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Distributional scalar curvature and Einstein metrics

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Abstract. In this paper, we study scalar curvature rigidity of non-smooth metrics on smooth manifolds with non-positive Yamabe invariant. We prove that if the scalar curvature is not less than the Yamabe invariant in the distributional sense, then the manifold must be isometric to an Einstein manifold. This result extends Theorem 1.4 in Jiang, Sheng and Zhang [27], from a special case where the manifolds have zero Yamabe invariant to general cases where the manifolds have non-positive Yamabe invariant.

§1 Introduction

Low-regularity geometry with weak curvature conditions has been appearing as an important theme in Riemannian geometry. Sectional curvature, Ricci curvature and scalar curvature are the most fundamental and the most important curvatures in Riemannian geometry. For sectional curvature lower bounds, Gromov, Perelman, etc., developed the Alexandrov spaces theory, which has great applications in the resolution of Poincaré conjecture, see [6], [39], etc. For Ricci curvature lower bounds, there is a profound theory developed by Cheeger, Colding, Tian, etc., see [8], [9], [10], [14], [15], [52] [24], [12], etc. An another theory for Ricci curvature lower bounds was developed by Lott, Villani, Sturm, etc., via an optimal transport approach, see [36], [51].

However, for scalar curvature lower bounds, it has not been well understood. Gromov proposed to study scalar curvature lower bounds in a weak sense, see [18, Page 1118]. And he pointed out that one could consider weak scalar curvature in the distributional sense (see [18, Page 1118, Line 11 from below]). Bamler and Burkhardt-Guim developed a notion of weak scalar curvature by using Ricci flow (see [1] and [7]). Jiang, Sheng and the author made a connection between the notion developed by Bamler and Burkhardt-Guim and the notion of distributional scalar curvature in [27]. In [29], Lee-LeFloch proved a positive mass theorem for distributional scalar curvature. In [26], Jiang, Sheng and the author improved some of

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the results in [29]. The author partially solved the Yamabe problem for distributional scalar curvature in [53]. For more work in this topic, see [50], [31], etc.

In particular, Schoen wanted to generalize the following theorem, involving scalar curvature lower bounds, from smooth metrics to non-smooth metrics:

Theorem 1.1 ([28], [43]). *Let M^n be a compact smmoth differentiable manifold with $\sigma(M) \leq 0$, where $\sigma(M)$ is the Yamabe invariant of M, and g be a smooth metric on M with unit volume such that $R_g \geq \sigma(M)$ pointwisely on M. Then g is Einstein with $R_g = \sigma(M)$.*

The Yamabe invariant here is an invariant defined on smooth differentiable manifolds, it appears to be the key concept in the resolvation of prescribing scalar curvature problem (see [28]). See (79) for its formal definition.

The paper [33] mentioned such a question of Schoen: In Theorem 1.1, if g admits a singular set, and $R_g \geq \sigma(M)$ only holds away from the singular set, can we still deduce that g is Einstein? In particular, he conjectured that if g is uniformly Euclidean near the singular set, $R_g \geq 0 \geq \sigma(M)$ away from the singular set, and the singular set is a submanifold of codimension at least 3, then g extends smoothly to the whole manifold to a Ricci flat metric. Actually, Schoen didn't propose this conjecture in some papers, but his students recorded that conjecture in lecture notes, which is unpublished.

This question is still open. [33] confirms Schoen's conjecture for 3-manifolds in the case of an isolated singularity. If the metric is $W^{1,p}(p > n)$ near the singular set and the singular set is small, [48] proved that the metric is Einstein away from the singular set, which almost solved Schoen's conjecture in $W^{1,p}(p > n)$ case. In [27], Jiang, Sheng and the author improved this result and gave an optimal condition of the singular set in the case $p = +\infty$. If the metric is $W^{1,n}$ or C^0 near the singular set and the codim of singular set is strictly greater than 2, Chu-Lee and Lee-Tam proved the metric is Einstein away from the singular set respectively in [13] and [30], which solved Schoen's conjecture in $W^{1,n}$ or C^0 case.

Motivated by Gromov's suggestion in [18] and Schoen's question in [33], we study a more radical case in this paper. In contrast with the work above, in which their metrics are still smooth away from a small singular set, our metrics, in this paper, could be non-smooth on the whole manifold. Correspondingly, the scalar curvature lower bounds are assumed in the distributional sense. Our main theorem is:

Theorem 1.2. *Let M^n be a compact manifold with $\sigma(M) \leq 0$, where $\sigma(M)$ is the Yamabe invariant of M, and g be a $W^{1,p}(n < p \leq \infty)$ metric on M with unit volume such that $R_g \geq \sigma(M)$ in the distributional sense. Then (M, g) is distance isometric to an Einstein manifold with scalar curvature equal to $\sigma(M)$.*

Remark 1.1. Theorem 1.2 confirms Schoen's conjecture for $W^{1,p}(n < p \leq \infty)$ metrics. In fact, if g is C^2 away from a closed subset Σ, whose Hausdorff measure satisfis $\mathcal{H}^{n-\frac{p}{p-1}}(\Sigma) < \infty$ for $n < p < \infty$ or $\mathcal{H}^{n-1}(\Sigma) = 0$ for $p = \infty$, and if $R_g \geq a$ pointwisely away from Σ, then we can deduce that $R_g \geq a$ in the distributional sense, see [26, Lemma 2.7].

Remark 1.2. Jiang, Sheng and the author have proved the special case $\sigma(M) = 0$, see [27, Theorem 1.4]. However, our work in [27] does not directly solve the general case $\sigma(M) \leq 0$. To prove the general case $\sigma(M) \leq 0$, in this paper, we must do more work. Particularly, we must improve the estimate in [27, Theorem 1.1] to a better estimate. This improvement is given in our Lemma 4.1. Moreover, we need to use a modified flow, rather than the Ricci flow in [27].

Remark 1.3. Since in this paper we extend the result by Jiang, Sheng and the author in [27] from $\sigma(M) = 0$ to $\sigma(M) \leq 0$. One might ask for similar results for $\sigma(M) > 0$. In fact, the corresponding question for $\sigma(M) > 0$ is nonsense, since the resolution of the prescribing scalar curvature problem tells that for any M with $\sigma(M) > 0$, any smooth function is the scalar curvature of some metrics on M, see [28].

Organization:

In section 2, we give some prelimilaries. In section 3, we construct an auxiliary function and give some estimates, which are of great importance in our proof of Theorem 1.2. In section 4, we study scalar curvature lower bounds along Ricci flow, where the initial metric only has scalar curvature lower bounds in the distributional sense. In section 5, we prove Theorem 1.2.

§2 Prelimilaries

2.1 Distributional scalar curvature

Due to the lack of the second order derivative, singular metrics do not have the concept of curvature in classical sense. However, inspired by the distributional theory (or theory of generalized functions), in which the derivative exists even for very weird functions, it is naturally to consider derivative and curvature in the distributional sense for singular metrics.

Let M^n be a compact smooth manifold. Fix an arbitrary smooth background metric h on M, for any tensor field T on M, its Sobolev norm $W^{k,q}(M)$ is defined naturally as:

$$\|T\|_{W^{k,q}(M)} := \sum_{s=0}^{k} \int_M |\tilde{\nabla}^k T|_h d\mu_h. \tag{1}$$

Here and below $\tilde{\nabla}$, $|\cdot|_h$ and $d\mu_h$ will denote the Levi-Civita connection, the norm and the volume form respectively taken with respect to h. Although the $W^{k,q}(M)$ norm depends on the background metric h, the norms for different h are all equivalent and the $W^{k,q}(M)$ space does not depend on it.

Therefore, a $W^{k,q}$ metric g, or a metric $g \in W^{k,q}(M)$, means a symmetric and positive definite $(0, 2)$ tensor field on M with finite $W^{k,q}(M)$ norm.

In this paper, we focus on $W^{1,p}(n < p \leq +\infty)$ metrics. For any metric $g \in W^{1,p}(M)$, its distributional scalar curvature R_g is defined as (see [27, 32, 29, 34], etc.):

$$\langle R_g, \varphi \rangle := \int_M \left(-V \cdot \tilde{\nabla} \left(\varphi \frac{d\mu_g}{d\mu_h} \right) + F\varphi \frac{d\mu_g}{d\mu_h} \right) d\mu_h, \forall \varphi \in C^\infty(M), \tag{2}$$

where V and F is a vector field and a function respectively, defined by:

$$\Gamma_{ij}^k := \frac{1}{2} g^{kl} \left(\tilde{\nabla}_i g_{jl} + \tilde{\nabla}_j g_{il} - \tilde{\nabla}_l g_{ij} \right), \tag{3}$$

$$V^k := g^{ij}\Gamma_{ij}^k - g^{ik}\Gamma_{ji}^j = g^{ij} g^{k\ell}(\tilde{\nabla}_j g_{i\ell} - \tilde{\nabla}_\ell g_{ij}), \tag{4}$$

$$F := \mathrm{tr}_g \mathrm{Ric}_h - \tilde{\nabla}_k g^{ij}\Gamma_{ij}^k + \tilde{\nabla}_k g^{ik}\Gamma_{ji}^i + g^{ij}\left(\Gamma_{k\ell}^k \Gamma_{ij}^\ell - \Gamma_{j\ell}^k \Gamma_{ik}^\ell \right), \tag{5}$$

where Ric_h is the Ricci curvature tensor of h.

This definition is independent of the background metric h, see [29]. Moreover, if g is smooth, this definition of $\langle R_g, \varphi \rangle$ is just the integral $\int_M R_g \varphi d\mu_g$. See also [26, 53], etc., for more properties of the distributional scalar curvature.

2.2 Mollification of the metric and estimate on the scalar curvature

Any $W^{1,p}$ metric could be mollified to a smooth metric family, which converges to it in $W^{1,p}$ topology. Concretely, we have the following lemma:

Lemma 2.1 (Lemma 4.1 in [20]). *Let M^n be a compact smooth manifold with a $W^{1,p}(n < p \leq \infty)$ metric g on it, then there exists a family of smooth metrics $g_\delta, \delta > 0$, such that g_δ converges to g in $W^{1,p}$ topology as $\delta \to 0^+$.*

Remark 2.1. In [20, Lemma 4.1], the lemma is only claimed for $W^{2,\frac{n}{2}}$ metrics. However, their proof indeed works for general $W^{k,q}$ case, especially for our $W^{1,p}$ case.

The following lemma established by Jiang, Sheng and the author shows that the distributional scalar curvature functional is continuously dependent on the metric in a certain sense:

Lemma 2.2 (Lemma 2.2 in [27]). *Let M^n be a compact smooth manifold. Suppose g_δ is a family of metrics which converges to g in $W^{1,p}$ topology as $\delta \to 0^+$, then we have that for any $\epsilon > 0$, there exists $\delta_0 = \delta_0(g) > 0$, such that*

$$|\langle R_{g_\delta}, u \rangle - \langle R_g, u \rangle| \leq \epsilon \|u\|_{W^{1,\frac{n}{n-1}}(M)}, \forall u \in C^\infty(M), \forall \delta \in (0, \delta_0).$$

where R_{g_δ} is the scalar curvature of g_δ.

2.3 Estimates on Ricci flow

The Ricci flow, introduced by Hamilton in [21], is defined as follows:

Definition 2.3 (Ricci flow). The Ricci flow on M is a family of metrics $g(t)$ such that

$$\frac{\partial}{\partial t} g(t) = -2\mathrm{Ric}_{g(t)},$$

where $\mathrm{Ric}_{g(t)}$ is the Ricci curvature tensor of $g(t)$.

Though Hamilton only introduced Ricci flow with smooth initial metrics, it is quite useful to consider Ricci flow with singular initial metrics. For our use, we mainly need the following theorem, given by Jiang, Sheng and the author, which considers Ricci flow with $W^{1,p}(n < p \leq +\infty)$ initial metrics:

Lemma 2.4 (Theorem 3.2 in [27]). *There exists an $\epsilon(n) > 0$ such that, for any compact n-manifold M and any $W^{1,p}(n < p \leq +\infty)$ metric $\hat{g}$ on M, there exists a $T_0 = T_0(n, \hat{g}) > 0$ and a Ricci flow $g(t) \in C^\infty(M \times (0, T_0])$, such that*

(1) $\lim_{t\to 0} d_{GH}((M, g(t)), (M, \hat{g})) = 0$.

(2) $|\mathrm{Rm}(g(t))|(t) \leq \frac{C(n,\hat{g},p)}{t^{\frac{n}{4p}+\frac{3}{4}}}$, $\forall t \in (0, T_0]$.

(3) $\int_0^{T_0} \int_M |\mathrm{Rm}(g(t))|^2 d\mu_{g(t)} dt \leq C(n, \hat{g}, p)$,

where d_{GH} is the Gromov-Hausdorff distance, and $C(n, \hat{g}, p)$ is a positive constant independent of t.

We also need to consider the h-flow, which is equivalent to the Ricci flow after a family of diffeomorphisms. It was firstly introduced by Simon in [49], in order to study the Ricci flow with C^0 initial two metrics.

Before we give the definition of h-flow, we firstly give the definition of $(1+\delta)$-fairness between metrics.

Definition 2.5. Given a constant $\delta \geq 0$, a metric h is called to be $(1+\delta)$-fair to g, if h is C^∞,

$$\sup_M |\tilde{\nabla}^j \mathrm{Rm}(h)| = L_j < \infty,$$

and

$$(1+\delta)^{-1} h \leq g \leq (1+\delta) h \quad on\ M.$$

Here and below, $\tilde{\nabla}$ means the covariant derivative taken with respect to h.

Definition 2.6. [h-flow] For any background smooth metrics h, the h-flow is a family of metrics $g(t)$ that satisfy

$$\frac{\partial}{\partial t} g_{ij}(t) = -2R_{g(t);ij} + \nabla_i V_j + \nabla_j V_i,$$

where the derivatives are taken with respect to $g(t)$, $R_{g(t);ij}$ is the Ricci curvature of $g(t)$,

$$V_j = g_{jk}(t) g^{pq}(t)(\Gamma^k_{pq}(t) - \tilde{\Gamma}^k_{pq}),$$

and $\Gamma(t)$ and $\tilde{\Gamma}$ are the Christoffel symbols of $g(t)$ and h respectively.

Simon proved such an existence results:

Lemma 2.7 (Theorem 1.1 in [49]). *There exists an $\epsilon(n) > 0$ such that, for any compact n-manifold M with a complete C^0 metric $\hat{g}$ and a C^∞ metric h which is $(1+\frac{\epsilon(n)}{2})$-fair to $\hat{g}$, there exists a $T_0 = T_0(n, k_0) > 0$ and a family of metrics $g(t) \in C^\infty(M \times (0, T_0]), t \in (0, T_0]$ which solves h-flow for $t \in (0, T_0]$, h is $(1+\epsilon(n))$-fair to $g(t)$, and*

(1)

$$\lim_{t\to 0^+} \sup_{x\in M} |g(x,t) - \hat{g}(x)| = 0,$$

(2)

$$\sup_M |\tilde{\nabla}^i g(t)| \leq \frac{c_i(n,h)}{t^{i/2}}, \forall t \in (0, T_0], i \geq 1,$$

where the derivatives and the norms are taken with respect to h.

To apply the flow to prove our results, we will let h be $(1 + \frac{\epsilon(n)}{2})$-fair to $\hat{g}$. The existence of such h is ensured by the following lemma:

Lemma 2.8 ([49]). *Let M be a compact manifold, for any C^0 metric g on M, and any $0 < \delta < 1$, there exists a C^∞ metric h which is $(1+\delta)$-fair to g.*

In the case of $W^{1,p}(n < p \leq +\infty)$ initial metric, Lemma 2.7 could be improved:

Lemma 2.9 (Theorem 3.11 in [27]). *In the condition of Lemma 2.7, if $\hat{g}$ is $W^{1,p}(n < p \leq +\infty)$ on M, then there exists a $T_0 = T_0(n, h, \|\hat{g}\|_{W^{1,p}(M)}, p)$, such that $g(t)$, $t \in (0, T_0]$ is the h-flow with initial metric $\hat{g}$, and*

(1) $\int_M |\tilde{\nabla} g(t)|^p d\mu_h \leq 10 \int_M |\tilde{\nabla} \hat{g}|^p d\mu_h$, $\forall t \in (0, T_0]$,

(2) $|\tilde{\nabla} g|(t) \leq \frac{C(n,h,\|\hat{g}\|_{W^{1,p}(M)},p)}{t^{\frac{n}{2p}}}$, $\forall t \in (0, T_0]$,

(3) $|\tilde{\nabla}^2 g|(t) \leq \frac{C(n,h,\|\hat{g}\|_{W^{1,p}(M)},p)}{t^{\frac{n}{4p}+\frac{3}{4}}}$, $\forall t \in (0, T_0]$,

where $C(n, h, \|\hat{g}\|_{W^{1,p}(M)}, p)$ is a positive constant depends only on $n, h, \|\hat{g}\|_{W^{1,p}(M)}, p$, and does not depend on t.

The Ricci flow in Lemma 2.4 just comes from the h-flow in Lemma 2.9. In fact, they are equivalent after a certain family of diffeomorphisms.

§3 An auxiliary function along Ricci flow

In this section, we consider a compact smooth Riemannian manifold $(M^n, \hat{g})$ and the Ricci flow $g(t)(t \in [0, T_0])$ with initial metric $\hat{g}$. To study a $W^{1,p}$ metric g, firstly we work with the smooth metrics g_δ given in Lemma 2.2. Thus, in this section, we consider Ricci flow with smooth initial metrics.

For any positive time $T \in (0, T_0]$ and any $\tilde{\varphi} \in C^\infty(M)$, we want to construct an auxiliary function $\varphi(x,t)$ defined on $M \times [0,T]$, such that $\varphi(T, \cdot) = \tilde{\varphi}(\cdot)$, and the integral

$$\int_M \left(R_{g(t)} - a_0 \left(1 - \frac{2a_0}{n} t\right)^{-1} \right) \varphi(\cdot, t) d\mu_{g(t)} \tag{6}$$

is monotone increasing with respect to t.

Moreover, we also need some estimate for $\varphi(x,t)$. In fact, we have the following lemma, which is the main result of this section:

Lemma 3.1. *Let $(M^n, \hat{g})$ be a compact smooth Riemannian manifold and $g(t)(t \in [0, T_0])$ be the Ricci flow with initial metric $\hat{g}$. For any non-positive constant a, any $T \in (0, T_0]$ and any $\tilde{\varphi} \in C^\infty(M)$, there exists a function $\varphi(x,t) \in C^\infty(M \times [0,T])$, such that*

(1) $\varphi(\cdot, T) = \tilde{\varphi}(\cdot)$, on M.

(2) $\int_M \left(R_{g(t)} - a_0 \left(1 - \frac{2a_0}{n} t\right)^{-1}\right) \varphi(\cdot, t) d\mu_{g(t)}$ is monotonously increasing with respect to t.

(3) $\varphi(\cdot, t) \le C(n, h, p, \|\hat{g}\|_{W^{1,p}(M)}, \|\tilde{\varphi}\|_{L^\infty}, a_0)$, $\forall t \in [0, T]$.

$\|\varphi(\cdot, t)\|_{W^{1,\frac{n}{n-1}}(M)} \le C(n, h, p, \|\hat{g}\|_{W^{1,p}(M)}, \|\tilde{\varphi}\|_{L^\infty}, a_0)$, $\forall t \in [0, T]$.

Remark 3.1. Though Lemma 3.1 only claims results for a non-positive, our proof essentially gives similar results for a_0 positive. The matter is that the term $\left(1 - \frac{2a_0}{n} t\right)^{-1}$ in item (2) would divergence to infinity at $t = \frac{n}{2a_0}$. This divergence phenomenon is expectable by noting that the Ricci flow with the round sphere with scalar curvature $a_0 > 0$ as its initial metric collapses to a single point at time $t = \frac{n}{2a_0}$, since we use Ricci flow in a short time, so we can restrict $T_0 < \frac{n}{2a_0}$. However, the case of a_0 non-positive is good enough for our use.

Now we are going to construct the function φ in the lemma above. Since $(M^n, \hat{g})$ is a compact smooth Riemannian manifold, it is known that the heat kernel along its Ricci flow $g(t)(t \in [0, T_0]$ exists. The heat kernel is a funciton $K(y, s; x, t)$, $y, x \in M$, $0 \le t < s \le T_0$, such that

$$(\partial_s - \Delta_{g(s);y}) K(y, s; x, t) = 0, \forall \text{ fixed } x \in M, t \in (0, T_0], \tag{7}$$

and

$$\lim_{s \to t^+} K(y, s; x, t) = \delta_x(y), \forall \text{ fixed } x \in M, \tag{8}$$

where $\Delta_{g(s);y}$ denotes the Laplacian taken to the variable y and with respect to the metric $g(s)$, $\delta_x(\cdot)$ denotes the Dirac Delta functional at x.

Moreover,

$$(\partial_t + \Delta_{g(t);x} - R_{g(t)}(x)) K(y, s; x, t) = 0, \forall \text{ fixed } y \in M, s \in (0, T_0], \tag{9}$$

and

$$\lim_{t \to s^-} K(y, s; x, t) = \delta_y(x), \forall \text{ fixed } y \in M, \tag{10}$$

where $R_{g(t)}$ is the scalar curvature of $g(t)$.

Now, for any $a_0 \in \mathbb{R}, T \in (0, T_0]$ and $\tilde{\varphi} \in C^\infty(M)$, we define a function φ on $M \times [0, T]$ as follows:

$$\varphi(x,t) = \left(1 - \frac{2a_0}{n} T\right)^{-2} \left(1 - \frac{2a_0}{n} t\right)^2 \int_M \tilde{\varphi}(y) K(y, T; x, t) d\mu_{g(T)}(y), \tag{11}$$

For convenience, we denote:

$$a(t) := a_0 \left(1 - \frac{2a_0}{n} t\right)^{-1}. \tag{12}$$

One of the advantages of φ defined above is that it satisfies an equation which is useful for our use:

Lemma 3.2. *Let $(M, g(t))$ be a Ricci flow, $K(y, s; x, t)$ be the heat kernel along $g(t)$, then the* *function φ defined in* (11) *satisfies*

$$\partial_t\varphi = -\Delta_{g(t)}\varphi + \left(R_{g(t)} - \frac{4}{n}a(t)\right)\varphi, \tag{13}$$

where $a(t)$ is the real function defined in (12).

Proof. Since

$$\varphi(x,t) = \left(1-\frac{2a_0}{n}T\right)^{-2}\left(1-\frac{2a_0}{n}t\right)^{2}\int_M \tilde{\varphi}(y)K(y,T;x,t)d\mu_{g(T)}(y), \tag{14}$$

we have

$$\begin{aligned}
&\partial_t\varphi(x,t)\\
=&\left(1-\frac{2a_0}{n}T\right)^{-2}\partial_t\left(1-\frac{2a_0}{n}t\right)^{2}\int_M \tilde{\varphi}(y)K(y,T;x,t)d\mu_{g(T)}(y)\\
&+\left(1-\frac{2a_0}{n}T\right)^{-2}\left(1-\frac{2a_0}{n}t\right)^{2}\partial_t\int_M \tilde{\varphi}(y)K(y,T;x,t)d\mu_{g(T)}(y)\\
=&-\left(1-\frac{2a_0}{n}T\right)^{-2}\frac{4a_0}{n}\left(1-\frac{2a_0}{n}t\right)\int_M \tilde{\varphi}(y)K(y,T;x,t)d\mu_{g(T)}(y)\\
&+\left(1-\frac{2a_0}{n}T\right)^{-2}\left(1-\frac{2a_0}{n}t\right)^{2}\partial_t\int_M \tilde{\varphi}(y)K(y,T;x,t)d\mu_{g(T)}(y)\\
=&-\frac{4}{n}a(t)\varphi(x,t)\\
&+\left(1-\frac{2a_0}{n}T\right)^{-2}\left(1-\frac{2a_0}{n}t\right)^{2}\partial_t\int_M \tilde{\varphi}(y)K(y,T;x,t)d\mu_{g(T)}(y).
\end{aligned} \tag{15}$$

By (9), we have

$$\partial_t K(y,T;x,t) = (-\Delta_{g(t);x} + R_{g(t)}(x))K(y,T;x,t). \tag{16}$$

Thus (15) becomes

$$\begin{aligned}
\partial_t\varphi(x,t) = -\frac{4}{n}a(t)\varphi(x,t) + \left(1-\frac{2a_0}{n}T\right)^{-2}\left(1-\frac{2a_0}{n}t\right)^{2}\int_M \tilde{\varphi}(y)(-\Delta_{g(t);x} + R_{g(t)}(x))\\
K(y,T;x,t)d\mu_{g(T)}(y).
\end{aligned} \tag{17}$$

On the other hand, we have

$$\Delta_{g(t)}\varphi(x,t) = \left(1-\frac{2a_0}{n}T\right)^{-2}\left(1-\frac{2a_0}{n}t\right)^{2}\int_M \tilde{\varphi}(y)\Delta_{g(t)}K(y,T;x,t)d\mu_{g(T)}(y). \tag{18}$$

Combining (17) and (18), we have

$$\begin{aligned}
(\partial_t + \Delta_{g(t)})\varphi(x,t) =& -\frac{4}{n}a(t)\varphi(x,t) + \left(1-\frac{2a_0}{n}T\right)^{-2}\left(1-\frac{2a_0}{n}t\right)^{2}\int_M \tilde{\varphi}(y)R_{g(t)}(x)\\
&K(y,T;x,t)d\mu_{g(T)}(y)\\
&= -\frac{4}{n}a(t)\varphi(x,t) + R_{g(t)}(x)\left(1-\frac{2a_0}{n}T\right)^{-2}\left(1-\frac{2a_0}{n}t\right)^{2}\int_M \tilde{\varphi}(y)\\
&K(y,T;x,t)d\mu_{g(T)}(y)\\
&= -\frac{4}{n}a(t)\varphi(x,t) + R_{g(t)}(x)\varphi(x,t),
\end{aligned} \tag{19}$$

which is just the equation (13), thus the lemma is proved. □

Now we can prove Lemma 3.1:

proof of Lemma 3.1. Let $K(y,s;x,t)$ be the heat kernel along $g(t)$, and φ be the function defined in (11).

For (1), it is known that the convolution $\int_M \tilde{\varphi}(y)(x)K(y,T;x,t)d\mu_{g(T)}(y)$ is smooth and

$$\int_M \tilde{\varphi}(y)(x)K(y,T;x,t)d\mu_{g(T)}(y)\Big|_{t=T} = \tilde{\varphi}(x). \tag{20}$$

On the other hand, we have

$$\left(1-\frac{2a_0}{n}T\right)^{-2}\left(1-\frac{2a_0}{n}t\right)^2|_{t=T} = 1. \tag{21}$$

Thus by (11) we have

$$\varphi(x,T) = 1\tilde{\varphi}(x) = \tilde{\varphi}(x), \tag{22}$$

which proves (1).

For (2), we calculate that

$$\begin{aligned}
&\partial_t \int_M \left(R_{g(t)} - a_0\left(1-\frac{2a_0}{n}t\right)^{-1}\right)\varphi(x,t)d\mu_{g(t)}\\
=&\partial_t \int_M \left(R_{g(t)} - a(t)\right)\varphi(x,t)d\mu_{g(t)}\\
=&\int_M \left(\left(\partial_t R_{g(t)} - a'(t)\right)\varphi(x,t) + \left(R_{g(t)} - a'(t)\right)\partial_t\varphi(x,t)\right)d\mu_{g(t)} + \int_M \left(R_{g(t)} - a(t)\right)\\
&\varphi(x,t)\partial_t d\mu_{g(t)}.
\end{aligned} \tag{23}$$

And we have

$$a'(t) = \frac{2a_0^2}{n}\left(1-\frac{2a_0}{n}t\right)^{-2} = \frac{2}{n}a^2(t). \tag{24}$$

Recall $g(t)$ is a Ricci flow, we have the standard evolution equation

$$\partial_t R_{g(t)} = \Delta_{g(t)}R_{g(t)} + 2|\mathrm{Ric}_{g(t)}|^2_{g(t)}, \tag{25}$$

and

$$\partial_t d\mu_{g(t)} = -R_{g(t)}d\mu_{g(t)}, \tag{26}$$

Combining (23)-(26) and the evolution equation of φ, (13), we have

$$\begin{aligned}
&\partial_t \int_M \left(R_{g(t)} - a_0\left(1-\frac{2a_0}{n}t\right)^{-1}\right)\varphi(x,t)d\mu_{g(t)}\\
=&\int_M \left(\Delta_{g(t)}R_{g(t)} + 2|\mathrm{Ric}_{g(t)}|^2_{g(t)} - \frac{2}{n}a^2(t)\right)\varphi(x,t)d\mu_{g(t)} +\\
&\int_M \left(R_{g(t)} - a(t)\right)\left(-\Delta_{g(t)}\varphi + \left(R_{g(t)} - \frac{4}{n}a(t)\right)\varphi(x,t)\right)d\mu_{g(t)}\\
&+\int_M \left(R_{g(t)} - a(t)\right)\varphi(x,t)\left(-R_{g(t)}\right)d\mu_{g(t)}
\end{aligned}$$

$$
\begin{aligned}
=&\int_M \left(\Delta_{g(t)}R_{g(t)}\varphi(x,t)-R_{g(t)}\Delta_{g(t)}\varphi(x,t)\right)d\mu_{g(t)}+\\
&\int_M\left(2|\mathrm{Ric}_{g(t)}|^2_{g(t)}-\frac{2}{n}a^2(t)-\frac{4}{n}a(t)R_{g(t)}+\frac{4}{n}a^2(t)\right)\varphi(x,t)d\mu_{g(t)}\\
&+\int_M\left(R_{g(t)}-a(t)\right)\varphi(x,t)\left(-R_{g(t)}\right)d\mu_{g(t)}+\int_M\left(R_{g(t)}-a(t)\right)\varphi(x,t)R_{g(t)}d\mu_{g(t)}\\
&+\int_M a(t)\Delta_{g(t)}\varphi(x,t)Rd\mu_{g(t)}\\
=&\int_M \left(\Delta_{g(t)}R_{g(t)}\varphi(x,t)-R_{g(t)}\Delta_{g(t)}\varphi(x,t)\right)d\mu_{g(t)}+\\
&\int_M\left(2|\mathrm{Ric}_{g(t)}|^2_{g(t)}-\frac{4}{n}a(t)R_{g(t)}+\frac{2}{n}a^2(t)\right)\varphi(x,t)d\mu_{g(t)}\\
&+\int_M a(t)\Delta_{g(t)}\varphi(x,t)Rd\mu_{g(t)}.
\end{aligned}\tag{27}
$$

By integration by parts, we have

$$\int_M \left(\Delta_{g(t)}R_{g(t)}\varphi(x,t)-R_{g(t)}\Delta_{g(t)}\varphi(x,t)\right)d\mu_{g(t)}=0,\tag{28}$$

and

$$\int_M a(t)\Delta_{g(t)}\varphi(x,t)Rd\mu_{g(t)}=a(t)\int_M\Delta_{g(t)}\varphi(x,t)Rd\mu_{g(t)}=0.\tag{29}$$

Combining (27)-(29), we have

$$
\begin{aligned}
&\partial_t\int_M\left(R_{g(t)}-a_0\left(1-\frac{2a_0}{n}t\right)^{-1}\right)\varphi(x,t)d\mu_{g(t)}\\
=&\int_M\left(\left(2|\mathrm{Ric}_{g(t)}|^2_{g(t)}-\frac{4}{n}a(t)R_{g(t)}+\frac{2}{n}a^2(t)\right)\varphi(x,t)\right)d\mu_{g(t)}.
\end{aligned}\tag{30}
$$

By Cauchy inequality and a direct calculus using a special coordinate, one has

$$2|\mathrm{Ric}_{g(t)}|^2_{g(t)}\geq\frac{2}{n}R^2_{g(t)}.\tag{31}$$

By the mean value inequality, we have

$$-\frac{4}{n}a(t)R_{g(t)}\geq-\frac{2}{n}R^2_{g(t)}-\frac{2}{n}a^2(t)\tag{32}$$

Combining (30)-(32), we have

$$
\begin{aligned}
&\partial_t\int_M\left(R_{g(t)}-a_0\left(1-\frac{2a_0}{n}t\right)^{-1}\right)\varphi(x,t)d\mu_{g(t)}\\
\geq&\int_M\left(\left(\frac{2}{n}R^2_{g(t)}-\frac{2}{n}R^2_{g(t)}-\frac{2}{n}a^2(t)+\frac{2}{n}a^2(t)\right)\varphi(x,t)\right)d\mu_{g(t)}\\
=&0.
\end{aligned}\tag{33}
$$

Thus $\int_M\left(R_{g(t)}-a_0\left(1-\frac{2a_0}{n}t\right)^{-1}\right)\varphi(\cdot,t)d\mu_{g(t)}$ is monotonously increasing with respect to t and (2) is proved.

For (3), by (11) we have

$$\varphi(t,x)\leq\left(1-\frac{2a_0}{n}T\right)^{-2}\left(1-\frac{2a_0}{n}t\right)^2\|\tilde{\varphi}\|_{L^\infty}\int_M K(y,T;x,t)d\mu_{g(T)}(y).\tag{34}$$

We denote $I(t,T)=\int_M K(y,T;x,t)d\mu_{g(T)}(y)$, then we have $\lim_{T\to t^+} I(t,T)=1$, and by the standard evolution equation $\partial_T d\mu_{g(T)}=-R_{g(T)}d\mu_{g(T)}$, we have

$$\partial_T I(t,T)=\int_M \left(\Delta_{g(T);y}K(y,T;x,t)-R_{g(T)}K(y,T;x,t)\right)d\mu_{g(T)}(y). \tag{35}$$

By the divergence theorem, we have

$$\int_M \Delta_y K(y,T;x,t)d\mu_{g(T)}(y)=0. \tag{36}$$

Combining Lemma 2.4 (2), (35) and (36), we have

$$\partial_T I(t,T)\le \frac{C(n,h,p,\|\hat g\|_{W^{1,p}(M)})}{T^{\frac{n}{4p}+\frac{3}{4}}}I(t,T). \tag{37}$$

Since $\lim_{T\to t^+} I(t,T)=1$ and $\frac{n}{4p}+\frac{3}{4}\in(0,1)$, by taking integral we have

$$I(t,T)\le C(n,h,p,\|\hat g\|_{W^{1,p}(M)}),\forall 0\le t<T\le T_0. \tag{38}$$

By (34) and (38), we have

$$\varphi(t,x)\le C(n,h,p,\|\hat g\|_{W^{1,p}(M)},\|\tilde\varphi\|_{L^\infty})\left(1-\frac{2a_0}{n}T\right)^{-2}\left(1-\frac{2a_0}{n}t\right)^2,\forall t\in[0,T]. \tag{39}$$

Let us estimate $\left(1-\frac{2a_0}{n}T\right)^{-2}\left(1-\frac{2a_0}{n}t\right)^2$.

If $a_0>0$, then we have assumed $t\le T\le T_0<\frac{n}{2a_0}$ without loss of generality. In this case $\left(1-\frac{2a_0}{n}t\right)^2$ is monotonously increasing with respect to t, thus

$$\left(1-\frac{2a_0}{n}T\right)^{-2}\left(1-\frac{2a_0}{n}t\right)^2\le 1,\ \text{if } a_0>0 \tag{40}$$

If $a_0<0$, then we have

$$\begin{aligned}\left(1-\frac{2a_0}{n}T\right)^{-2}\left(1-\frac{2a_0}{n}t\right)^2 &\le (1)^{-2}\left(1-\frac{2a_0}{n}T_0\right)^2\le C(a_0,T_0)\\ &=C(n,h,p,\|\hat g\|_{W^{1,p}(M)},a_0),\ \text{if } a_0<0\end{aligned} \tag{41}$$

If $a_0=0$, then we have

$$\left(1-\frac{2a_0}{n}T\right)^{-2}\left(1-\frac{2a_0}{n}t\right)^2\equiv 1,\ \text{if } a_0=0 \tag{42}$$

In all cases, we always have

$$\left(1-\frac{2a_0}{n}T\right)^{-2}\left(1-\frac{2a_0}{n}t\right)^2\le C(n,h,p,\|\hat g\|_{W^{1,p}(M)},a_0). \tag{43}$$

Combining (39) and (43), we have

$$\varphi(x,t)\le C(n,h,p,\|\hat g\|_{W^{1,p}(M)},\|\tilde\varphi\|_{L^\infty},a_0),\forall (x,t)\in M\times[0,T], \tag{44}$$

which proves (3).

For (4), we consider a simpler function

$$\psi(x,t):=\int_M \tilde\varphi(y)K(y,T;x,t)d\mu_{g(T)}(y)=\left(1-\frac{2a_0}{n}T\right)^2\left(1-\frac{2a_0}{n}t\right)^{-2}\varphi(x,t), \tag{45}$$

firstly.

Now we want to estimate $\int_M|\nabla_{g(t)}\psi(\cdot,t)|^2_{g(t)}d\mu_{g(t)}$, which we denote as $E(t)$. We calculate that

$$\begin{aligned}\partial_t E(t) =&\partial_t \int_M \psi(\cdot,t)_i \psi(\cdot,t)_j g(t)^{ij} d\mu_{g(t)} \\ =&2\int_M \partial_t\psi(\cdot,t)_i \psi(\cdot,t)_j g(t)^{ij} d\mu_{g(t)} + \int_M \psi(\cdot,t)_i \psi(\cdot,t)_j \partial_t g(t)^{ij} d\mu_{g(t)} + \\ &\int_M |\nabla_{g(t)}\psi(\cdot,t)|^2_{g(t)} \partial_t d\mu_{g(t)} \\ =&\int_M \left(2\langle \nabla_{g(t)}\partial_t\psi(\cdot,t), \nabla_{g(t)}\psi(\cdot,t)\rangle_{g(t)}\right) d\mu_{g(t)} + \int_M \psi(\cdot,t)_i \psi(\cdot,t)_j \partial_t g(t)^{ij} d\mu_{g(t)} + \\ &\int_M (-R_{g(t)}) |\nabla_{g(t)}\psi(\cdot,t)|^2_{g(t)} d\mu_{g(t)}. \end{aligned} \tag{46}$$

From the Ricci flow equation

$$\partial_t g(t)_{ij} = -2\mathrm{Ric}_{g(t);ij}, \tag{47}$$

we have

$$\partial_t g(t)^{ij} = 2\mathrm{Ric}^{ij}_{g(t)}. \tag{48}$$

Thus (46) becomes

$$\begin{aligned} &\partial_t E(t) \\ =&\int_M \Big(2\langle \nabla_{g(t)}\partial_t\psi(\cdot,t), \nabla_{g(t)}\psi(\cdot,t)\rangle_{g(t)} + 2\mathrm{Ric}_{g(t)}(\nabla_{g(t)}\psi(\cdot,t), \nabla_{g(t)}\psi(\cdot,t)) - \\ &R_{g(t)}|\nabla_{g(t)}\psi(\cdot,t)|^2_{g(t)}\Big) d\mu_{g(t)}. \end{aligned} \tag{49}$$

By (11), we have

$$\begin{aligned} &\int_M \langle \nabla_{g(t)}\partial_t\psi(\cdot,t), \nabla_{g(t)}\psi(\cdot,t)\rangle_{g(t)} d\mu_{g(t)} \\ =&\int_M -\langle \nabla_{g(t)}(\Delta_{g(t)}\psi(\cdot,t) - R_{g(t)}\psi(\cdot,t)), \nabla_{g(t)}\psi(\cdot,t)\rangle_{g(t)} d\mu_{g(t)} \\ =&\int_M \left(-\langle \nabla_{g(t)}\Delta_{g(t)}\psi(\cdot,t), \nabla_{g(t)}\psi(\cdot,t)\rangle_{g(t)} + \langle \nabla_{g(t)}(R_{g(t)}\psi(\cdot,t)), \nabla_{g(t)}\psi(\cdot,t)\rangle_{g(t)}\right) d\mu_{g(t)}. \end{aligned} \tag{50}$$

Using the Bochner formula, (50) becomes

$$\begin{aligned} &\int_M \langle \nabla_{g(t)}\partial_t\psi(\cdot,t), \nabla_{g(t)}\psi(\cdot,t)\rangle_{g(t)} d\mu_{g(t)} \\ =&\int_M \Big(-\frac{1}{2}\Delta_{g(t)}|\nabla_{g(t)}\psi(\cdot,t)|^2_{g(t)} + |\nabla^2_{g(t)}\psi(\cdot,t)|^2_{g(t)} + \mathrm{Ric}_{g(t)}(\nabla_{g(t)}\psi(\cdot,t), \nabla_{g(t)}\psi(\cdot,t)) \\ &+ \langle \nabla_{g(t)}(R_{g(t)}\psi(\cdot,t)), \nabla_{g(t)}\psi(\cdot,t)\rangle_{g(t)}\Big) d\mu_{g(t)}. \end{aligned} \tag{51}$$

By integration by parts, we have

$$\int_M \frac{1}{2}\Delta_{g(t)}|\nabla_{g(t)}\psi(\cdot,t)|^2_{g(t)} d\mu_{g(t)} = 0. \tag{52}$$

Combining (51) and (52), we have

$$\int_M \langle \nabla_{g(t)}\partial_t\psi(\cdot,t), \nabla_{g(t)}\psi(\cdot,t)\rangle_{g(t)} d\mu_{g(t)}$$

$$= \int_M \left(\left(|\nabla^2_{g(t)}\psi(\cdot,t)|^2_{g(t)} + \mathrm{Ric}_{g(t)}(\nabla_{g(t)}\psi(\cdot,t), \nabla_{g(t)}\psi(\cdot,t)) \right) - R_{g(t)}\psi(\cdot,t)\Delta_{g(t)}\psi(\cdot,t) \right) d\mu_{g(t)}. \tag{53}$$

Since $|\Delta_{g(t)}\psi(\cdot,t)|^2_{g(t)} \le C_1(n)|\nabla^2_{g(t)}\psi(\cdot,t)|^2$, using Cauchy inequality, we have

$$\begin{aligned}\int_M \left(-R_{g(t)}\psi(\cdot,t)\Delta_{g(t)}\psi(\cdot,t)\right) d\mu_{g(t)} \ge& -\frac{1}{2C_1(n)}\int_M |\Delta_{g(t)}\psi(\cdot,t)|^2_{g(t)} d\mu_{g(t)} \\ &- \frac{C_1(n)}{2}\int_M R^2_{g(t)}\psi(\cdot,t)^2 d\mu_{g(t)} \\ \ge& -\frac{1}{2}\int_M |\nabla^2_{g(t)}\psi(\cdot,t)|^d\mu_{g(t)} - \frac{C_1(n)}{2}\int_M R^2_{g(t)}\psi(\cdot,t)^2 d\mu_{g(t)}\end{aligned} \tag{54}$$

Combining (53) and (54) we have

$$\begin{aligned}\int_M \langle \nabla_{g(t)}\partial_t\psi(\cdot,t), \nabla_{g(t)}\psi(\cdot,t)\rangle_{g(t)} d\mu_{g(t)} \ge \int_M \big(&\mathrm{Ric}_{g(t)}(\nabla_{g(t)}\psi(\cdot,t), \nabla_{g(t)}\psi(\cdot,t)) - \\ &C(n)R^2_{g(t)}\psi(\cdot,t)^2\big) d\mu_{g(t)}.\end{aligned} \tag{55}$$

Combining (49) and (55), and by Cauchy inequality, we have

$$\begin{aligned}&\partial_t E(t) \\ \ge& \int_M \left(4\mathrm{Ric}_{g(t)}(\nabla_{g(t)}\psi(\cdot,t), \nabla_{g(t)}\psi(\cdot,t)) - R_{g(t)}|\nabla_{g(t)}\psi(\cdot,t)|^2_{g(t)} - C(n)R^2_{g(t)}\psi(\cdot,t)^2\right) d\mu_{g(t)} \\ \ge& \int_M \left(4|\mathrm{Ric}_{g(t)}|_{g(t)} - R_{g(t)}\right)|\nabla_{g(t)}\psi(\cdot,t)|^2 d\mu_{g(t)} - C(n)\int_M R^2_{g(t)}\psi(\cdot,t)^2 d\mu_{g(t)}.\end{aligned} \tag{56}$$

By Cauchy inequality and a direct calculus using a special coordinate, one has

$$2|\mathrm{Ric}_{g(t)}|^2_{g(t)} \ge \frac{2}{n}R^2_{g(t)}. \tag{57}$$

By (56) and (57), we have

$$\partial_t E(t) \ge -\int_M |\mathrm{Ric}_{g(t)}||\nabla_{g(t)}\psi(\cdot,t)|^2 d\mu_{g(t)} - C(n)\int_M R^2_{g(t)}\psi(\cdot,t)^2 d\mu_{g(t)}. \tag{58}$$

By Lemma 2.4 (2) and Lemma 3.1 (1) proved above, we have

$$\begin{aligned}\partial_t E(t) \ge -\frac{C(n,h,p,\|\hat{g}\|_{W^{1,p}(M)})}{t^{\frac{n}{4p}+\frac{3}{4}}}\int_M |\nabla_{g(t)}\psi(\cdot,t)|^2 d\mu_{g(t)} - \\ C(n,h,p,\|\hat{g}\|_{W^{1,p}(M)},\tilde{\varphi},a_0)\int_M R^2_{g(t)} d\mu_{g(t)}.\end{aligned}$$

In order to avoid the vanishing case, we consider $E(t)+1$, and we have

$$\begin{aligned}&\partial_t\left(E(t)+1\right) \\ \ge& -\frac{C(n,h,p,\|\hat{g}\|_{W^{1,p}(M)})}{t^{\frac{n}{4p}+\frac{3}{4}}}\left(E(t)+1\right) - C(n,h,p,\|\hat{g}\|_{W^{1,p}(M)},\tilde{\varphi},a_0)\int_M R^2_{g(t)} d\mu_{g(t)}.\end{aligned}$$

Dividing both sides by $(E(t)+1)$, we have

$$\partial_t \log\left(E(t)+1\right) \ge -\frac{C(n,h,p,\|\hat{g}\|_{W^{1,p}(M)})}{t^{\frac{n}{4p}+\frac{3}{4}}} - C(n,h,p,\|\hat{g}\|_{W^{1,p}(M)},\tilde{\varphi},a_0)\int_M R^2_{g(t)} d\mu_{g(t)}.$$

By Lemma 2.4 (3), $\int_M R^2_{g(t)} d\mu_{g(t)}$ is integrable on $(0,T)$ and the integral is controlled by

$C(n,h,p,\|\hat{g}\|_{W^{1,p}(M)})$. Thus taking integral we have

$$\log(E(T)+1)-\log(E(t)+1)\geq C(n,h,p,\|\hat{g}\|_{W^{1,p}(M)},\tilde{\varphi},a_0),\forall t\in[0,T]. \tag{59}$$

Recall that $\psi_T=\tilde{\varphi}$, since Ricci flow and h-flow are equivalent after a family of diffeomorphisms, we have

$$E(T)=\int_M|\nabla_{g(T)}\tilde{\varphi}(\cdot)|^2_{g(T)}d\mu_{g(T)}=\int_M|\nabla_{\bar{g}(T)}\tilde{\varphi}(\cdot)|^2_{\bar{g}(T)}d\mu_{\bar{g}(T)}, \tag{60}$$

where $\bar{g}(T)$ is the metric at time T of the h-flow $\bar{g}(t)$ given in Lemma 2.9. By Lemma 2.9, h is $1+\epsilon(n)$-fair to $\bar{g}(t)$, thus we have

$$E(T)\leq C(n)\int_M|\nabla_h\tilde{\varphi}(\cdot)|^2_h d\mu_h\leq C(n,h,\tilde{\varphi}) \tag{61}$$

Combining (59) and (61), we have

$$E(t)\leq C(n,h,p,\|\hat{g}\|_{W^{1,p}(M)},\tilde{\varphi},a_0),\forall t\in[0,T]. \tag{62}$$

By Lemma 3.1, (62) and Hölder inequality, we have

$$\|\varphi(\cdot,t)\|_{W^{1,\frac{p}{p-1}}(M)}\leq\|\varphi(\cdot,t)\|_{C^0(M)}+C(n,h)E^{1/2}(t)\leq C(n,h,p,\|\hat{g}\|_{W^{1,p}(M)},\tilde{\varphi},a_0),\forall t\in[0,T]. \tag{63}$$

which proves (4), thus the lemma is proved. □

§4 Ricci flow and scalar curvature lower bounds

In this section, we study the scalar curvature lower bounds along Ricci flow. The main result in this section is:

Lemma 4.1. *Let M^n be a compact smooth manifold with a metric $\hat{g}\in W^{1,p}(M)$ $(n<p\leq\infty)$. Suppose $R_{\hat{g}}\geq a_0$ in distirbutional sense for some constant a_0, and let $g(t),t\in(0,T_0]$ be the Ricci flow given in Lemma 2.4. Then for any $t\in(0,T_0]$, there holds $R_{g(t)}\geq a_0\left(1-\frac{2a_0}{n}t\right)^{-1}$ pointwisely on M.*

Proof. Let $\hat{g}_\delta$ be the family of smooth metrics constructed in Lemma 2.1, such that $\hat{g}_\delta$ converges to $\hat{g}$ in $W^{1,p}$-norm. For each smooth metric $\hat{g}_\delta$ we consider the Ricci flow $g_\delta(t)$ given in Lemma 2.4 with initial metric $\hat{g}_\delta$. It is known that by letting δ converge to 0^+, $g_\delta(t)(0\in(0,T_0])$ converge to a Ricci flow $g(t)(0\in(0,T_0])$ such that $\lim_{t\to0}d_{GH}((M,g(t)),(M,\hat{g}))=0$, where d_{GH} is the Gromov-Hausdorff distance.

For any $T\in(0,T_0]$ and any nonnegative $\tilde{\varphi}\in C^\infty(M)$, we will prove

$$\int_M\left(R_{g(T)}-a_0\left(1-\frac{2a_0}{n}t\right)^{-1}\right)\tilde{\varphi}d\mu_{g(T)}\geq0, \tag{64}$$

which is sufficient to give $R_{g(t)}\geq a_0\left(1-\frac{2a_0}{n}t\right)^{-1}$ pointwise on M.

To do this, for each $\delta\in(0,\delta_0]$ we consider the auxiliary functions φ_δ given in Lemma 3.1, such that

(1) $\varphi_\delta(\cdot,T)=\tilde{\varphi}(\cdot)$, on M.

(2) For any constant a_0, $\int_M\left(R_{g_\delta(t)}-a_0\left(1-\frac{2a_0}{n}t\right)^{-1}\right)\varphi_\delta(\cdot,t)d\mu_{g_\delta(t)}$ is monotonously increasing with respect to t (if $a_0>0$, then we require $t\leq T\leq T_0<\frac{n}{2a_0}$).

(3) $\varphi_\delta(\cdot,t) \le C(n,h,p,\|\hat{g}_\delta\|_{W^{1,p}(M)}, \|\tilde{\varphi}\|_{L^\infty}, a_0)$, $\forall t \in [0,T]$.

(4) $\|\varphi_\delta(\cdot,t)\|_{W^{1,\frac{n}{n-1}}(M)} \le C(n,h,p,\|\hat{g}_\delta\|_{W^{1,p}(M)}, \|\tilde{\varphi}\|_{L^\infty}, a_0)$, $\forall t \in [0,T]$.

Since $\hat{g}_\delta$ converges to $\hat{g}$ in $W^{1,p}$ norm , we have

$$\|\hat{g}_\delta\|_{W^{1,p}(M)} \le 2\|\hat{g}\|_{W^{1,p}(M)}. \tag{65}$$

Recall $h = \hat{g}_{\delta_0}$ only depends on $\hat{g}$. Thus by (65), the estimate above could be uniformized as

$$\varphi_\delta(\cdot,t) \le C(n,p,\hat{g},\tilde{\varphi},a_0), \forall t \in [0,T], \tag{66}$$

and

$$\|\varphi_\delta(\cdot,t)\|_{W^{1,\frac{n}{n-1}}(M)} \le C(n,p,\hat{g},\tilde{\varphi},a_0), \forall t \in [0,T]. \tag{67}$$

Let us estimate the integral

$$\int_M \left(R_{g_\delta(t)} - a_0\left(1-\frac{2a_0}{n}t\right)^{-1}\right)\varphi_\delta(\cdot,t)d\mu_{g_\delta(t)}, \tag{68}$$

firstly.

By the monotonicity, we have

$$\int_M \left(R_{g_\delta(t)} - a_0\left(1-\frac{2a_0}{n}t\right)^{-1}\right)\varphi_\delta(\cdot,t)d\mu_{g_\delta(t)} \ge \int_M \left(R_{\hat{g}_\delta} - a_0\right)\varphi_\delta(\cdot,0)d\mu_{\hat{g}_\delta}. \tag{69}$$

To estimate $\int_M \left(R_{\hat{g}_\delta} - a_0\right)\varphi_\delta(\cdot,0)d\mu_{\hat{g}_\delta}$, by Lemma 2.2, we have

$$\left|\int_M R_{\hat{g}_\delta}\varphi_\delta(\cdot,0)d\mu_{\hat{g}_\delta} - \langle R_{\hat{g}}, \varphi_\delta(\cdot,0)\rangle\right| \le \Psi(\delta|\hat{g})\|\varphi_\delta(\cdot,0)\|_{W^{1,\frac{n}{n-1}}(M)}, \forall\varphi \in C^\infty(M), \tag{70}$$

where $\Psi(\delta|\hat{g})$ is a positive function such that $\lim_{\delta\to 0^+}\Psi(\delta|\hat{g}) = 0$ for any fixed $\hat{g}$, and $\Psi(\delta|\hat{g})$ varies from line to line.

Moreover, by Sobolev embedding we have $\lim_{\delta\to 0^+}\left\|\frac{d\mu_{\hat{g}_\delta}}{d\mu_{\hat{g}}} - 1\right\|_{C^0(M)} = 0$, thus by Hölder inequality, we have

$$\begin{aligned}
\left|\int_M \varphi_\delta(\cdot,0)d\mu_{\hat{g}_\delta} - \int_M \varphi_\delta(\cdot,0)d\mu_{\hat{g}}\right| &= \left|\int_M \varphi_\delta(\cdot,0)\left(\frac{d\mu_{\hat{g}_\delta}}{d\mu_{\hat{g}}} - 1\right)d\mu_{\hat{g}}\right| \\
&\le \left\|\frac{d\mu_{\hat{g}_\delta}}{d\mu_{\hat{g}}} - 1\right\|_{C^0(M)} \int_M |\varphi_\delta(\cdot,0)|d\mu_{\hat{g}} \\
&\le C(n,\hat{g})\left\|\frac{d\mu_{\hat{g}_\delta}}{d\mu_{\hat{g}}} - 1\right\|_{C^0(M)} \|\varphi_\delta(\cdot,0)\|_{W^{1,\frac{n}{n-1}}(M)} \\
&\le \Psi(\delta|\hat{g})\|\varphi_\delta(\cdot,0)\|_{W^{1,\frac{n}{n-1}}(M)}.
\end{aligned} \tag{71}$$

By triangular inequality,

$$\begin{aligned}
&\left|\int_M (R_{\hat{g}_\delta} - a_0)\varphi_\delta(\cdot,0)d\mu_{\hat{g}_\delta} - \langle R_{\hat{g}} - a_0, \varphi_\delta(\cdot,0)\rangle\right| \\
\le &\left|\int_M R_{\hat{g}_\delta}\varphi_\delta(\cdot,0)d\mu_{\hat{g}_\delta} - \langle R_{\hat{g}}, \varphi_\delta(\cdot,0)\rangle\right| + |a_0|\left|\int_M \varphi_\delta(\cdot,0)d\mu_{\hat{g}_\delta} - \int_M \varphi_\delta(\cdot,0)d\mu_{\hat{g}}\right|.
\end{aligned} \tag{72}$$

Combining (70)-(72), we have

$$\begin{aligned}&\left|\int_M (R_{\hat{g}_\delta} - a_0)\varphi_\delta(\cdot,0)d\mu_{\hat{g}_\delta} - \langle R_{\hat{g}} - a_0, \varphi_\delta(\cdot,0)\rangle\right| \\ \leq &\Psi(\delta|\hat{g})\|\varphi_\delta(\cdot,0)\|_{W^{1,\frac{n}{n-1}}(M)}, \forall\tilde{\varphi} \in C^\infty(M), \tilde{\varphi} \geq 0 \forall\delta \in (0,\delta_0].\end{aligned} \tag{73}$$

Since we have assumed $R_{\hat{g}} \geq a$ in the distributional sense, we have

$$\langle R_{\hat{g}} - a_0, \varphi_\delta(\cdot,0)\rangle \geq 0, \forall\varphi_\delta(\cdot,0) \in C^\infty(M), \varphi_\delta(\cdot,0) \geq 0. \tag{74}$$

Combining (69), (73) and (74), we have

$$\begin{aligned}\int_M \left(R_{g_\delta(t)} - a_0\left(1 - \frac{2a_0}{n}t\right)^{-1}\right)\varphi_\delta(\cdot,t)d\mu_{g_\delta(t)} &\geq \int_M (R_{\hat{g}_\delta} - a_0)\varphi_\delta(\cdot,0)d\mu_{\hat{g}_\delta} \\ &\geq -\Psi(\delta|\hat{g})\|\varphi_\delta(\cdot,0)\|_{W^{1,\frac{n}{n-1}}(M)}.\end{aligned} \tag{75}$$

By (67) and (75), we have

$$\begin{aligned}\int_M \left(R_{g_\delta(t)} - a_0\left(1 - \frac{2a_0}{n}t\right)^{-1}\right)\varphi_\delta(\cdot,t)d\mu_{g_\delta(t)} &\geq -C(n,p,\hat{g},\tilde{\varphi},a_0)\Psi(\delta|\hat{g}) \\ &\geq -\Psi(\delta|n,p,\hat{g},\tilde{\varphi},a_0), \forall t \in [0,T], \delta \in (0,\delta_0].\end{aligned} \tag{76}$$

In particular, letting $t = T$ in (76), we have

$$\int_M \left(R_{g_\delta(T)} - a_0\left(1 - \frac{2a_0}{n}T\right)^{-1}\right)\tilde{\varphi}d\mu_{g_\delta(T)} \geq -\Psi(\delta|n,p,\hat{g},\tilde{\varphi},a_0), \forall\delta \in (0,\delta_0]. \tag{77}$$

where $\Psi(\delta|n,p,\hat{g},\tilde{\varphi},a_0)$ denotes a positive function such that $\lim_{\delta\to 0^+}\Psi(\delta|n,p,\hat{g},\tilde{\varphi},a_0) = 0$ for any fixed $n, p, \hat{g}, \tilde{\varphi}$ and a_0.

By Simon's estimate, Lemma 2.7, as δ tends to 0, $g_\delta(T)$ smoothly converges to $g(T)$. Thus taking limit in (77), we have

$$\int_M (R_{g(T)} - a_0\left(1 - \frac{2a_0}{n}T\right)^{-1})\tilde{\varphi}d\mu_{g(T)} \geq 0, \forall T \in (0,T_0], \forall\tilde{\varphi} \in C^\infty(M), \tilde{\varphi} \geq 0. \tag{78}$$

Recall that $g(t)$ is a smooth metric for $t \in (0,T_0]$ and $R_{g(t)}$ is well defined pointwisely on M, thus by (78) we have $R_{g(t)} \geq a_0\left(1 - \frac{2a_0}{n}t\right)^{-1}$ pointwisely on M for any $t \in (0,T_0]$, which completes the proof of the theorem. □

§5 Proof of Theorem 1.2

In this section, we prove Theorem 1.2. Let us restate it as follows:

Theorem 5.1. *Let M^n be a compact manifold with $\sigma(M) \leq 0$ and $\hat{g}$ be a $W^{1,p}(n < p \leq \infty)$ metric on M with unit volume such that $R_{\hat{g}} \geq \sigma(M)$ in the distributional sense. Then $(M,\hat{g})$ is isometric to an Einstein manifold with scalar curvature equal to $\sigma(M)$.*

For any smooth manifold M, its Yamabe invariant $\sigma(M)$ is defined as:

$$\sigma(M) := \sup_{\mathcal{C}} \inf_{g\in\mathcal{C}} \frac{\int_M R_g d\mu_g}{(\mathrm{Vol}(M,g))^{(n-2)/2}}, \tag{79}$$

where $\mathcal{C}$ is the set that consists of every conformal class of Riemannian metrics on M, R_g is the scalar curvature of g, and $\mathrm{Vol}(M, g)$ is the volume of (M, g).

Roughly speaking, the basic idea of proving Theorem 5.1 is to flow the initial metric $\hat{g}$. The flow at positive time is smooth, thus we can prove they are Einstein by using Theorem 1.1.

Proof. **Step A: we consider a normalized Ricci flow $\mathring{g}(t)$ which keeps unit volume and prove that $R_{\mathring{g}(t)} \geq \sigma(M)$.**

In order to apply Theorem 1.1 to a flow, the volume needs to be preserved along the flow. Therefore, we consider such a normalized Ricci flow:

$$\mathring{g}(t) = (\mathrm{Vol}(M, g(t)))^{-2/n} g(t), t \in (0, T_0], \tag{80}$$

where $g(t)(t \in (0, T_0])$ is the Ricci flow with initial metric $\hat{g}$ (see Lemma 2.4).

Then by a standard calculation, we have

$$\mathrm{Vol}(M, \mathring{g}(t)) \equiv 1, t \in (0, T_0], \tag{81}$$

and

$$R_{\mathring{g}(t)} = (\mathrm{Vol}(M, g(t)))^{2/n} R_{g(t)}, t \in (0, T_0]. \tag{82}$$

We want to prove $R_{\mathring{g}(t)} \geq \sigma(M)$. To do this, recall that we have assumed $R_{\hat{g}} \geq \sigma(M)$ in the distributional sense, by Lemma 4.1, we have

$$R_{g(t)} \geq \sigma(M) \left(1 - \frac{2\sigma(M)}{n} t\right)^{-1}, \tag{83}$$

pointwisely on M.

Thus, we need to compare $(\mathrm{Vol}(M, g(t)))^{2/n}$ with $\left(1 - \frac{2\sigma(M)}{n} t\right)$.

By (26), we calculate that

$$\begin{aligned} \frac{d}{dt}\mathrm{Vol}(M, g(t)) &= \frac{d}{dt} \int_M d\mu_{g(t)} \\ &= \int_M (-R_{g(t)}) d\mu_{g(t)}. \end{aligned} \tag{84}$$

Combining (83) and (84), we have

$$\begin{aligned} \frac{d}{dt}\mathrm{Vol}(M, g(t)) &\leq -\int_M \sigma(M) \left(1 - \frac{2\sigma(M)}{n} t\right)^{-1} d\mu_{g(t)} \\ &= -\sigma(M) \left(1 - \frac{2\sigma(M)}{n} t\right)^{-1} \mathrm{Vol}(M, g(t)). \end{aligned} \tag{85}$$

Then we have

$$\frac{d}{dt} \log \mathrm{Vol}(M, g(t) \leq -\sigma(M) \left(1 - \frac{2\sigma(M)}{n} t\right)^{-1}. \tag{86}$$

Taking integral on $(0, t)$ we have

$$\begin{aligned} \log \mathrm{Vol}(M, g(t) - \log \mathrm{Vol}(M, \hat{g}) &\leq -\sigma(M) \frac{n}{2\sigma(M)} \log \left(1 - \frac{2\sigma(M)}{n} t\right) \\ &= \frac{n}{2} \log \left(1 - \frac{2\sigma(M)}{n} t\right). \end{aligned} \tag{87}$$

Recall that we have assumed $\hat{g}$ with unit volume, thus $\log \mathrm{Vol}(M, \hat{g}) = 0$, and we have

$$\mathrm{Vol}(M, g(t)) \leq \left(1 - \frac{2\sigma(M)}{n} t\right)^{n/2}. \tag{88}$$

Combining (82), (83) and (88), we have

$$R_{\mathring{g}(t)} \geq \sigma(M), \tag{89}$$

which completes the step A.

Step B: we prove the normalized Ricci flow $\mathring{g}(t)$ is independent of t.

Since $\mathring{g}(t)$ is a smooth Riemannian metric on M, and $\sigma(M)$ is nonpositive, by the classical theorem, Theorem 1.1, we have that $\mathring{g}(t)$ is Einstein with $\mathrm{Ric}_{\mathring{g}(t)} = \frac{\sigma(M)}{n} \mathring{g}(t)$.

Thus by (80), we have

$$\mathrm{Ric}_{g(t)} = \mathrm{Ric}_{\mathring{g}(t)} = \frac{\sigma(M)}{n} \mathring{g}(t). \tag{90}$$

On the other hand, by (80) we have

$$\begin{aligned} \frac{\partial}{\partial t} \mathring{g}(t) &= -\frac{2}{n} \left(\mathrm{Vol}(M, g(t))\right)^{-2/n-1} \frac{d}{dt} \mathrm{Vol}(M, g(t)) g(t) + \frac{\partial}{\partial t} g(t) \\ &= -\frac{2}{n} \left(\mathrm{Vol}(M, g(t))\right)^{-2/n-1} \frac{d}{dt} \mathrm{Vol}(M, g(t)) g(t) - 2\mathrm{Ric}_{g(t)}. \end{aligned} \tag{91}$$

By (90) and (91), we have

$$\begin{aligned} \frac{\partial}{\partial t} \mathring{g}(t) &= -\frac{2}{n} \left(\mathrm{Vol}(M, g(t))\right)^{-2/n-1} \frac{d}{dt} \mathrm{Vol}(M, g(t)) g(t) - \frac{2\sigma(M)}{n} \mathring{g}(t) \\ &= f(t) g(t), \end{aligned} \tag{92}$$

where $f(t) = -\frac{2}{n} \left(\left(\mathrm{Vol}(M, g(t))\right)^{-2/n-1} \frac{d}{dt} \mathrm{Vol}(M, g(t)) + \sigma(M) \right)$ is a constant on M for any fixed $t \in (0, T_0]$.

Thus $\mathring{g}(t)$ must be self-similar, that is, we have

$$\mathring{g}(t_1) = F(t_1, t_2) \mathring{g}(t_2), \ \forall t_1, t_2 \in (0, T_0], \tag{93}$$

where F is a constant on M depends only on t_1, t_2.

Thus, their Ricci curvature satisfies

$$\mathrm{Ric}_{\mathring{g}(t_1)} = \mathrm{Ric}_{\mathring{g}(t_2)} = \frac{\sigma(M)}{n} \mathring{g}(t). \tag{94}$$

By (90) and (94), we have

$$\frac{\sigma(M)}{n} \mathring{g}(t_1) = \mathrm{Ric}_{\mathring{g}(t_1)} = \mathrm{Ric}_{\mathring{g}(t_2)} = \frac{\sigma(M)}{n} \mathring{g}(t_2), \tag{95}$$

which gives

$$\mathring{g}(t_1) = \mathring{g}(t_2), \ \forall t_1, t_2 \in (0, T_0], \tag{96}$$

which completes the step B.

Step C: we prove the initial metric is isometric to an Einstein manifold with scalar curvature $\sigma(M)$.

Thus, by (92) and (96), the function $f(t)$ in (92) is identically zero on $M \times (0, T_0]$.

Recall $f(t) = -\frac{2}{n} \left(\left(\mathrm{Vol}(M, g(t))\right)^{-2/n-1} \frac{d}{dt} \mathrm{Vol}(M, g(t)) + \sigma(M) \right)$, thus $f \equiv 0$ gives an ODE:

$$\left(\mathrm{Vol}(M, g(t))\right)^{-2/n-1} \frac{d}{dt} \mathrm{Vol}(M, g(t)) + \sigma(M) = 0, \ t \in (0, T_0]. \tag{97}$$

Since we have assumed that $\hat{g}$ has unit volume, by Lemma 2.4 (1), we have

$$\lim_{t\to 0^+} \mathrm{Vol}(M, g(t)) = \mathrm{Vol}(M, \hat{g}) = 1. \tag{98}$$

Solving the ODEs (97) and (98), we have

$$\mathrm{Vol}(M, g(t)) = \left(1 - \frac{2\sigma(M)}{n} t\right)^{n/2}. \tag{99}$$

By (80), we have

$$\mathring{g}(t) = \left(1 - \frac{2\sigma(M)}{n} t\right)^{-1} g(t). \tag{100}$$

Since Lemma 2.4 (1) tells that

$$\lim_{t\to 0} d_{GH}((M, g(t)), (M, \hat{g})) = 0. \tag{101}$$

By (100) and (101), we have $\mathring{g}(t)$ converges to

$$\lim_{t\to 0} d_{GH}((M, \mathring{g}(t)), (M, \hat{g})) = 0. \tag{102}$$

However, by (96), $\mathring{g}(t)$ does not depend on t, thus, (102) gives

$$d_{GH}((M, \mathring{g}(t)), (M, \hat{g})) = 0, \ \forall t \in (0, T_0]. \tag{103}$$

Thus, by (94) and (103), we have that $\hat{g}$ is isometric to an Einstein manifold with scalar curvature $\sigma(M)$, which completes the proof of the theorem. □

Declarations

Conflict of interest The authors declare no conflict of interest.

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