Appl. Math. J. Chinese Univ.
2025, 40(2): 443-457

Hölder's inequalities for a class of analytic functions connected with a certain hybrid-type convolution operator

H. M. Srivastava[1,2,3,4,5,6,*] Sheza M. El-Deeb[7]

Abstract. By using a certain hybrid-type convolution operator, we first introduce a new subclass of normalized analytic functions in the open unit disk. For members of this analytic function class, we then derive several properties and characteristics including (for example) the modified Hadamard products, Hölder's inequalities and convolution properties as well as some closure properties under a general family of integral transforms.

§1 Introduction, Definitions and Preliminaries

Let $\mathcal{A}$ denote the class of functions of the form:

$$f(z) = z + \sum_{j=2}^{\infty} a_j\, z^j, \tag{1.1}$$

which are analytic in the open unit disk

$$\mathbb{U} = \{z : z \in \mathbb{C} \quad \text{and} \quad |z| < 1\}.$$

We denote by $\mathcal{S}$ the subclass of all functions in $\mathcal{A}$ which are univalent in $\mathbb{U}$ and indeed normalized by

$$f(0) = 0 = f'(0) - 1.$$

Suppose also that $\mathcal{S}^*(\gamma)$ and $\mathcal{C}(\delta)$ are the subclasses of $\mathcal{S}$ consisting of all functions which are, respectively, starlike and convex of order γ $(0 \leqq \gamma < 1)$ in $\mathbb{U}$. Thus, by definition, we have

$$\mathcal{S}^*(\gamma) := \left\{ f : f \in \mathcal{S} \quad \text{and} \quad \Re\left(\frac{zf'(z)}{f(z)}\right) > \gamma \qquad (0 \leqq \gamma < 1;\ z \in \mathbb{U}) \right\} \tag{1.2}$$

and

$$\mathcal{C}(\gamma) := \left\{ f : f \in \mathcal{S} \quad \text{and} \quad \Re\left(1 + \frac{zf''(z)}{f'(z)}\right) > \gamma \qquad (0 \leqq \gamma < 1;\ z \in \mathbb{U}) \right\}. \tag{1.3}$$

Received: 2022-07-01. Revised: 2022-09-26.
MR Subject Classification: 05A30, 30C45, 30A10, 30C20.
Keywords: analytic functions, univalent functions, binomial series: hybrid-type convolution operator, Hölder's inequalities, modified Hadamard product, convolution properties, closure properties under integral transforms.
Digital Object Identifier(DOI): https://doi.org/10.1007/s11766-025-4780-1.

*Corresponding author.

The function classes $\mathcal{S}^*(\gamma)$ and $\mathcal{C}\,(\gamma)$ were introduced by Reberston [14] and satisfy the following equivalence relation:

$$f(z) \in \mathcal{C}(\gamma) \iff zf'(z) \in \mathcal{S}^*(\gamma) \qquad (0 \leqq \gamma < 1;\ z \in \mathbb{U}). \tag{1.4}$$

We note also that

$$\mathcal{S}^*(0) =: \mathcal{S}^* \quad \text{and} \quad \mathcal{C}(0) =: \mathcal{C}$$

for the relatively more familiar classes $\mathcal{S}^*$ and $\mathcal{C}$ of starlike and convex functions in $\mathbb{U}$.

Definition 1. For functions $f, g \in \mathcal{A}$, we say that the function f is subordinate to g, written as $f(z) \prec g(z)$, if there exists a *Schwarz function* ϖ, which is analytic in $\mathbb{U}$, with

$$\varpi(0) = 0 \quad \text{and} \quad |\varpi(z)| < 1 \qquad (\forall\ z \in \mathbb{U}),$$

such that

$$f(z) = g\big(\varpi(z)\big) \qquad (z \in \mathbb{U}).$$

Furthermore, if the function g is univalent in $\mathbb{U}$, then we have the following equivalence (see, for details, [9]; see also [3]):

$$f(z) \prec g(z) \iff f(0) = g(0) \quad \text{and} \quad f(\mathbb{U}) \subset g(\mathbb{U}).$$

We now let $\mathcal{T}$ denote the subclass of the univalent function class $\mathcal{S}$ consisting of functions of the form:

$$f\,(z) = z - \sum_{j=2}^{\infty} a_j z^j \qquad \big(a_j \geqq 0;\ z \in \mathbb{U}\big). \tag{1.5}$$

Recently, Nishiwaki *et al.* [12] investigated some Hölder-type inequalities for a subclass of uniformly starlike functions in $\mathbb{U}$. We also recall the generalized convolutions, studied by Choi *et al.* [6], which are defined for the functions $f_\ell(z) \in \mathcal{T}$ given by

$$f_\ell(z) = z - \sum_{j=2}^{\infty} a_{j,\ell}\ z^j \qquad (a_{j,\ell} \geqq 0;\ \ell = 1, 2, 3, \cdots, n)\,. \tag{1.6}$$

We define these modified Hadamard products (or convolutions) $\mathcal{G}_n(z)$ and $\mathcal{H}_n(z)$ by

$$\mathcal{G}_n(z) := z - \sum_{j=2}^{\infty} \left(\prod_{\ell=1}^{n} a_{j,\ell} \right) z^j \tag{1.7}$$

and

$$\mathcal{H}_n(z) := z - \sum_{j=2}^{\infty} \left(\prod_{\ell=1}^{n} (a_{j,\ell})^{p_\ell} \right) z^j \qquad \big(p_\ell > 0;\ \ell = 1, 2, 3, \cdots, n\big), \tag{1.8}$$

respectively. Clearly, $\mathcal{G}_n(z)$ denotes the modified Hadamard product (or convolution) of $f_\ell(z)$ $(\ell = 1, 2, 3, \cdots, n)$ given by (1.6). Therefore, $\mathcal{H}_n(z)$ is a generalization of the modified Hadamard product (or convolution) $\mathcal{G}_n(z)$.

We note each of the following special cases:

(i) For $n = 2$, it is easily seen that

$$\mathcal{G}_2(z) = (f_1 * f_2)(z),$$

where the functions $f_1, f_2 \in \mathcal{T}$ are given by (1.6).

(ii) For $p_\ell = 1$ $(\ell = 1, 2, 3, \cdots, n)$, we have

$$\mathcal{H}_n(z) = \mathcal{G}_n(z).$$

Next, for functions $f_\ell(z)$ $(\ell = 1,2,3,\cdots,n)$ given by (1.6), the familiar Hölder inequality assumes the following form (see [7,10,11,15]):

$$\sum_{j=2}^{\infty}\left(\prod_{\ell=1}^{n} a_{j,\ell}\right) \leqq \prod_{\ell=1}^{n}\left(\sum_{j=2}^{\infty}(a_{j,\ell})^{p_\ell}\right)^{\frac{1}{p_\ell}} \tag{1.9}$$

$$\left(p_\ell \geqq 1;\ \ell = 1,2,3,\cdots,n;\ \sum_{\ell=1}^{n}\frac{1}{p_\ell} \geqq 1\right).$$

The operator $\mathcal{L}_\lambda^\sigma : \mathbb{U} \to \mathbb{U}$, studied by Babalola [1], is defined by

$$\mathcal{L}_\lambda^\sigma f\,(z) := \left(\rho_\sigma * \rho_{\sigma,\lambda}^{-1} * f\right)(z)\,, \tag{1.10}$$

where

$$\rho_{\sigma,\lambda}\,(z) = \frac{z}{(1-z)^{\sigma-\lambda+1}}, \quad \sigma-\lambda+1>0, \quad \text{and} \quad \rho_\sigma = \rho_{\sigma,0}.$$

and $\rho_{\sigma,\lambda}^{-1}$ is given by

$$\left(\rho_{\sigma,\lambda} * \rho_{\sigma,\lambda}^{-1}\right)(z) = \frac{z}{1-z} \qquad (\sigma,\lambda \in \mathbb{N} := \{1,2,3,\cdots\})\,.$$

If the function $f \in \mathcal{A}$ is given by (1.1), then (1.10) is equivalent to

$$\mathcal{L}_\lambda^\sigma f\,(z) = z + \sum_{j=2}^{\infty}\left(\frac{\Gamma(\sigma+j)}{\Gamma(\sigma+1)}\cdot\frac{(\sigma-\lambda)!}{(\sigma+j-\lambda-1)!}\right) a_j\ z^j \qquad (z \in \mathbb{U}).$$

Making use the binomial series:

$$(1-\delta)^n = \sum_{\ell=0}^{n}\binom{n}{\ell}(-\delta)^\ell \qquad (n \in \mathbb{N}_0 := \mathbb{N}\cup\{0\} = \{0,1,2,\cdots\})\,,$$

for $f \in \mathcal{A}$, we introduce the linear derivative operator defined successively as follows:

$$\mathcal{D}_{n,\delta,\lambda}^{\sigma,0} f\,(z) = f\,(z)\,,$$

$$\begin{aligned}
\mathcal{D}_{n,\delta,\lambda}^{\sigma,1} f\,(z) &= \mathcal{D}_{n,\delta,\lambda}^{\sigma} f\,(z) \\
&= (1-\delta)^n\,\mathcal{L}_\lambda^\sigma f\,(z) + [1-(1-\delta)^n]\,z\,(\mathcal{L}_\lambda^\sigma f)'\,(z) \\
&= z + \sum_{j=2}^{\infty}\left[1+(j-1)\,\mathfrak{c}_n^\delta\right]\left(\frac{\Gamma(\sigma+j)}{\Gamma(\sigma+1)}\cdot\frac{(\sigma-\lambda)!}{(\sigma+j-\lambda-1)!}\right) a_j\ z^j,
\end{aligned}$$

and, in general,

$$\begin{aligned}
\mathcal{D}_{n,\delta,\lambda}^{\sigma,m} f\,(z) &= \mathcal{D}_{n,\delta,\lambda}^{\sigma}\left(\mathcal{D}_{n,\delta,\lambda}^{\sigma,m-1} f\,(z)\right) \\
&= (1-\delta)^n\,\mathcal{D}_{n,\delta,\lambda}^{\sigma,m-1} f\,(z) + [1-(1-\delta)^n]\,z\left(\mathcal{D}_{n,\delta,\lambda}^{\sigma,m-1} f\,(z)\right)' \\
&= z + \sum_{j=2}^{\infty}\left[1+(j-1)\,\mathfrak{c}_n^\delta\right]^m\left(\frac{\Gamma(\sigma+j)}{\Gamma(\sigma+1)}\cdot\frac{(\sigma-\lambda)!}{(\sigma+j-\lambda-1)!}\right) a_j\ z^j \\
&= z + \sum_{j=2}^{\infty}\varphi_j\ a_j\ z^j \qquad (\delta>0;\ n,\sigma,\lambda \in \mathbb{N};\ m \in \mathbb{N}_0 := \mathbb{N}\cup\{0\})\,,
\end{aligned} \tag{1.11}$$

where

$$\varphi_j = \left[1+(j-1)\,\mathfrak{c}_n^\delta\right]^m\left(\frac{\Gamma(\sigma+j)}{\Gamma(\sigma+1)}\cdot\frac{(\sigma-\lambda)!}{(\sigma+j-\lambda-1)!}\right) \tag{1.12}$$

and (for convenience)

$$\mathfrak{c}_n^{\delta} := -\sum_{\ell=1}^{n} \binom{n}{\ell} (-\delta)^{\ell} = 1 - (1-\delta)^n \qquad (n \in \mathbb{N}).$$

It follows from (1.11) that

$$\mathfrak{c}_n^{\delta}\ z\ \left(\mathcal{D}_{n,\delta,\lambda}^{\sigma,m} f(z)\right)' = \mathcal{D}_{n,\delta,\lambda}^{\sigma,m+1} f(z) - \left(1-\mathfrak{c}_n^{\delta}\right)\mathcal{D}_{n,\delta,\lambda}^{\sigma,m} f(z). \tag{1.13}$$

By using the hybrid-type convolution operator $\mathcal{D}_{n,\delta,\lambda}^{\sigma,m}$, we define a subclass

$$\mathcal{H}_{n,\delta,\lambda}^{\sigma,m,\kappa}(\beta,\alpha,\eta,A,B)$$

of the class $\mathcal{T}$ as follows.

Definition 2. For $\delta > 0$, $n,\sigma,\lambda \in \mathbb{N}$, $m \in \mathbb{N}_0$, $0 \leqq \kappa \leqq 1$, $0 < \alpha \leqq 1$, $0 \leqq \beta < 1$, $-1 \leqq B < A \leqq 1$ and $0 \leqq \eta \leqq 1$, let $\mathcal{H}_{n,\delta,\lambda}^{\sigma,m,\kappa}(\beta,\alpha,\eta,A,B)$ denote the subclass of the class $\mathcal{T}$ consisting of functions of the form (1.5) and satisfying the following condition:

$$\left|\frac{\mathcal{F}_{n,\delta,\lambda}^{\sigma,m,\kappa}(z)-1}{\eta\,(B-A)\left(\mathcal{F}_{n,\delta,\lambda}^{\sigma,m,\kappa}(z)-\beta\right)-B\left(\mathcal{F}_{n,\delta,f}^{\sigma,m,\kappa}(z)-1\right)}\right| < \alpha \tag{1.14}$$

where

$$\mathcal{F}_{n,\delta,\lambda}^{\sigma,m,\kappa}(z) = \frac{z\big(\mathcal{D}_{n,\delta,\lambda}^{\sigma,m} f(z)\big)' + \kappa z^2\big(\mathcal{D}_{n,\delta,\lambda}^{\sigma,m} f(z)\big)''}{(1-\kappa)\mathcal{D}_{n,\delta,\lambda}^{\sigma,m} f(z) + \kappa z\big(\mathcal{D}_{n,\delta,\lambda}^{\sigma,m} f(z)\big)'} \qquad (0 \leqq \rho \leqq 1). \tag{1.15}$$

The constants A and B $(-1 \leqq B < A \leqq 1)$ were indeed used in defining the Janowski-type analytic functions by means of the principle of subordination (see Definition 1).

For functions f in the subclass $\mathcal{H}_{n,\delta,\lambda}^{\sigma,m,\kappa}(\beta,\alpha,\eta,A,B)$ of the function class $\mathcal{T}$, which is defined above (see Definition 2) by using the hybrid-type convolution operator in (1.11), we propose to derive several properties and characteristics including (for example) the modified Hadamard products, Hölder's inequalities and convolution properties as well as some closure properties under a general family of integral transforms.

§2 A Set of Coefficient Estimates

Unless otherwise mentioned, we shall assume in the remainder of this paper that $\delta > 0$, $n,\sigma,\lambda \in \mathbb{N}$, $m \in \mathbb{N}_0$, $0 \leqq \kappa \leqq 1$, $0 < \alpha \leqq 1$, $0 \leqq \beta < 1$, $-1 \leqq B < A \leqq 1$, $0 \leqq \eta \leqq 1$ and $z \in \mathbb{U}$, and the powers are understood to be the principle values.

In the following theorem, we obtain the necessary and sufficient conditions for functions f in the class $\in \mathcal{H}_{n,\delta,\lambda}^{\sigma,m,\kappa}(\beta,\alpha,\eta,A,B)$.

Theorem 1. *Let the function f be defined by* (1.5). *Then the function f is in the class $\mathcal{H}_{n,\delta,\lambda}^{\sigma,m,\kappa}(\beta,\alpha,\eta,A,B)$ if and only if*

$$\sum_{j=2}^{\infty} C_j\ a_j \leqq \alpha\,(B-A)\,\eta\,(1-\beta), \tag{2.1}$$

where

$$C_j := (1+j\kappa-\kappa)[(1-B\alpha)\,(j-1)+(B-A)\,\eta\alpha\,(j-\beta)]\varphi_j \qquad (j \in \mathbb{N}\setminus\{1\}) \tag{2.2}$$

and φ_j *is given by* (1.12).

Proof. We first assume that the inequality (2.1) holds true. Then we find from (1.5) and (1.14) that

$$\begin{aligned}
&\left|\mathcal{F}^{\sigma,m,\kappa}_{n,\delta,\lambda}(z)-1\right|-\alpha\left|\eta\,(B-A)\left(\mathcal{F}^{\sigma,m,\kappa}_{n,\delta,\lambda}(z)-\beta\right)-B\left(\mathcal{F}^{\sigma,m,\kappa}_{n,\delta,\lambda}(z)-1\right)\right|\\
&\quad=\left|\sum_{j=2}^{\infty}(1+j\kappa-\kappa)\,(j-1)\,\varphi_j a_j z^j\right|\\
&\qquad-\left|\alpha\,(B-A)\,\eta\,(1-\beta)\,z\right.\\
&\qquad\left.+\sum_{j=2}^{\infty}\alpha(1+j\kappa-\kappa)\,[\eta\,(B-A)\,(j-\beta)-B\,(j-1)]\,\varphi_j a_j z^j\right|,
\end{aligned}$$

that is, that

$$\begin{aligned}
&\left|\mathcal{F}^{\sigma,m,\kappa}_{n,\delta,\lambda}(z)-1\right|-\alpha\left|\eta\,(B-A)\left(\mathcal{F}^{\sigma,m,\kappa}_{n,\delta,\lambda}(z)-\beta\right)-B\left(\mathcal{F}^{\sigma,m,\kappa}_{n,\delta,\lambda}(z)-1\right)\right|\\
&\quad\leqq\sum_{j=2}^{\infty}(1+j\kappa-\kappa)\,(j-1)\,\varphi_j a_j\ z^{j-1}r^k-\alpha\,(B-A)\,|\eta|\,(1-\beta)\,r\\
&\qquad+\sum_{j=2}^{\infty}\alpha(1+j\kappa-\kappa)\,[\eta\,(B-A)\,(j-\beta)-B\,(j-1)]\,\varphi_j a_j\ r^j\\
&\quad\leqq\sum_{j=2}^{\infty}(1+j\kappa-\kappa)[(1-B\alpha)\,(j-1)+(B-A)\,|\eta|\,\alpha\,(j-\beta)]\varphi_j a_j\\
&\qquad-\alpha\,(B-A)\,|\eta|\,(1-\beta)\leqq 0\qquad(z\in\mathbb{U}).
\end{aligned}$$

Hence, by the *Maximum Modulus Theorem*, we have $f(z)\in\mathcal{H}^{\sigma,m,\kappa}_{n,\delta,\lambda}(\beta,\alpha,\eta,A,B)$.

Conversely, we suppose that

$$\begin{aligned}
&\left|\frac{\mathcal{F}^{\sigma,m,\kappa}_{n,\delta,\lambda}(z)-1}{\eta\,(B-A)\left(\mathcal{F}^{\sigma,m,\kappa}_{n,\delta,\lambda}(z)-\beta\right)-B\left(\mathcal{F}^{\sigma,m,\kappa}_{n,\delta,\lambda}(z)-1\right)}\right|\\
&\quad=\left|\frac{\sum\limits_{j=2}^{\infty}(1+j\kappa-\kappa)\,(j-1)\,\varphi_j z^j}{(B-A)\,\eta\,(1-\beta)\,z-\sum\limits_{j=2}^{\infty}(1+j\kappa-\kappa)\,[\eta\,(B-A)\,(j-\beta)-B\,(j-1)]\,\varphi_j a_j z^j}\right|\\
&\quad<\alpha\qquad(z\in\mathbb{U}).
\end{aligned}$$

Since $\Re(z)\leqq|z|$ for all z, we find that

$$\begin{aligned}
&\Re\left(\frac{\sum\limits_{j=2}^{\infty}(1+j\kappa-\kappa)\,(j-1)\,\varphi_j z^j}{(B-A)\,\eta\,(1-\beta)\,z-\sum\limits_{j=2}^{\infty}(1+j\kappa-\kappa)\,[\eta\,(B-A)\,(j-\beta)-B\,(j-1)]\,\varphi_j a_j z^j}\right)\\
&\quad<\alpha. \qquad (2.3)
\end{aligned}$$

We now choose values of z on the real axis in the complex z-plane so that $f'(z)$ is real. Then, upon clearing the denominator in (2.3) and letting $z \to 1-$ through real values, we have

$$\sum_{j=2}^{\infty}(1+j\kappa-\kappa)[(1-B\alpha)(j-1)+(B-A)\eta\alpha(j-\beta)]\varphi_j a_j - \alpha(B-A)\eta(1-\beta) \leqq 0.$$

This completes the proof of Theorem 1. □

Upon setting $\kappa = 0$ and $\kappa = 1$ in Theorem 1, we get the following corollaries.

Corollary 1. *Let the function f be defined by (1.5). Then the function f is in the class $\mathcal{H}_{n,\delta,\lambda}^{\sigma,m,0}(\beta,\alpha,\eta,A,B)$ if and only if*

$$\sum_{j=2}^{\infty}[(1-B\alpha)(j-1)+(B-A)\eta\alpha(j-\beta)]\varphi_j a_j \leqq \alpha(B-A)\eta(1-\beta).$$

Corollary 2. *Let the function f be defined by (1.5). Then the function f is in the class $\mathcal{H}_{n,\delta,\lambda}^{\sigma,m,1}(\beta,\alpha,\eta,A,B)$ if and only if*

$$\sum_{j=2}^{\infty}j[(1-B\alpha)(j-1)+(B-A)\eta\alpha(j-\beta)]\varphi_j a_j \leqq \alpha(B-A)\eta(1-\beta).$$

§3 Convolution Properties for Functions in the Class $\mathcal{H}_{n,\delta,\lambda}^{\sigma,m,\kappa}(\beta,\alpha,\eta,A,B)$

In this section, by using the techniques of Schild and Silverman [13], we obtain some convolution properties for functions f in the class $\mathcal{H}_{n,\delta,\lambda}^{\sigma,m,\kappa}(\beta,\alpha,\eta,A,B)$.

Theorem 2. *Let the function f_1 defined by (1.6) be in the class $\mathcal{H}_{n,\delta,\lambda}^{\sigma,m,\kappa}(\beta_1,\alpha,\eta,A,B)$ and let the function f_2 defined by (1.6) be in the class $\mathcal{H}_{n,\delta,\lambda}^{\sigma,m,\kappa}(\beta_2,\alpha,\eta,A,B)$. If the coefficient sequence $\{C_j\}_{j=2}^{\infty}$ given by (2.2) is non-decreasing, then $f_1 * f_2 \in \mathcal{H}_{n,\delta,\lambda}^{\sigma,m,\kappa}(\beta^*,\alpha,\eta,A,B)$, where*

$$\beta^* = 1 - \frac{\alpha\eta(B-A)(1-\beta_1)(1-\beta_2)[1-B\alpha+\eta\alpha(B-A)]}{(1+\kappa)\sqrt{\Theta_2(\beta_1,\alpha,\eta,A,B)}\sqrt{\Theta_2(\beta_2,\alpha,\eta,A,B)}\,\varphi_2-[\alpha\eta(B-A)]^2(1-\beta_1)(1-\beta_2)}, \tag{3.1}$$

where

$$\Theta_2(\beta_1,\alpha,\eta,A,B) = [(1-B\alpha)+(B-A)\eta\alpha(2-\beta_1)] \tag{3.2}$$

and

$$\Theta_2(\beta_2,\alpha,\eta,A,B) = [(1-B\alpha)+(B-A)\eta\alpha(2-\beta_2)], \tag{3.3}$$

and φ_2 is given by (1.12) for $j = 2$.

Proof. In order to demonstrate Theorem 2, it is sufficient to show that

$$\sum_{j=2}^{\infty}\frac{(1+j\kappa-\kappa)[(1-B\alpha)(j-1)+(B-A)\eta\alpha(j-\beta^*)]\varphi_j\ a_{j,1}a_{j,2}}{\alpha\eta(B-A)(1-\beta^*)} \leqq 1, \tag{3.4}$$

where β^* is defined by (3.1). Since $f_1 \in \mathcal{H}_{n,\delta,\lambda}^{\sigma,m,\kappa}(\beta_1,\alpha,\eta,A,B)$, we have

$$\sum_{j=2}^{\infty}\frac{(1+j\kappa-\kappa)\lambda(\beta_1,\alpha,\eta,A,B)\varphi_j\ a_{j,1}}{\alpha\eta(B-A)(1-\beta_1)} \leqq 1. \tag{3.5}$$

Furthermore, since $f_2 \in \mathcal{H}_{n,\delta,\lambda}^{\sigma,m,\kappa}(\beta_2, \alpha, \eta, A, B)$, we have

$$\sum_{j=2}^{\infty} \frac{(1+j\kappa-\kappa)\lambda(\beta_2, \alpha, \eta, A, B)\varphi_j a_{j,2}}{\alpha\eta\,(B-A)\,(1-\beta_2)} \leqq 1. \tag{3.6}$$

In view of (3.5) and (3.6), we observe that

$$\Theta_j(\beta_1, \alpha, \eta, A, B) = [(1-B\alpha)\,(j-1) + (B-A)\,\eta\alpha\,(j-\beta_1)]$$

and

$$\Theta_j(\beta_2, \alpha, \eta, A, B) = [(1-B\alpha)\,(j-1) + (B-A)\,\eta\alpha\,(j-\beta_2)].$$

On the other hand, under the hypotheses of Theorem 2, and by the Cauchy-Schwarz inequality, we get

$$\sum_{j=2}^{\infty} \frac{(1+j\kappa-\kappa)\sqrt{\Theta_j(\beta_1, \alpha, \eta, A, B)}\sqrt{\Theta_j(\beta_2, \alpha, \eta, A, B)}}{\alpha\eta\,(B-A)\,\sqrt{(1-\beta_1)\,(1-\beta_2)}}\,\varphi_j\sqrt{a_{j,1}a_{j,2}} \leqq 1. \tag{3.7}$$

Thus, clearly, we find from (3.5) and (3.6) that

$$\sum_{k=2}^{\infty} \frac{(1+j\kappa-\kappa)^2\ \Theta_j(\beta_1, \alpha, \eta, A, B)\ \Theta_j(\beta_2, \alpha, \eta, A, B)\ \varphi_j^2 a_{j,1}a_{j,2}}{[\alpha\eta\,(B-A)]^2\,(1-\beta_1)\,(1-\beta_2)} \leqq 1. \tag{3.8}$$

We now turn toward finding the largest β^* such that

$$\sum_{k=2}^{\infty} \frac{(1+j\kappa-\kappa)[(1-B\alpha)\,(j-1) + (B-A)\,\eta\alpha\,(j-\beta^*)]\varphi_j\ a_{j,1}a_{j,2}}{\alpha\eta\,(B-A)\,(1-\beta^*)}$$
$$\leqq \sum_{k=2}^{\infty} \frac{(1+j\kappa-\kappa)\sqrt{\Theta_j(\beta_1, \alpha, \eta, A, B)}\sqrt{\Theta_j(\beta_2, \alpha, \eta, A, B)}}{\alpha\eta\,(B-A)\,\sqrt{(1-\beta_1)\,(1-\beta_2)}}\,\varphi_j\ \sqrt{a_{j,1}a_{j,2}}.$$

Indeed we have

$$\sqrt{a_{k,1}a_{k,2}} \leqq \frac{(1-\beta^*)\,\sqrt{\Theta_j(\beta_1, \alpha, \eta, A, B)}\sqrt{\Theta_j(\beta_2, \alpha, \eta, A, B)}}{\sqrt{(1-\beta_1)\,(1-\beta_2)}\ [(1-B\alpha)\,(j-1) + (B-A)\,\eta\alpha\,(j-\beta^*)]}.$$

By applying (3.7), it is sufficient to find the largest β^* such that

$$\frac{\alpha\eta\,(B-A)\,\sqrt{(1-\beta_1)\,(1-\beta_2)}}{(1+j\kappa-\kappa)\sqrt{\Theta_j(\beta_1, \alpha, \eta, A, B)}\sqrt{\Theta_j(\beta_2, \alpha, \eta, A, B)}\ \varphi_j}$$
$$\leqq \frac{(1-\beta^*)\,\sqrt{\Theta_j(\beta_1, \alpha, \eta, A, B)}\sqrt{\Theta_j(\beta_2, \alpha, \eta, A, B)}}{\sqrt{(1-\beta_1)\,(1-\beta_2)}[(1-B\alpha)\,(j-1) + (B-A)\,\eta\alpha\,(j-\beta^*)]},$$

which leads us to

$$\beta^* \leqq 1 - \frac{\alpha\eta\,(B-A)\,(1-\beta_1)\,(1-\beta_2)\,(j-1)\,[1-B\alpha+\eta\alpha\,(B-A)]}{(1+j\kappa-\kappa)\sqrt{\Theta_j(\beta_1, \alpha, \eta, A, B)}\sqrt{\Theta_j(\beta_2, \alpha, \eta, A, B)}\varphi_j - [\alpha\eta\,(B-A)]^2\,(1-\beta_1)\,(1-\beta_2)}.$$

Let us now put

$$\Phi(j) := \frac{\alpha\eta\,(B-A)\,(1-\beta_1)\,(1-\beta_2)\,(j-1)\,[1-B\alpha+\eta\alpha\,(B-A)]}{(1+j\kappa-\kappa)\sqrt{\Theta_j(\beta_1, \alpha, \eta, A, B)}\sqrt{\Theta_j(\beta_2, \alpha, \eta, A, B)}\varphi_j - [\alpha\eta\,(B-A)]^2\,(1-\beta_1)\,(1-\beta_2)}.$$

Since $\Phi(j)$ is a non-decreasing function of j $(j \geqq 2)$, we have $\beta^* \leqq 1 - \Phi(j)$, that is,

$$\beta^* \leqq 1 - \frac{\alpha\eta\,(B-A)\,(1-\beta_1)\,(1-\beta_2)\,[1-B\alpha+\eta\alpha\,(B-A)]}{(1+\kappa)\sqrt{\Theta_2(\beta_1, \alpha, \eta, A, B)}\sqrt{\Theta_2(\beta_2, \alpha, \eta, A, B)}\varphi_2 - [\alpha\eta\,(B-A)]^2\,(1-\beta_1)\,(1-\beta_2)}.$$

This completes the proof of Theorem 2. □

Theorem 3. *Let the function* f_ℓ $(\ell = 1,2)$ *defined by* (1.6) *be in the class* $\mathcal{H}^{\sigma,m,\kappa}_{n,\delta,\lambda}(\beta,\alpha,\eta,A,B)$. *If the coefficient sequence* $\{C_j\}_{j=2}^{\infty}$ *given by* (2.2) *is non-decreasing, then the function* $h(z)$ *given by*

$$h(z) = z - \sum_{j=2}^{\infty}\left(a_{j,1}^2 + a_{j,2}^2\right) z^j \tag{3.9}$$

belongs to the class $\mathcal{H}^{\sigma,m,\kappa}_{n,\delta,\lambda}(\chi,\alpha,\eta,A,B)$, *where*

$$\chi = 1 - \frac{2\eta\alpha\,(B-A)\,(1-\beta)^2\,[1-B\alpha+\eta\alpha\,(B-A)]}{(1+\kappa)[1-B\alpha+\eta\alpha\,(B-A)\,(2-\beta)]^2\varphi_2 - 2\,[\eta\alpha\,(B-A)\,(1-\beta)]^2} \tag{3.10}$$

and φ_2 *is given by* (1.12).

Proof. From Theorem 1, it is sufficient to prove that

$$\sum_{j=2}^{\infty}\frac{(1+j\kappa-\kappa)[(1-B\alpha)\,(j-1)+(B-A)\,\eta\alpha\,(j-\chi)]\varphi_j}{\alpha\,(B-A)\,\eta\,(1-\chi)}\left(a_{j,1}^2+a_{j,2}^2\right) \leqq 1.$$

Since the functions f_ℓ $(\ell = 1,2)$ are in the class $\mathcal{H}^{\sigma,m,\kappa}_{n,\delta,\lambda}(\beta,\alpha,\eta,A,B)$, we have

$$\begin{aligned}&\sum_{j=2}^{\infty}\left\{\frac{(1+j\kappa-\kappa)[(1-B\alpha)\,(j-1)+(B-A)\,\eta\alpha\,(j-\beta)]\varphi_j}{\alpha\,(B-A)\,\eta\,(1-\beta)}\right\}^2 a_{j,1}^2\\ &\qquad\leqq \sum_{j=2}^{\infty}\left\{\frac{(1+j\kappa-\kappa)[(1-B\alpha)\,(j-1)+(B-A)\,\eta\alpha\,(j-\beta)]\varphi_j}{\alpha\,(B-A)\,\eta\,(1-\beta)}a_{j,1}\right\}^2 \leqq 1\end{aligned} \tag{3.11}$$

and

$$\begin{aligned}&\sum_{j=2}^{\infty}\left\{\frac{(1+j\kappa-\kappa)[(1-B\alpha)\,(j-1)+(B-A)\,\eta\alpha\,(j-\beta)]\varphi_j}{\alpha\,(B-A)\,\eta\,(1-\beta)}\right\}^2 a_{j,2}^2\\ &\qquad\leqq \sum_{j=2}^{\infty}\left\{\frac{(1+j\kappa-\kappa)[(1-B\alpha)\,(j-1)+(B-A)\,\eta\alpha\,(j-\beta)]\varphi_j}{\alpha\,(B-A)\,\eta\,(1-\beta)}a_{j,2}\right\}^2 \leqq 1,\end{aligned} \tag{3.12}$$

It follows from (3.11) and (3.12) that

$$\frac{1}{2}\sum_{j=2}^{\infty}\left(\frac{(1+j\kappa-\kappa)[(1-B\alpha)\,(j-1)+(B-A)\,\eta\alpha\,(j-\beta)]\varphi_j}{\alpha\,(B-A)\,\eta\,(1-\beta)}\right)^2\left(a_{j,1}^2+a_{j,2}^2\right) \leqq 1.$$

Therefore, we need to find the largest χ such that

$$\begin{aligned}&\frac{(1+j\kappa-\kappa)[(1-B\alpha)\,(j-1)+(B-A)\,\eta\alpha\,(j-\chi)]\varphi_j}{\alpha\,(B-A)\,\eta\,(1-\chi)}\\ &\qquad\leqq \frac{1}{2}\left(\frac{(1+j\kappa-\kappa)[(1-B\alpha)\,(j-1)+(B-A)\,\eta\alpha\,(j-\beta)]\varphi_j}{\alpha\,(B-A)\,\eta\,(1-\beta)}\right)^2 \qquad (j \geqq 2),\end{aligned}$$

that is, that

$$\chi = 1 - \frac{2\eta\alpha\,(B-A)\,(1-\beta)^2\,(j-1)\,[1-B\alpha+\eta\alpha\,(B-A)]}{(1+j\kappa-\kappa)[(1-B\alpha)\,(j-1)+(B-A)\,\eta\alpha\,(j-\beta)]^2\varphi_j - 2\,[\eta\alpha\,(B-A)\,(1-\beta)]^2}.$$

We now let

$$\Psi(j) := \frac{2\eta\alpha\,(B-A)\,(1-\beta)^2\,(j-1)\,[1-B\alpha+\eta\alpha\,(B-A)]}{(1+j\kappa-\kappa)[(1-B\alpha)\,(j-1)+\eta\alpha\,(B-A)\,(j-\beta)]^2\varphi_j - 2\,[\eta\alpha\,(B-A)\,(1-\beta)]^2}.$$

Since $\Psi(j)$ is a non-decreasing function of $j\ \ (j \geqq 2)$, we have $\chi \leqq 1 - \Psi(j)$, that is,

$$\chi \leqq 1 - \frac{2\eta\alpha\,(B-A)\,(1-\beta)^2\,[1-B\alpha+\eta\alpha\,(B-A)]}{(1+\kappa)[1-B\alpha+\eta\alpha\,(B-A)\,(2-\beta)]^2\varphi_2 - 2\,[\eta\alpha\,(B-A)\,(1-\beta)]^2}.$$

This completes the proof of Theorem 3. □

§4 Hölder's Inequalities

Our main result on Hölder's Inequalities is given by Theorem 4 below.

Theorem 4. *Let the functions $f_\ell\ \ (\ell = 1,2,3,\cdots,n)$ defined by (1.6) be in the class $\mathcal{H}^{\sigma,m,\kappa}_{n,\delta,\lambda}(\beta_\ell,\alpha,\eta,A,B)$. Then $\mathcal{H}_n(z)$ is in the class $\mathcal{H}^{\sigma,m,\kappa}_{n,\delta,\lambda}(\chi,\alpha,\eta,A,B)$ such that*

$$\chi \leqq 1 - \frac{[\eta\alpha\,(B-A)]^{r-1}\,[1-B\alpha+\eta\alpha\,(B-A)]\prod\limits_{\ell=1}^{n}(1-\beta_\ell)^{p_\ell}}{[(1+\kappa)\varphi_2]^{r-1}\prod\limits_{\ell=1}^{n}[(1-B\alpha)+\eta\alpha\,(B-A)\,(2-\beta_\ell)]^{p_\ell} - [\eta\alpha\,(B-A)]^r\prod\limits_{\ell=1}^{n}[(1-\beta_\ell)]^{p_\ell}}, \tag{4.1}$$

where

$$r = \sum_{\ell=1}^{n} p_\ell \geqq 1, \quad p_\ell \geqq \frac{1}{q_\ell}, \quad \sum_{\ell=1}^{n}\frac{1}{q_\ell} \geqq 1 \quad \textit{and} \quad q_\ell > 1 \qquad (\ell = 1,2,3,\cdots,n)$$

and φ_2 is given by (1.12).

Proof. For $f_\ell(z) \in \mathcal{H}^{\sigma,m,\kappa}_{n,\delta,\lambda}(\beta_\ell,\alpha,\eta,A,B)\ \ (\ell = 1,2,3,\cdots,n)$, we have

$$\sum_{j=2}^{\infty}\frac{(1+j\kappa-\kappa)[(1-B\alpha)\,(j-1)+\eta\alpha\,(B-A)\,(j-\beta_\ell)]\varphi_j}{\eta\alpha\,(B-A)\,(1-\beta_\ell)}a_{j,\ell} \leqq 1, \tag{4.2}$$

which implies that

$$\left(\sum_{j=2}^{\infty}\frac{(1+j\kappa-\kappa)[(1-B\alpha)\,(j-1)+\eta\alpha\,(B-A)\,(j-\beta_\ell)]\varphi_j}{\eta\alpha\,(B-A)\,(1-\beta_\ell)}a_{j,\ell}\right)^{\frac{1}{q_\ell}} \leqq 1, \tag{4.3}$$

with

$$q_\ell > 1 \quad (\ell = 1,2,3,\cdots,n) \quad \text{and} \quad \sum_{\ell=1}^{n}\frac{1}{q_\ell} \geqq 1.$$

From (4.3), we have

$$\prod_{\ell=1}^{n}\left(\sum_{j=2}^{\infty}\frac{(1+j\kappa-\kappa)[(1-B\alpha)\,(j-1)+\eta\alpha\,(B-A)\,(j-\beta_\ell)]\varphi_j}{\eta\alpha\,(B-A)\,(1-\beta_\ell)}a_{j,\ell}\right)^{\frac{1}{q_j}} \leqq 1.$$

Thus, by applying Hölder's inequality (1.9), we find that

$$\sum_{j=2}^{\infty}\left[\prod_{\ell=1}^{n}\left(\frac{(1+j\kappa-\kappa)[(1-B\alpha)\,(j-1)+\eta\alpha\,(B-A)\,(j-\beta_\ell)]\varphi_j}{\eta\alpha\,(B-A)\,(1-\beta_\ell)}\right)^{\frac{1}{q_i}} a_{j,\ell}^{\frac{1}{q_\ell}}\right] \leqq 1.$$

We now have to determine the largest χ such that

$$\sum_{j=2}^{\infty} \frac{(1+j\kappa-\kappa)[(1-B\alpha)(j-1)+\eta\alpha(B-A)(j-\chi)]\varphi_j}{\eta\alpha(B-A)(1-\chi)} \left(\prod_{\ell=1}^{n} a_{j,\ell}^{p_\ell}\right) \leqq 1,$$

that is, that

$$\sum_{j=2}^{\infty} \frac{(1+j\kappa-\kappa)[(1-B\alpha)(j-1)+\eta\alpha(B-A)(j-\chi)]\varphi_j}{\eta\alpha(B-A)(1-\chi)} \left(\prod_{\ell=1}^{n} a_{j,\ell}^{p_\ell}\right)$$
$$\leqq \sum_{j=2}^{\infty} \left[\prod_{\ell=1}^{n} \left(\frac{(1+j\kappa-\kappa)[(1-B\alpha)(j-1)+\eta\alpha(B-A)(j-\beta_\ell)]\varphi_j}{\eta\alpha(B-A)(1-\beta_\ell)}\right)^{\frac{1}{q_\ell}} a_{j,\ell}^{\frac{1}{q_\ell}}\right].$$

Therefore, we need to find the largest χ such that

$$\frac{(1+j\kappa-\kappa)[(1-B\alpha)(j-1)+\eta\alpha(B-A)(j-\chi)]\varphi_j}{\eta\alpha(B-A)(1-\chi)} \left(\prod_{\ell=1}^{n} a_{j,\ell}^{p_\ell-\frac{1}{q_\ell}}\right)$$
$$\leqq \prod_{\ell=1}^{n} \left(\frac{(1+j\kappa-\kappa)[(1-B\alpha)(j-1)+\eta\alpha(B-A)(j-\beta_\ell)]\varphi_j}{\alpha(B-A)\eta(1-\beta_\ell)}\right)^{\frac{1}{q_\ell}},$$

for $j \geqq 2$. Since

$$\prod_{\ell=1}^{n} \left(\frac{(1+j\kappa-\kappa)[(1-B\alpha)(j-1)+(B-A)\eta\alpha(j-\chi)]\varphi_j}{\eta\alpha(B-A)(1-\chi)}\right)^{p_\ell-\frac{1}{q_\ell}} a_{j,\ell}^{p_\ell-\frac{1}{q_\ell}} \leqq 1$$
$$\left(p_\ell - \frac{1}{q_\ell} \geqq 0 \ \ (\ell = 1,2,3,\cdots,n)\right),$$

we see that

$$\prod_{\ell=1}^{n} a_{j,\ell}^{p_\ell-\frac{1}{q_\ell}}$$
$$\leqq \frac{1}{\prod\limits_{\ell=1}^{n} \left(\frac{(1+j\kappa-\kappa)[(1-B\alpha)(j-1)+\eta\alpha(B-A)(j-\beta_\ell)]\varphi_j}{\eta\alpha(B-A)(1-\beta_\ell)}\right)^{p_\ell-\frac{1}{q_\ell}}},$$

which leads us to the following inequality:

$$\frac{(1+j\kappa-\kappa)[(1-B\alpha)(j-1)+\eta\alpha(B-A)(j-\chi)]\varphi_j}{\eta\alpha(B-A)(1-\chi)}$$
$$\leqq \frac{\prod\limits_{\ell=1}^{n} [(1+j\kappa-\kappa)((1-B\alpha)(j-1)+\eta\alpha(B-A)(j-\beta_\ell))\varphi_j]^{p_\ell}}{\prod\limits_{\ell=1}^{n} [\eta\alpha(B-A)(1-\beta_\ell)]^{p_\ell}} \tag{4.4}$$

or, equivalently,

$$\chi \leqq$$
$$1 - \frac{(j-1)\Lambda_\ell \ [1-B\alpha+\eta\alpha(B-A)]}{\eta\alpha(B-A)\left[\prod\limits_{\ell=1}^{n} [(1+j\kappa-\kappa)\varphi_j]^{p_\ell-1} [(1-B\alpha)(j-1)+\eta\alpha(B-A)(j-\beta_\ell)]^{p_\ell} - \Lambda_\ell\right]}, \tag{4.5}$$

where

$$\Lambda_\ell := \prod_{\ell=1}^{n} [\eta\alpha(B-A)(1-\beta_\ell)]^{p_\ell} \qquad (\ell = 1,2,3,\cdots,n).$$

Let

$$\Xi(j) := 1 - \frac{(j-1)\,\Lambda_\ell\ \left[1 - B\alpha + \eta\alpha\,(B-A)\right]}{\eta\alpha\,(B-A)\left[\prod\limits_{\ell=1}^{n}\left[(1+j\kappa-\kappa)\varphi_j\right]^{p_\ell-1}\left[(1-B\alpha)\,(j-1)+\eta\alpha\,(B-A)\,(j-\beta_\ell)\right]^{p_\ell} - \Lambda_\ell\right]}, \tag{4.6}$$

which is an increasing function in $j \geqq 2$. Hence we have

$$\chi \leqq \Xi(2) = 1 - \frac{\left[\eta\alpha\,(B-A)\right]^{r-1}\left[1-B\alpha+\eta\alpha\,(B-A)\right]\prod\limits_{\ell=1}^{n}(1-\beta_\ell)^{p_\ell}}{\left[\left[(1+\kappa)\varphi_2\right]^{r-1}\prod\limits_{\ell=1}^{n}\left[(1-B\alpha)+\eta\alpha\,(B-A)\,(2-\beta_\ell)\right]^{p_\ell} - \left[\eta\alpha\,(B-A)\right]^{r}\prod\limits_{\ell=1}^{n}\left[(1-\beta_\ell)\right]^{p_\ell}\right]}. \tag{4.7}$$

This completes the proof of Theorem 4. □

If we set $p_\ell = 1$ $(\ell = 1, 2, 3, \cdots, m)$ in Theorem 4, we obtain of the following corollary:

Corollary 3. *Let the functions* f_ℓ $(\ell = 1, 2, 3, \cdots, m)$ *defined by* (1.6) *be in the class* $\mathcal{H}^{\sigma,m,\kappa}_{n,\delta,\lambda}(\beta_\ell, \alpha, \eta, A, B)$. *Then* $\mathcal{G}_m(z)$ *is in the class* $\mathcal{H}^{\sigma,m,\kappa}_{n,\delta,\lambda}(\varkappa, \alpha, \eta, A, B)$ *for*

$$\varkappa \leqq 1 - \frac{\left[\eta\alpha\,(B-A)\right]^{m-1}\left[1-B\alpha+\eta\alpha\,(B-A)\right]\prod\limits_{\ell=1}^{m}(1-\beta_\ell)}{\left[(1+\kappa)\varphi_2\right]^{m-1}\prod\limits_{\ell=1}^{m}\left[(1-B\alpha)+\eta\alpha\,(B-A)\,(2-\beta_\ell)\right] - \left[\eta\alpha\,(B-A)\right]^{m}\prod\limits_{\ell=1}^{m}(1-\beta_\ell)}. \tag{4.8}$$

§5 Closure Properties Under Integral Transforms

In this section, we first recall the following integral transform (see [11]):

$$\mathcal{I}_\omega(f)(z) := \int_0^1 \omega(t)\ \frac{f(zt)}{t}\ dt, \tag{5.1}$$

where ω is a real-valued and non-negative weight function which is normalized such that

$$\int_0^1 \omega(t)\ dt = 1,$$

for which $\mathcal{I}_\omega$ is known as the Bernardi operator (see [2]). Moreover, when

$$\omega(t) = \frac{(\mu+1)^\sigma}{\Gamma(\sigma)}\ t^\mu\ \left[\log\left(\frac{1}{t}\right)\right]^{\sigma-1} \qquad (\mu > -1;\ \sigma \geqq 0), \tag{5.2}$$

we are led to the Komatu operator (see [8]).

Theorem 5. *Let the function* f, *which is defined by* (1.5), *belong to the class* $\mathcal{H}^{\sigma,m,\kappa}_{n,\delta,\lambda}(\beta, \alpha, \eta, A, B)$. *Then* $\mathcal{I}_\omega(f)$ *is in the class* $\mathcal{H}^{\sigma,m,\kappa}_{n,\delta,\lambda}(\beta, \alpha, \eta, A, B)$.

Proof. From (5.1) and (5.2), we have

$$\begin{aligned}\mathcal{I}_\omega(f)(z) &= \frac{(-1)^{\sigma-1}(\mu+1)^\sigma}{\Gamma(\sigma)}\int_0^1 t^\mu(\log t)^{\sigma-1}\left(z-\sum_{j=2}^{\infty}a_j\ z^j t^{j-1}\right)dt \\ &= z-\sum_{j=2}^{\infty}\left(\frac{\mu+1}{\mu+j}\right)^\sigma a_j z^j. \end{aligned}\tag{5.3}$$

Now, in order to prove that $\mathcal{I}_\omega(f)\in\mathcal{H}_{n,\delta,\lambda}^{\sigma,m,\kappa}(\beta,\alpha,\eta,A,B)$, it is sufficient to show that

$$\sum_{j=2}^{\infty}\frac{(1+j\kappa-\kappa)[(1-B\alpha)(j-1)+(B-A)\eta\alpha(j-\beta)]\varphi_j}{\alpha(B-A)\eta(1-\beta)}\left(\frac{\mu+1}{\mu+j}\right)^\sigma a_j \leqq 1. \tag{5.4}$$

Since, in view of Theorem 1, $f\in\mathcal{H}_{n,\delta,\lambda}^{\sigma,m,\kappa}(\beta,\alpha,\eta,A,B)$ if and only if

$$\sum_{j=2}^{\infty}\frac{(1+j\kappa-\kappa)[(1-B\alpha)(j-1)+(B-A)\eta\alpha(j-\beta)]\varphi_j}{\alpha(B-A)\eta(1-\beta)}a_j \leqq 1, \tag{5.5}$$

the assertion (5.4) holds true because

$$\frac{\mu+1}{\mu+j}<1 \qquad (j\geqq 2).$$

This completes the proof of Theorem 5. □

Theorem 6. *Let the function f, which is defined by (1.5), belong to the class $\mathcal{H}_{n,\delta,\lambda}^{\sigma,m,\kappa}(\beta,\alpha,\eta,A,B)$. Then $\mathcal{I}_\omega(f)$ is starlike of order η $(0\leqq\eta<1)$ in $|z|<\mathcal{R}_1$, where*

$$\begin{aligned}\mathcal{R}_1 := \inf_j\Bigg\{&\left(\frac{\mu+j}{\mu+1}\right)^\sigma \\ &\cdot\frac{(1-\eta)(1+j\kappa-\kappa)[(1-B\alpha)(j-1)+(B-A)\eta\alpha(j-\beta)]\varphi_j}{\alpha(j-\eta)(B-A)\eta(1-\beta)}\Bigg\}^{\frac{1}{j-1}},\end{aligned}\tag{5.6}$$

in which φ_j is given by (1.12).

Proof. It is sufficient to prove that

$$\left|\frac{z(\mathcal{I}_\omega(f)(z))'}{\mathcal{I}_\omega(f)(z)}-1\right|<1-\eta \qquad (|z|<\mathcal{R}_1),$$

where $\mathcal{R}_1$ is given by (5.6). It is easily observed that

$$\left|\frac{z(\mathcal{I}_\omega(f)(z))'}{\mathcal{I}_\omega(f)(z)}-1\right|\leqq\frac{\sum\limits_{j=2}^{\infty}(j-1)\left(\frac{\mu+1}{\mu+j}\right)^\sigma a_j|z|^{j-1}}{1-\sum\limits_{j=2}^{\infty}\left(\frac{\mu+1}{\mu+j}\right)^\sigma a_j|z|^{j-1}},$$

so that

$$\left|\frac{z(\mathcal{I}_\omega(f)(z))'}{\mathcal{I}_\omega(f)(z)}-1\right|<1-\eta,$$

if

$$\sum_{j=2}^{\infty}\left(\frac{j-\eta}{1-\eta}\right)\left(\frac{\mu+1}{\mu+j}\right)^\sigma a_j|z|^{j-1}\leqq 1. \tag{5.7}$$

Since $f \in \mathcal{H}_{n,\delta,\lambda}^{\sigma,m,\kappa}(\beta,\alpha,\eta,A,B)$, we find from (5.5) and (5.7) that

$$\left(\frac{j-\eta}{1-\eta}\right)\left(\frac{\mu+1}{\mu+j}\right)^{\sigma}|z|^{j-1} \leqq \frac{(1+j\kappa-\kappa)[(1-B\alpha)(j-1)+(B-A)\eta\alpha(j-\beta)]\varphi_j}{\alpha(B-A)\eta(1-\beta)}.$$

Thus we have

$$|z| \leqq \left\{\left(\frac{\mu+j}{\mu+1}\right)^{\sigma}\frac{(1-\eta)(1+j\kappa-\kappa)[(1-B\alpha)(j-1)+(B-A)\eta\alpha(j-\beta)]\varphi_j}{\alpha(j-\eta)(B-A)\eta(1-\beta)}\right\}^{\frac{1}{j-1}}. \tag{5.8}$$

This completes the proof of Theorem 6. □

Finally, we present Theorem 7 below.

Theorem 7. *Suppose that the function f, which is defined by* (1.5), *belongs to the class* $\mathcal{H}_{n,\delta,\lambda}^{\sigma,m,\kappa}(\beta,\alpha,\eta,A,B)$.
Then $\mathcal{I}_{\omega}(f)$ is convex of order η $(0 \leqq \eta < 1)$ in $|z| < \mathcal{R}_2$, where

$$\mathcal{R}_2 := \inf_j\left\{\left(\frac{\mu+j}{\mu+1}\right)^{\sigma} \cdot \frac{(1-\eta)(1+j\kappa-\kappa)[(1-B\alpha)(j-1)+(B-A)\eta\alpha(j-\beta)]\varphi_j}{\alpha j(j-\eta)(B-A)\eta(1-\beta)}\right\}^{\frac{1}{j-1}}, \tag{5.9}$$

in which φ_j is given by (1.12).

Proof. The proof is similar to the proof of theorem 6, so it is being omitted here. □

§6 Concluding Remarks and Observations

In our present investigation, we have made use of a certain hybrid-type convolution operator with a view to introducing a new subclass of normalized analytic functions in the open unit disk. For functions belonging to this newly-defined analytic function class, we have derived a number of properties and characteristics including (for example) the modified Hadamard products, Hölder's inequalities and convolution properties as well as some closure properties under a general family of integral transforms. The interested reader will find some recent developments on Hölder's inequalities in [4] and [5].

Declarations

Conflict of interest The authors declare no conflict of interest.

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[1]Department of Mathematics and Statistics, University of Victoria, Victoria, British Columbia V8W 3R4, Canada.

[2]Department of Medical Research, China Medical University Hospital, China Medical University, Taichung 40402, China.

[3]Center for Converging Humanities, Kyung Hee University, 26 Kyungheedae-ro, Dongdaemun-gu, Seoul 02447, South Korea.
[4]Department of Applied Mathematics, Chung Yuan Christian University, Chung-Li, Taoyuan City 320314, China.
[5]Department of Mathematics and Informatics, Azerbaijan University, 71 Jeyhun Hajibeyli Street, AZ1007 Baku, Azerbaijan.
[6]Section of Mathematics, International Telematic University Uninettuno, I-00186 Rome, Italy.
Email: harimsri@math.uvic.ca
[7]Department of Mathematics, College of Science, Qassim University, Buraydah 51452, Saudi Arabia.
Email: s.eldeeb@qu.edu.sa