left\{-\frac{4}{5} r |\ln \epsilon|\right\}. \tag{61}$$

Thus, in view of (61), one obtains (59). $\square$

By the partial normal form constructed in Theorem 4.1, one can obtain the long time stability result.

Theorem 4.3. Long time stability result
Consider the partial BNF constructed in Theorem 4.1 (see (49)). If the initial datum $q(0)$ satisfies $\|q(0)\|_\sigma \le \epsilon$, then one has

$$\|q(t)\|_\sigma \le 2\epsilon, \quad \forall\ |t| \le T_\epsilon, \tag{62}$$

where

(1) $$T_\epsilon \ge \exp\left\{\frac{4}{5} r \left|\ln \epsilon\right|\right\}, \quad \text{for } \psi(\mathbf{j}, \sigma) = \exp\left\{\sigma \ln \langle \mathbf{j} \rangle\right\}; \tag{63}$$

(2) $$T_\epsilon \ge \exp\left\{\frac{1}{2} \left|\ln \epsilon\right|^{1+\beta}\right\}, \quad \text{for } \psi(\mathbf{j}, \sigma) = \exp\left\{\sigma \ln^2 [\mathbf{j}]\right\}; \tag{64}$$

(3) $$T_\epsilon \ge \exp\left\{-\frac{C_* \left|\ln \epsilon\right|^{4/3}}{\ln |\ln \epsilon|}\right\}, \quad \text{for } \psi(\mathbf{j}, \sigma) = \exp\left\{\sigma \sqrt{|\mathbf{j}|}\right\}; \tag{65}$$

where $0 < \beta < 1/7$, $0 < C_* \le 1/200$.

Proof. In view of (23) and (54), one has the transformation Φ_χ^1 is close to the identity, i.e. $q = \Phi_\chi^1(\breve{q}) = \breve{q} + O(\|\breve{q}\|_\sigma^2)$. Thus it suffices to prove that if $\|\breve{q}(0)\|_\sigma \le \epsilon$, then

$$\|\breve{q}(t)\|_\sigma \le 2\epsilon, \quad \forall\ |t| \le T_\epsilon.$$

Without loss of generality, we only need to prove (63). Let

$$\breve{I}(t) = \sum_{\mathbf{j} \in \mathbb{Z}^d} |\breve{q}_{\mathbf{j}}|^2 e^{2\sigma \ln \langle \mathbf{j} \rangle}.$$

On one hand, one has by (49)

$$\left|\frac{\mathrm{d}}{\mathrm{d}t} \breve{I}(t)\right| = \left|\left\{\breve{I}, \mathbf{W} + \mathbf{R}\right\}\right| \le 2 \left\|\breve{q}(t)\right\|_\sigma \left(\|X_{\mathbf{W}}\|_\sigma + \|X_{\mathbf{R}}\|_\sigma\right).$$

On the other hand, using Newton-Leibnitz formula

$$\breve{I}(t)-\breve{I}(0)=\int_0^t \frac{\mathrm{d}\breve{I}}{\mathrm{d}s}(s)\mathrm{d}s,$$

we have

$$|\breve{I}(t)|\leq|\breve{I}(0)|+\int_0^t 2\left\|\breve{q}(s)\right\|_\sigma\left(\left\|X_{\mathbf{W}}\right\|_\sigma+\left\|X_{\mathbf{R}}\right\|_\sigma\right)\mathrm{d}s,$$

where the last inequality is based on (66).

Define

$$T^*=\inf\left\{|t|:\left\|\breve{q}(t)\right\|_\sigma=2\epsilon\right\}$$

and we will prove that

$$T^*\geq\exp\left\{\frac{4}{5}r\left|\ln\epsilon\right|\right\}. \tag{66}$$

If (66) does not hold, then by using (59)

$$4\epsilon^2=|\breve{I}(T^*)|\leq|\breve{I}(0)|_\sigma+4\epsilon\int_0^{T^*}\left(\left\|X_{\mathbf{W}}\right\|_\sigma^{2\epsilon}+\left\|X_{\mathbf{R}}\right\|_\sigma^{2\epsilon}\right)\mathrm{d}s\leq\epsilon^2+2\epsilon^2=3\epsilon^2,$$

which is impossible. This completes the proof of (63).

Similarly, following the proof of (63), we obtain the estimates (64) and (65). □

§5 The proof of main result

5.1 Proof of Theorem 1.2

In this subsection, we will prove (10) and (11) in Theorem 1.2 by using Theorem 4.3. To this end, it suffices to estimate the tame norm of the Hamiltonian P (see (15)). Given any two vectors $q,q'\in\ell^2_\sigma(\mathbb{C})$, define the convolution $q*q'\in\mathbb{C}^{\mathbb{Z}^d}$ by

$$(q*q')_{\mathbf{j}}:=\sum_{\mathbf{l}\in\mathbb{Z}^d}q_{\mathbf{j}-\mathbf{l}}\cdot q'_{\mathbf{l}}. \tag{67}$$

Then we will give the proof of Theorem 1.2 as follows.

Proof. Firstly, in view of 12 and 15, one has $X_P(u)=|u|^2u$, and then

$$\lfloor\mathcal{X}_P\rceil\left(u^{(1)},u^{(2)},u^{(3)}\right)=\frac{1}{6}\sum_\tau a(x)u^{\tau(1)}u^{\tau(2)}u^{\tau(3)},$$

where τ are all the permutations of the first 3 integers.

Secondly, write

$$u^{(i)}=\sum_{\mathbf{j}\in\mathbb{Z}^d}q^{(i)}_{\mathbf{j}}\phi_{\mathbf{j}},\quad i=1,2,3,$$

then one has

$$u^{(1)}u^{(2)}u^{(3)}=q^{(1)}*q^{(2)}*q^{(3)}. \tag{68}$$

Finally, in view of (15), Definition 1, Definition 2 and (80), there exists a positive constant $C(\sigma)$ depending on σ only such that

$$|P|^T_{\sigma,R}\leq C(\sigma)R^3. \tag{69}$$

Then using Theorem 4.3, we finish the proof of (10) and (11) in Theorem 1.2. □

5.2 Proof of Theorem 1.1

Following the proof of Theorem 1.2, we can deduce (8) in Theorem 1.1. Consequently, in this subsection, we will focus solely on proving (9) in Theorem 1.1. To this end, it suffices to prove the tame-like property

$$\|q * q' * q''\|_\sigma \le \widetilde{C}(d)\left(\frac{3}{4}\right)^\sigma \|q\|_\sigma \|q'\|_\sigma \|q''\|_\sigma \tag{70}$$

for any $\sigma > d$, where constant $\widetilde{C}(d)$ depending on d only. Then, we will proceed with the proof of Theorem 1.1 as follows.

Proof. Firstly, we will prove (70) as follows.

Given any $\mathbf{j}_1, \mathbf{j}_2, \mathbf{j}_3 \in \mathbb{Z}^d$ and letting $\mathbf{j} = \mathbf{j}_1 - \mathbf{j}_2 + \mathbf{j}_3$, then one has

$$\lfloor \mathbf{j} \rfloor \le 3\max\{\lfloor \mathbf{j}_1 \rfloor, \lfloor \mathbf{j}_2 \rfloor, \lfloor \mathbf{j}_3 \rfloor\}. \tag{71}$$

Without loss of generality, we assume that

$$\max\{\lfloor \mathbf{j}_1 \rfloor, \lfloor \mathbf{j}_2 \rfloor, \lfloor \mathbf{j}_3 \rfloor\} = \lfloor \mathbf{j}_1 \rfloor. \tag{72}$$

Noting that $\lfloor \mathbf{j}_2 \rfloor, \lfloor \mathbf{j}_3 \rfloor \ge 2$, then using (71) and (72), one has

$$\lfloor \mathbf{j} \rfloor^{2\sigma} \le 3^{2\sigma} \lfloor \mathbf{j}_1 \rfloor^{2\sigma} \le 9^d \left(\frac{3}{4}\right)^{2(\sigma-d)} \lfloor \mathbf{j}_1 \rfloor^{2\sigma} \lfloor \mathbf{j}_2 \rfloor^{2(\sigma-d)} \lfloor \mathbf{j}_3 \rfloor^{2(\sigma-d)}. \tag{73}$$

Hence we have

$$\begin{aligned}
\|q * q' * q''\|_\sigma^2 &= \sum_{\mathbf{j}\in\mathbb{Z}^d} \left(\left| \sum_{\substack{\mathbf{j}_1,\mathbf{j}_2,\mathbf{j}_3\in\mathbb{Z}^d \\ \mathbf{j}_1-\mathbf{j}_2+\mathbf{j}_3=\mathbf{j}}} q_{\mathbf{j}_1} q'_{\mathbf{j}_2} q''_{\mathbf{j}_3} \right| \right)^2 \lfloor \mathbf{j} \rfloor^{2\sigma} \\
&\le 9^d \left(\frac{3}{4}\right)^{2(\sigma-d)} \sum_{\mathbf{j}\in\mathbb{Z}^d} \left(\sum_{\substack{\mathbf{j}_1,\mathbf{j}_2,\mathbf{j}_3\in\mathbb{Z}^d \\ \mathbf{j}_1-\mathbf{j}_2+\mathbf{j}_3=\mathbf{j}}} \left(\lfloor \mathbf{j}_1 \rfloor^{\sigma} |q_{\mathbf{j}_1}|\right) \left(\lfloor \mathbf{j}_2 \rfloor^{\sigma-d} |q'_{\mathbf{j}_2}|\right) \left(\lfloor \mathbf{j}_3 \rfloor^{\sigma-d} |q''_{\mathbf{j}_3}|\right) \right)^2 \\
&\quad \text{(in view of (73))} \\
&\le 9^d \left(\frac{3}{4}\right)^{2(\sigma-d)} \left(\sum_{\mathbf{j}_1\in\mathbb{Z}^d} \lfloor \mathbf{j}_1 \rfloor^{2\sigma} |q_{\mathbf{j}_1}|^2 \right) \left(\sum_{\mathbf{j}_2\in\mathbb{Z}^d} \lfloor \mathbf{j}_2 \rfloor^{\sigma-d} |q'_{\mathbf{j}_2}| \right)^2 \left(\sum_{\mathbf{j}_3\in\mathbb{Z}^d} \lfloor \mathbf{j}_3 \rfloor^{\sigma-d} |q''_{\mathbf{j}_3}| \right)^2 \\
&\quad \text{(using } \|a * b\|_{\ell^2} \le \|a\|_{\ell^2} \|b\|_{\ell^1},\ a \in \ell^2, b \in \ell^1\text{)} \\
&\le 9^d \left(\frac{3}{4}\right)^{2(\sigma-d)} C^2(d) \|q\|_\sigma^2 \|q'\|_\sigma^2 \|q''\|_\sigma^2,
\end{aligned}$$

where the last inequality uses Cauchy inequality and

$$C(d) = \sum_{\mathbf{l}\in\mathbb{Z}^d} \lfloor \mathbf{l} \rfloor^{-2d}.$$

Taking $\widetilde{C}(d) = 4^d C(d)$, we finish the proof of (70).

Finally, in view of (68) and (70), one has

$$\|X_P\|_\sigma^{2\epsilon} \le \widetilde{C}(d)\left(\frac{3}{4}\right)^\sigma (2\epsilon)^3 \le 2\epsilon^2 \left(\frac{3}{4}\right)^\sigma$$

where using the fact that $0 < \epsilon < \epsilon_0(d)$ sufficiently small. And we finish the proof of (9). □

Lemma 5.1. Assuming $W = (w_{ij})_{1\le i,j\le d}$ is a $d\times d$ invertible matrix, then there exists a constant $C = C(W,d) > 1$ depending only on W and d such that

$$C^{-1}\,|\mathbf{j}| \le |W\mathbf{j}| \le C|\mathbf{j}|. \tag{74}$$

Lemma 5.2. Given any two numbers $x_1, x_2 \in \mathbb{R}$ with $x_2 \ge x_1 > 1$, then for any $0 < s < 1/2$, one has

$$x_2^{2s} - x_1^{2s} \le \frac{2(x_2 - x_1)}{x_1^{s_*}}, \tag{75}$$

where $s_* = \min\{2s, 1-2s\}$ with $0 < s_* \le 1/2$.

Proof. **Case 1**: $0 < s \le 1/4$.

Then one has

$$x_2^{2s} - x_1^{2s} = \frac{x_2^{4s} - x_1^{4s}}{x_2^{2s} + x_1^{2s}} \le \frac{x_2 - x_1}{x_2^{2s} + x_1^{2s}}, \tag{76}$$

where by using the fact

$$x_2^{4s} - x_1^{4s} \le x_2 - x_1.$$

Case 2: $1/4 < s < 1/2$.

Then one has

$$x_2^{2s} - x_1^{2s} \le \frac{x_2 - x_1 + x_2^{2s}x_1^{1-2s} - x_1^{2s}x_2^{1-2s}}{x_2^{1-2s} + x_1^{1-2s}} \le \frac{2(x_2 - x_1)}{x_2^{1-2s} + x_1^{1-2s}}, \tag{77}$$

where the last inequality uses the fact

$$x_2^{2s}x_1^{1-2s} - x_1^{2s}x_2^{1-2s} \le x_2 - x_1.$$

Thus, combining (76) and (77), we finish the proof of (75). □

Lemma 5.3. For any $\sigma > \rho \ge 0$ and any $\mathbf{j}, \mathbf{l} \in \mathbb{Z}^d$, we assume that

$$\psi(\mathbf{j},\sigma) \le \psi(\mathbf{j}-\mathbf{l},\sigma)\cdot\psi(\mathbf{l},\rho) \quad \text{when} \quad |\mathbf{j}-\mathbf{l}| \ge |\mathbf{l}|\,, \tag{78}$$

$$\psi(\mathbf{j},\sigma) \le \psi(\mathbf{j}-\mathbf{l},\rho)\cdot\psi(\mathbf{l},\sigma) \quad \text{when} \quad |\mathbf{j}-\mathbf{l}| < |\mathbf{l}|\,. \tag{79}$$

Then there exists σ' satisfying $\sigma > \sigma' > \rho$ such that

$$\|q * q'\|_\sigma \le C(\sigma',\rho)\left(\|q\|_\sigma \|q'\|_{\sigma'} + \|q\|_{\sigma'}\|q'\|_\sigma\right), \tag{80}$$

where $C(\sigma',\rho) > 0$ is a constant depending on σ' and ρ only.

Proof. In view of (67), (78) and (79), one has

$$\begin{aligned}
\|q*q'\|_\sigma^2 \;&\le\; \sum_{\mathbf{j}\in\mathbb{Z}^d}\left|\sum_{\mathbf{l}\in\mathbb{Z}^d} q_{\mathbf{j}-\mathbf{l}}\cdot q'_{\mathbf{l}}\right|^2 \psi(\mathbf{j}-\mathbf{l},\sigma)^2\cdot\psi(\mathbf{l},\rho)^2 \\
&\quad + \sum_{\mathbf{j}\in\mathbb{Z}^d}\left|\sum_{\mathbf{l}\in\mathbb{Z}^d} q_{\mathbf{j}-\mathbf{l}}\cdot q'_{\mathbf{l}}\right|^2 \psi(\mathbf{j}-\mathbf{l},\rho)^2\cdot\psi(\mathbf{l},\sigma)^2 \\
&\le\; C(\sigma',\rho)\left(\|q\|_\sigma\|q'\|_{\sigma'} + \|q\|_{\sigma'}\|q'\|_\sigma\right),
\end{aligned}$$

where the last inequality uses Young's inequality

$$\|a*b\|_{\ell^2} \le \|a\|_{\ell^2}\,\|b\|_{\ell^1}\,, \quad a\in\ell^2, b\in\ell^1, \tag{81}$$

and Cauchy's inequality. □

Next, we will demonstrate that there exists $\rho \geq 0$ such that the conditions (78) and (79) in the lemma 5.3 are satisfied.

Remark 5.1. According to the fact that

$$\mathbf{j}^{\sigma} \leq 2^{\sigma} \cdot (\mathbf{j}-\mathbf{l})^{\sigma} \cdot \mathbf{l}^{0} \quad \text{when} \quad |\mathbf{j}-\mathbf{l}| \geq |\mathbf{l}|\,,$$
$$\mathbf{j}^{\sigma} \leq 2^{\sigma} \cdot \mathbf{l}^{\sigma} \cdot (\mathbf{j}-\mathbf{l})^{0} \quad \text{when} \quad |\mathbf{j}-\mathbf{l}| < |\mathbf{l}|\,,$$

one gets in Lemma 5.3

$$\rho = 0 \quad \text{when} \quad \psi(\mathbf{j},\sigma) = \exp\{\sigma \ln \langle \mathbf{j} \rangle\}\,.$$

Moreover, from Lemma 6.1 in [19] and Lemma 5.1 in [20], one gets in Lemma 5.3

$$\rho = 3\sigma/4 \quad \text{when} \quad \psi(\mathbf{j},\sigma) = \exp\left\{\sigma \ln^2 [\mathbf{j}]\right\},$$
$$\rho = (\sqrt{2}-1)\sigma \quad \text{when} \quad \psi(\mathbf{j},\sigma) = \exp\left\{\sigma \sqrt{|\mathbf{j}|}\right\}.$$

Declarations

Conflict of interest The authors declare no conflict of interest.

References

[1] J Asch, P Duclos, P Exner. *Stability of driven systems with growing gaps, quantum rings, and wannier ladders*, J Stat Phys, 1998, 92: 1053-1070.

[2] D Bambusi. *Nekhoroshev theorem for small amplitude solutions in nonlinear Schrödinger equations*, Math Z, 1999, 230(2): 345-387.

[3] M Berti, J M Delort. *Almost global solutions of capillary gravity water waves equations on the circle*, Springer, 2018.

[4] D Bambusi, J M Delort, B Grébert, et al. *Almost global existence for Hamiltonian semilinear Klein-Gordon equations with small Cauchy data on Zoll manifolds*, Comm Pure Appl Math, 2007, 60(11): 1665-1690.

[5] J Bernier, E Faou, B Grébert. *Rational normal forms and stability of small solutions to nonlinear Schrödinger equations*, Ann PDE, 2020, 6(2): 14-65.

[6] D Bambusi, R Feola, R Montalto. *Almost global existence for some Hamiltonian PDEs with small Cauchy data on general tori*, Comm Math Phys, 2024, 405(1): 15-50.

[7] D Bambusi, B Grébert. *Birkhoff normal form for partial differential equations with tame modulus*, Duke Math J, 2006, 135(3): 507-567.

[8] J Bernier, B Grébert. *Long time dynamics for generalized Korteweg-de Vries and Benjamin-Ono equations*, Arch Ration Mech Anal, 2021, 241(3): 1139-1241.

[9] D Bambusi, B Langella, R Montalto. *Growth of Sobolev norms for unbounded perturbations of the Schrödinger equation on flat tori*, J Differ Equ, 2022, 318: 344-358.

[10] D Bambusi, B Langella, R Montalto. *Spectral asymptotics of all the eigenvalues of Schrödinger operators on flat tori*, Nonlinear Anal, 2022, 216: 112679.

[11] M Berti, A Maspero. *Long time dynamics of Schrödinger and wave equations on flat tori*, J Differ Equ, 2019, 367(2): 1167-1200.

[12] G E B Moraes, H Borluk, G de Loreno, et al. *Orbital stability of periodic standing waves for the cubic fractional nonlinear Schrödinger equation*, J Differ Equ, 2019, 341: 263-291.

[13] L Biasco, J E Massetti, M Procesi. *An abstract Birkhoff normal form theorem and exponential type stability of the 1D NLS*, Comm Math Phys, 2020, 375(3): 2089-2153.

[14] D Bambusi, N N Nekhoroshev. *A property of exponential stability in nonlinear wave equations near the fundamental linear mode*, Phys D, 1998, 122(1-4): 73-104.

[15] J Bourgain. *Construction of approximative and almost periodic solutions of perturbed linear Schrödinger and wave equations*, Geom Funct Anal, 1996, 6(2): 201-230.

[16] J Bourgain. *Growth of Sobolev norms in linear Schrödinger equations with quasi-periodic potential*, Comm Math Phys, 1999, 204: 207-247.

[17] J Bourgain. *On growth of Sobolev norms in linear Schrödinger equations with smooth time dependent potential*, J Anal Math, 1999, 77: 315-348.

[18] M Berti, M Procesi. *Nonlinear wave and Schrödinger equations on compact Lie groups and homogeneous spaces*, Duke Math J, 2011, 159(3): 479-538.

[19] Q Chen, H Cong, L Meng, et al. *Long time stability result for 1-dimensional nonlinear Schrödinger equation*, J Differ Equ, 2022, 315: 90-121.

[20] H Cong, C Liu, P Wang. *A Nekhoroshev type theorem for the nonlinear wave equation*, J Differ Equ, 2020, 269(4): 3853-3889.

[21] H Cong, S Li, X Wu. *Long time stability result for d-dimensional nonlinear Schrödinger equation*, J Differ Equ, 2024, 394: 174-208.

[22] H Cong, L Mi, Y Shi. *Super-exponential stability estimate for the nonlinear Schrödinger equation*, J Funct Anal, 2022, 283(12): 109682.

[23] H Cong, L Mi, P Wang. *A Nekhoroshev type theorem for the derivative nonlinear Schrödinger equation*, J Differ Equ, 2020, 268(9): 5207-5256.

[24] J Delort. *Growth of Sobolev norms of solutions of linear Schrödinger equations on some compact manifolds*, Int Math Res Not, 2010, 2010: 2305-2328.

[25] J M Delort. *Quasi-linear perturbations of Hamiltonian Klein-Gordon equations on spheres*, Mem Amer Math Soc, 2015, 234: 1103.

[26] V D Dinh. *Strichartz estimates for the fractional Schrödinger and wave equations on compact manifolds without boundary*, J Differ Equ, 2017, 263(12): 8804-8837.

[27] J M Delort, J Szeftel. *Long-time existence for small data nonlinear Klein-Gordon equations on tori and spheres*, Int Math Res Not, 2004, 37: 1897-1966.

[28] J M Delort, J Szeftel. *Long-time existence for semi-linear Klein Gordon equations with small Cauchy data on Zoll manifolds*, Amer J Math, 2006, 128(5): 1187-1218.

[29] L Eliasson, S Kuksin. *On reducibility of Schrödinger equation with quasi-periodic in time potential*, Comm Math Phys, 2009, 286: 125-135.

[30] E Faou, B Grébert. *A Nekhoroshev-type theorem for the nonlinear Schrödinger equation on the torus*, Anal PDE, 2013, 6(6): 1243-1262.

[31] R Feola, F Giuliani, S Pasquali. *On the integrability of Degasperis-Procesi equation: control of the Sobolev norms and Birkhoff resonances*, J Differ Equ, 2019, 266(6): 3390-3437.

[32] R Feola, J E Massetti. *Sub-exponential stability for the beam equation*, J Differ Equ, 2023, 356: 188-242.

[33] B Grébert, R Imekraz, E Paturel. *Normal forms for semilinear quantum harmonic oscillators*, Comm Math Phys, 2009, 291(3): 763-798.

[34] M Kapoor, N A Shah, W Weera. *Analytical solution of time-fractional Schrödinger equations via Shehu Adomian decomposition method*, AIMS Math, 2022, 7(10): 19562-19596.

[35] S B Kuksin. *Fifteen years of KAM for PDE. In Geometry, topology, and mathematical physics*, Amer Math Soc Transl, 2004, 212: 237-258.

[36] N Laskin. *Fractional Schrödinger equation*, Phys Rev E, 2002, 66(5): 056108.

[37] H Lin, S Liu, H Zhang, et al. *Stability for a system of the 2D incompressible magneto-micropolar fluid equations with partial mixed dissipation*, Nonlinearity, 2024, 37(5): 055001.

[38] Y Sun, S Li, X Wu. *Exponential and sub-exponential stability times for the derivative wave equation*, Discrete Contin Dyn Syst, 2023, 43(6): 2186-2212.

[39] Y Sun. *Quasi-periodic solutions of derivative beam equation on flat tori*, Qual Theory Dyn Syst, 2022, 21(4): 121.

[40] W M Wang. *Logarithmic bounds on Sobolev norms for time dependent linear Schrödinger equations*, Comm Partial Differ Equ, 2008, 33(12): 2149-2179.

[41] S Wang, M Xu. *Generalized fractional Schrödinger equation with space-time fractional derivatives*, J Math Phys, 2007, 48(4): 043502.

[1]School of Mathematical Sciences, Dalian University of Technology, Dalian 116024, China.
Emails: 15998408006@163.com, a1253560634@163.com

[2]School of Mathematical Sciences, Fudan University, Shanghai 200433, China.
Email: 21110180025@m.fudan.edu.cn