

Certain Subclasses of Bi-univalent Functions Associated with (u, v) -Chebyshev Polynomials

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Abstract. For the new subclass $\mathfrak{B}$ of the bi-univalent functions constructed with the help of the (u, v) -Chebyshev polynomials of the second type, we get estimates for the first two initial coefficients and upper bounds of the Fekete-Szegő functional.

§1 Introduction

Legendre discovered orthogonal polynomials in 1784 [22], and they have been researched extensively since then. Orthogonal polynomials are frequently used in the mathematical study of model problems to discover solutions to ordinary differential equations given particular criteria imposed by the model. Orthogonal polynomials have unquestionable relevance in modern mathematics, as well as a wide range of commercial applications and physics. The relevance of these polynomials in approximation theory issues is well-known. In differential and integral equations theory, along with mathematical statistics, they can be found. Quantum physics, signal analysis, automated control, scattering theory and axially symmetric potential theory [13, 16] have all made use of them. Firstly, bi-univalent functions were introduced by Lewin [23]. However, this subject attracted the attention of many mathematicians after Srivastava et al. [35] studied coefficient bounds for certain subclasses of analytic and bi-univalent functions (also see [29] and [31]). Later, by defining many new subclasses of analytic bi-univalent functions with the help of some special polynomials, some special functions and some special operators, coefficient bounds for these classes and Fekete-Szegő functionals were studied. For example, Srivastava et al. obtained general coefficient bounds for these classes with the help of Faber polynomials by introducing the new class of analytic and by using bi-univalent functions and by using the Tremblay fractional derivative operator in [30] and the q -integral operator in [34]. Additionally, Srivastava et al. [28] studied a new subclass of bi-univalent functions using Horadam polynomials. In recent years, bi-univalent functions and their various applications are among the most studied topics in geometric function theory (see [33], [36], [37], [38]).

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§2 Definitions and preliminaries

We let $\mathcal{U} = \{z : z \in \mathcal{C}, |z| < 1\}$ be a unit disk, $\Re = (-\infty, \infty)$, $\mathfrak{N} := \mathfrak{N}_0 \setminus \{0\}$ and let $\mathcal{A}$ denote the class of holomorphic functions of the form

$$\mu(z) = z + \sum_{r=2}^{\infty} \tau_r z^r \quad (z \in \mathcal{U}) \quad (1)$$

normalized by the condition $\mu(0) = \mu'(0) - 1 = 0$. Let $\mathcal{S} \subset \mathcal{A}$ be the class of holomorphic and univalent function in $\mathcal{U}$. Lewin [23] presented the class of bi-univalent functions as a subclass of $\mathcal{A}$ and identified certain coefficient bounds for the class. He showed that $|\tau_2| \leq 1.15$. Furthermore, the Koebe 1/4 theorem (see [17]) states that the range of every function $f \in \mathcal{S}$ contains the disc $d_\omega = \{\omega : |\omega| < 0.25\}$, hence, for all $\mu \in \mathcal{S}$ with its inverse μ^{-1} , such that

$$\mu^{-1}(\mu(z)) = z \quad (z \in \mathcal{U})$$

and

$$\mu(\mu^{-1}(\omega)) = \omega, \quad (\omega : |\omega| < r_0(\mu); r_0(\mu) \geq 0.25)$$

where $\mu^{-1}(\omega)$ is expressed as

$$G(\omega) = \omega - \tau_2 \omega^2 + (2\tau_2^2 - \tau_3) \omega^3 - (5\tau_2^3 - 5\tau_2 \tau_3 + \tau_4) \omega^4 + \cdots \quad (2)$$

So, a function $\mu \in \mathcal{A}$ is said to be bi-univalent in $\mathcal{U}$ if both $\mu(z)$ and $G(z)$ are univalent in $\mathcal{U}$. Let $\mathfrak{B}$ denote the class of holomorphic and bi-univalent functions in $\mathcal{U}$.

We know that some familiar functions $f \in \mathcal{S}$ such as the Koebe function $\kappa(z) = z/(1-z)^2$, its rotation function $\kappa_\zeta(z) = z/(1-e^{i\zeta}z)^2$, $\mu(z) = z - z^2/2$ and $\mu(z) = z/(1-z^2)$ are not members of $\mathfrak{B}$. Also some functions $\mu \in (\mathcal{S} \cap \mathfrak{B})$ includes $\mu(z) = z$, $\mu(z) = 1/2 \log[(1-z)/(1+z)]$, $z/(1-z)$. For more details see [6–8, 12, 27, 29, 30, 32, 34, 40].

From [17], let $s(z), S(z) \in \mathcal{A}$, then $s(z) \prec S(z)$, $z \in \mathcal{U}$, suppose w holomorphic in $\mathcal{U}$, such that $w(0) = 0$, $|w(z)| < 1$ and $s(z) = S(w(z))$. If the function $S(z)$ is univalent in $\mathcal{U}$ then $s(z) \prec S(z) \Rightarrow s(0) = S(0)$ and $s(\mathcal{U}) \subset S(\mathcal{U})$. The (u, v) -Chebyshev polynomials of the second class have the following recurrence relations:

$$V_m(t, a, u, v) = (u^m + v^m)tV_{m-1}(t, a, u, v) + (u, v)^{m-1}aV_{m-2}(t, a, u, v) \quad (3)$$

where

$$\begin{aligned} V_0(t, a, u, v) &= 1 \\ V_1(t, a, u, v) &= (u + v)t \\ V_2(t, a, u, v) &= ((u^2 + v^2)(u + v)t^2 + uva) \quad (m \geq 2, 0 < v < u \leq 1) \end{aligned}$$

t is a variable. We shall discuss the following intriguing points in light of this recurrence relation.

(1) The Jacobsthal polynomials denoted by $\mathcal{J}_{m+1}(y)$ are obtained when $t = 1/2$, $a = 2y$, $u = 1$, $v = 1$.

(2) The Jacobsthal numbers denoted by $\mathcal{J}_{m+1}$ are obtained when $t = 1/2$, $a = 2$, $u = 1$, $v = 1$.

(3) The Pell polynomials denoted by $\mathcal{P}_{m+1}(t)$ are obtained when $a = 1$, $u = 1$, $v = 1$.

(4) The Pell numbers denoted by $\mathcal{P}_{m+1}$ are obtained when $t = 1$, $a = 1$, $u = 1$, $v = 1$.

(5) The Fibonacci polynomials denoted by $\mathcal{F}_{m+1}(t)$ are obtained when $t = \frac{t}{2}$, $a = 1$, $u = 1$, $v = 1$.

- (6) The Fibonacci numbers denoted by $\mathcal{F}_{m+1}$ are obtained when $t = \frac{1}{2}$, $a = 1$, $u = 1$, $v = 1$.
 (7) The (u, v) -Fibonacci polynomials denoted by $\mathcal{F}_m(t, a, u, v)$ are obtained when $t = \frac{t}{2}$.
 (8) The second kind of Chebyshev polynomials denoted by $V_n(t)$ are obtained when $a = -1$, $u = 1$, $v = 1$.

These recursively defined polynomials over integers have more detailed information about the phenomenon properties and have been researched a lot. (see [1–5, 9–11, 14, 15, 18, 19, 21, 24–26, 39]) and closely related references cited therein. Kizilates et al. [20] recently defined the first and second kinds of (u, v) -Chebyshev polynomials and these polynomials have precise formulae, generating functions, and other notable characteristics.

The second-order (u, v) -Chebyshev polynomials have the following generating function:

$$\begin{aligned}\mathfrak{G}_{u,v}(z) &= \frac{1}{1 - (uvaz^2\zeta_{u,v} + tvz\zeta_v + tuz\zeta_u)} \\ &= \sum_{m=0}^{\infty} V_m(t, a, u, v)z^m \quad (z \in \mathcal{U}),\end{aligned}$$

where $V_m(t, a, u, v)$ is given by (3). This section begins with the definition of the class $\mathfrak{J}_{\sigma,\gamma}^{\mathfrak{B},\beta}(u, v)$ as follows:

Definition 1. For $\beta \geq 0$, $\sigma \geq 1$, $|\gamma| \leq 1$ but $\gamma \neq 1$, a function $\mu \in \mathfrak{B}$ is said to be in the class $\mathfrak{J}_{\sigma,\gamma}^{\mathfrak{B},\beta}(u, v)$ if the following subordination conditions are fulfilled:

$$\frac{((1-\gamma)z)^{1-\beta}(\mu'(z))^\sigma}{(\mu(z) - \mu(\gamma z))^{1-\beta}} \prec \mathfrak{G}_{u,v}(z) \quad (4)$$

and

$$\frac{((1-\gamma)\omega)^{1-\beta}(G'(\omega))^\sigma}{(G(\omega) - G(\gamma\omega))^{1-\beta}} \prec \mathfrak{G}_{u,v}(\omega). \quad (5)$$

By choosing special values for β, γ and σ , the class $\mathfrak{J}_{\sigma,\gamma}^{\mathfrak{B},\beta}(u, v)$ yields some interesting new classes as remarked below:

Remark 1. For $\sigma = 1$, we have the new class $\mathfrak{J}_{1,\gamma}^{\mathfrak{B},\beta}(u, v)$. The class $\mathfrak{J}_{1,\gamma}^{\mathfrak{B},\beta}(u, v)$ consists of the functions of $\mu \in \mathfrak{B}$ fulfilled by

$$\frac{((1-\gamma)z)^{1-\beta}\mu'(z)}{(\mu(z) - \mu(\gamma z))^{1-\beta}} \prec \mathfrak{G}_{u,v}(z) \quad (6)$$

and

$$\frac{((1-\gamma)\omega)^{1-\beta}G'(\omega)}{(G(\omega) - G(\gamma\omega))^{1-\beta}} \prec \mathfrak{G}_{u,v}(\omega). \quad (7)$$

Remark 2. For $\beta = 0$, we have the new class

$$\mathfrak{J}_{\sigma,\gamma}^{\mathfrak{B},0}(u, v) = \mathfrak{J}_{\sigma,\gamma}^{\mathfrak{B}}(u, v).$$

For this class $\mu \in \mathfrak{B}$ if

$$\frac{z(1-\gamma)(\mu'(z))^\sigma}{\mu(z) - \mu(\gamma z)} \prec \mathfrak{G}_{u,v}(z) \quad (8)$$

and

$$\frac{\omega(1-\gamma)(G'(\omega))^\sigma}{G(\omega) - G(\gamma\omega)} \prec \mathfrak{G}_{u,v}(\omega). \quad (9)$$

Also,

(1) Choosing $\sigma = 1$ in the class $\mathfrak{J}_{\sigma,\gamma}^{\mathfrak{B}}(u, v)$ we have a new class

$$\mathfrak{J}_{1,\gamma}^{\mathfrak{B}}(u, v) = \mathfrak{J}_{\gamma}^{\mathfrak{B}}(u, v).$$

For this class $\mu \in \mathfrak{B}$, if

$$\frac{z(1-\gamma)\mu'(z)}{f(z)-f(\gamma z)} \prec \mathfrak{G}_{u,v}(z) \quad (10)$$

and

$$\frac{\omega(1-\gamma)G'(\omega)}{G(\omega)-G(\gamma\omega)} \prec \mathfrak{G}_{u,v}(\omega). \quad (11)$$

(2) Choosing $\sigma = 1$ and fixing $\gamma = -1$ in the class $\mathfrak{J}_{\sigma,\gamma}^{\mathfrak{B}}(u, v)$ we have new class

$$\mathfrak{J}_{1,-1}^{\mathfrak{B}}(u, v) = \mathfrak{J}_S^{\mathfrak{B}}(u, v).$$

For this class $\mu \in \mathfrak{B}$, if

$$\frac{2z\mu'(z)}{\mu(z)-\mu(-z)} \prec \mathfrak{G}_{u,v}(z) \quad (12)$$

and

$$\frac{2\omega G'(\omega)}{G(\omega)-G(-\omega)} \prec \mathfrak{G}_{u,v}(\omega). \quad (13)$$

(3) Choosing $\gamma = 0$ in the class $\mathfrak{J}_{\sigma}^{\mathfrak{B}}(u, v)$ we have the class

$$\mathfrak{J}_{\sigma,0}^{\mathfrak{B}}(u, v) = \mathfrak{J}_{\sigma}^{\mathfrak{B}}(u, v).$$

For this class $\mu \in \mathfrak{B}$, if

$$\frac{z(\mu'(z))^{\sigma}}{\mu(z)} \prec \mathfrak{G}_{u,v}(z) \quad (14)$$

and

$$\frac{\omega(G'(\omega))^{\sigma}}{G(\omega)} \prec \mathfrak{G}_{u,v}(\omega). \quad (15)$$

(4) Choosing $\sigma = 2$ in the class $\mathfrak{J}_{\sigma}^{\mathfrak{B}}(u, v)$ we have the class

$$\mathfrak{J}_2^{\mathfrak{B}}(u, v) = \mathfrak{J}^{\mathfrak{B}}(u, v).$$

The class consists of the function $\mu \in \mathfrak{B}$ satisfying

$$\mu'(z) \frac{z\mu'(z)}{\mu(z)} \prec \mathfrak{G}_{u,v}(z) \quad (16)$$

and

$$G'(\omega) \frac{\omega G'(\omega)}{G(\omega)} \prec \mathfrak{G}_{u,v}(\omega). \quad (17)$$

For $\beta = 1$, we have the new class $\mathfrak{J}_{\sigma,\gamma}^{\mathfrak{B},1}(u, v)$. The class $\mathfrak{J}_{\sigma,\gamma}^{\mathfrak{B},1}(u, v)$ consists of the functions of $\mu \in \mathfrak{B}$ satisfying

$$(\mu'(z))^{\sigma} \prec \mathfrak{G}_{u,v}(z) \quad (18)$$

and

$$(G'(\omega))^{\sigma} \prec \mathfrak{G}_{u,v}(\omega). \quad (19)$$

Also,

Choosing $\sigma = 1$ in the class $\mathfrak{J}_{\sigma}^{\mathfrak{B},1}(u, v)$ we have the class

$$\mathfrak{J}_1^{\mathfrak{B},1}(u, v).$$

For this class $\mu \in \mathfrak{B}$, if

$$\mu'(z) \prec \mathfrak{G}_{u,v}(z) \quad (20)$$

and

$$G'(\omega) \prec \mathfrak{G}_{u,v}(\omega). \quad (21)$$

Now, we give our main result.

Theorem 1. Let $\mu(z) \in \mathfrak{J}_{\sigma,\gamma}^{\mathfrak{B},\beta}(u,v)$. Then

$$|\tau_2| \leq \frac{(u+v)t\sqrt{(u+v)t}}{\sqrt{\left| \left[(\beta-1)[G_1] + \sigma(2\sigma+1) \right] (u+v)^2 t^2 - G_2 \right|}}, \quad (22)$$

$$|\tau_3| \leq (u+v)t \left[\frac{(u+v)t}{|[2\sigma + (\beta-1)(1+\gamma)]^2|} + \frac{1}{|3\sigma + (\beta-1)(1+\gamma+\gamma^2)|} \right], \quad (23)$$

and for any real number χ ,

$$|\tau_3 - \chi\tau_2^2| \leq \begin{cases} \frac{(u+v)t}{|3\sigma + (\beta-1)(1+\gamma+\gamma^2)|}, & |\chi-1| \leq A \\ \frac{|1-\chi|(u+v)^3 t^3}{|(u+v)^2 t^2 \Omega - [2\sigma + (\beta-1)(1+\gamma)]^2 ((u^2+v^2)(u+v)t^2 + uva)|}, & |\chi-1| \geq A, \end{cases}$$

where

$$A = \frac{1}{|3\sigma + (\beta-1)(1+\gamma+\gamma^2)|} \left| \Omega - [2\sigma + (\beta-1)(1+\gamma)]^2 \left(\frac{u^2+v^2}{u+v} + \frac{uva}{(u+v)^2 t^2} \right) \right|,$$

$$\Omega = (\beta-1) \left[\frac{(\beta-2)(1+\gamma)^2}{2} + 2\sigma(1+\gamma) + 1 + \gamma + \gamma^2 \right] + \sigma(2\sigma+1),$$

$$G_1 = \frac{(\beta-2)(1+\gamma)^2}{2} + 2\sigma(1+\gamma) + (1+\gamma+\gamma^2),$$

$$G_2 = [2\sigma + (\beta-1)(1+\gamma)]^2 ((u^2+v^2)(u+v)t^2 + uva).$$

Proof. Let $\mu(z) \in \mathfrak{J}_{\sigma,\gamma}^{\mathfrak{B},\beta}(u,v)$. Then, from (4) and (5), we have

$$\frac{((1-\gamma)z)^{1-\beta}(\mu'(z))^\sigma}{(\mu(z) - \mu(\gamma z))^{1-\beta}} = \mathfrak{G}_{u,v}(\Phi(z))$$

and

$$\frac{((1-\gamma)\omega)^{1-\beta}(G'(\omega))^\sigma}{(G(\omega) - G(\gamma\omega))^{1-\beta}} = \mathfrak{G}_{u,v}(\Upsilon(\omega)),$$

for some holomorphic functions Φ, Υ such that $\Phi(0) = \Upsilon(0) = 0$, $|\Phi(z)| < 1$, $|\Upsilon(\omega)| < 1$, $z, \omega \in \mathcal{U}$, we can state

$$\begin{aligned} |\Phi(z)| &= |\varrho_1 z + \varrho_2 z^2 + \varrho_3 z^3 + \cdots| < 1 \quad (z \in \mathcal{U}) \\ |\Upsilon(\omega)| &= |\mu_1 \omega + \mu_2 \omega^2 + \mu_3 \omega^3 + \cdots| < 1 \quad (\omega \in \mathcal{U}) \end{aligned}$$

by equivalence

$$\frac{((1-\gamma)z)^{1-\beta}(\mu'(z))^\sigma}{(\mu(z) - \mu(\gamma z))^{1-\beta}} = V_0(t, a, u, v) + V_1(t, a, u, v)\Phi(z) + V_2(t, a, u, v)\Phi^2(z) + \cdots \quad (24)$$

and

$$\frac{((1-\gamma)\omega)^{1-\beta}(G'(\omega))^\sigma}{(G(\omega) - G(\gamma\omega))^{1-\beta}} = V_0(t, a, u, v) + V_1(t, a, u, v)\Upsilon(\omega) + V_2(t, a, u, v)\Upsilon^2(\omega) + \cdots \quad (25)$$

From (24) and (25), yields

$$\begin{aligned} \frac{((1-\gamma)z)^{1-\beta}(\mu'(z))^\sigma}{(\mu(z)-\mu(\gamma z))^{1-\beta}} &= 1 + V_1(t, a, u, v)\varrho_1 z \\ &+ \left[V_1(t, a, u, v)\varrho_2 + V_2(t, a, u, v)\varrho_1^2 \right] z^2 + \cdots \end{aligned} \quad (26)$$

and

$$\begin{aligned} \frac{((1-\gamma)\omega)^{1-\beta}(G'(\omega))^\sigma}{(G(\omega)-G(\gamma\omega))^{1-\beta}} &= 1 + V_1(t, a, u, v)\mu_1 \omega \\ &+ \left[V_1(t, a, u, v)\mu_2 + V_2(t, a, u, v)\mu_1^2 \right] \omega^2 + \cdots \end{aligned} \quad (27)$$

When the corresponding coefficients in (26) and (27) are compared, we get

$$[2\sigma + (\beta - 1)(1 + \gamma)]\tau_2 = V_1(t, a, u, v)\varrho_1, \quad (28)$$

$$\begin{aligned} [3\sigma + (\beta - 1)(1 + \gamma + \gamma^2)]\tau_3 + \left[\frac{(\beta - 1)(\beta - 2)}{2}(1 + \gamma)^2 + 2\sigma(\sigma - 1 + (\beta - 1)(1 + \gamma)) \right] \tau_2^2 \\ = V_1(t, a, u, v)\varrho_2 + V_2(t, a, u, v)\varrho_1^2, \end{aligned} \quad (29)$$

$$-[2\sigma + (\beta - 1)(1 + \gamma)]\tau_2 = V_1(t, a, u, v)\mu_1, \quad (30)$$

$$\begin{aligned} \left[2(\beta - 1)(1 + \gamma + \gamma^2) + \frac{(\beta - 1)(\beta - 2)}{2}(1 + \gamma)^2 + 2\sigma(\sigma + 2 + (\beta - 1)(1 + \gamma)) \right] \tau_2^2 \\ - [3\sigma + (\beta - 1)(1 + \gamma + \gamma^2)]\tau_3 = V_1(t, a, u, v)\mu_2 + V_2(t, a, u, v)\mu_1^2. \end{aligned} \quad (31)$$

From (28) and (30)

$$\varrho_1 = -\mu_1, \quad (32)$$

$$\begin{aligned} 2[2\sigma + (\beta - 1)(1 + \gamma)]^2\tau_2^2 = V_1^2(t, a, u, v)(\varrho_1^2 + \mu_1^2) \quad \text{or} \\ (\varrho_1^2 + \mu_1^2) = \frac{2[2\sigma + (\beta - 1)(1 + \gamma)]^2\tau_2^2}{V_1^2(t, a, u, v)} \quad \text{or} \quad \tau_2^2 = \frac{V_1^2(t, a, u, v)(\varrho_1^2 + \mu_1^2)}{2[2\sigma + (\beta - 1)(1 + \gamma)]^2}. \end{aligned} \quad (33)$$

Summation of (29) and (31) gives

$$\begin{aligned} \left[2(\beta - 1) \left[\frac{(\beta - 2)(1 + \gamma)^2}{2} + 2\sigma(1 + \gamma) + (1 + \gamma + \gamma^2) \right] + 2\sigma(2\sigma + 1) \right] \tau_2^2 \\ = V_1(t, a, u, v)(\varrho_2 + \mu_2) + V_2(t, a, u, v)(\varrho_1^2 + \mu_1^2) = V_1(t, a, u, v)(\varrho_2 + \mu_2) \\ + V_2(t, a, u, v) \left[\frac{2[2\sigma + (\beta - 1)(1 + \gamma)]^2\tau_2^2}{V_1^2(t, a, u, v)} \right]. \end{aligned} \quad (34)$$

Applying (33) in (34), yields

$$\begin{aligned} \left[V_1^2(t, a, u, v) \left[2(\beta - 1) \left[\frac{(\beta - 2)(1 + \gamma)^2}{2} + 2\sigma(1 + \gamma) + (1 + \gamma + \gamma^2) \right] \right. \right. \\ \left. \left. + 2\sigma(2\sigma + 1) \right] - 2V_2(t, a, u, v)[2\sigma + (\beta - 1)(1 + \gamma)]^2 \right] \tau_2^2 = V_1^3(t, a, u, v)(\varrho_2 + \mu_2) \end{aligned} \quad (35)$$

which gives

$$|\tau_2| \leq \frac{(u+v)t\sqrt{(u+v)t}}{\sqrt{\left| \left[(\beta-1)[G_1] + \sigma(2\sigma+1) \right] (u+v)^2 t^2 - G_2 \right|}}.$$

Hence, (29) minus (31) gives us

$$2[3\sigma + (\beta-1)(1+\gamma+\gamma^2)](\tau_3 - \tau_2^2) = V_1(t, a, u, v)(\varrho_2 - \mu_2) + V_2(t, a, u, v)(\varrho_1^2 - \mu_1^2). \quad (36)$$

Then, by using (32) and (33) in (36), we get

$$\tau_3 = \tau_2^2 + \frac{V_1(t, a, u, v)(\varrho_2 - \mu_2)}{2[3\sigma + (\beta-1)(1+\gamma+\gamma^2)]} \quad (37)$$

$$= \frac{V_1^2(t, a, u, v)(\varrho_1^2 + \mu_1^2)}{2[2\sigma + (\beta-1)(1+\gamma)]^2} + \frac{V_1(t, a, u, v)(\varrho_2 - \mu_2)}{2[3\sigma + (\beta-1)(1+\gamma+\gamma^2)]}. \quad (38)$$

Applying (3), we have

$$|\tau_3| \leq (u+v)t \left[\frac{(u+v)t}{|[2\sigma + (\beta-1)(1+\gamma)]^2|} + \frac{1}{|3\sigma + (\beta-1)(1+\gamma+\gamma^2)|} \right].$$

Furthermore, From (35) and (36), we get

$$\begin{aligned} \tau_3 - \chi\tau_2^2 = V_1(t, a, u, v) & \left[\left(\zeta(\chi) + \frac{1}{2[3\sigma + (\beta-1)(1+\gamma+\gamma^2)]} \right) \varrho_2 \right. \\ & \left. + \left(\zeta(\chi) - \frac{1}{2[3\sigma + (\beta-1)(1+\gamma+\gamma^2)]} \right) \mu_2 \right] \end{aligned}$$

where

$$\zeta(\chi) = \frac{V_1^2(t, a, u, v)(1-\chi)}{V_1^2(t, a, u, v)[G_3] - V_2(t, a, u, v)[2\sigma + (\beta-1)(1+\gamma)]^2}.$$

where

$$G_3 = \left[(\beta-1) \left(\frac{(\beta-2)(1+\gamma)^2}{2} + \sigma(1+\gamma) + (1+\gamma+\gamma^2) \right) + \sigma(2\sigma+1) \right].$$

Thus, according to (3), we have

$$|\tau_3 - \chi\tau_2^2| \leq \begin{cases} \frac{(u+v)t}{|3\sigma + (\beta-1)(1+\gamma+\gamma^2)|}, & 0 \leq |\zeta(\chi)| \leq \frac{1}{|2[3\sigma + (\beta-1)(1+\gamma+\gamma^2)]|} \\ 2|\zeta(\chi)|(u+v)t, & |\zeta(\chi)| \geq \frac{1}{|2[3\sigma + (\beta-1)(1+\gamma+\gamma^2)]|} \end{cases}$$

where

$$\zeta(\chi) = \frac{V_1^2(t, a, u, v)(1-\chi)}{V_1^2(t, a, u, v)[G_3] - V_2(t, a, u, v)[2\sigma + (\beta-1)(1+\gamma)]^2}.$$

Hence,

$$|\tau_3 - \chi\tau_2^2| \leq \begin{cases} \frac{(u+v)t}{|3\sigma + (\beta-1)(1+\gamma+\gamma^2)|}, & |\chi-1| \leq A \\ \frac{|1-\chi|(u+v)^3 t^3}{|(u+v)^2 t^2 \Omega - [2\sigma + (\beta-1)(1+\gamma)]^2 ((u^2+v^2)(u+v)t^2 + uva)|}, & |\chi-1| \geq A. \end{cases}$$

□

§3 Corollaries

We get the following results by specializing the parameters γ, β, σ , in Theorem 1.

Corollary 1. For $\sigma = 1$, let $\mu \in \mathfrak{J}_{1,\gamma}^{\mathfrak{B},\beta}(u, v)$. Then

$$|\tau_2| \leq \frac{(u+v)t\sqrt{(u+v)t}}{\sqrt{|G_4|}}, \quad (39)$$

$$|\tau_3| \leq (u+v)t \left[\frac{(u+v)t}{|[2 + (\beta - 1)(1 + \gamma)]^2|} + \frac{1}{|3 + (\beta - 1)(1 + \gamma + \gamma^2)|} \right], \quad (40)$$

and for any real number χ ,

$$|\tau_3 - \chi\tau_2^2| \leq \begin{cases} \frac{\frac{(u+v)t}{|3 + (\beta - 1)(1 + \gamma + \gamma^2)|},}{\frac{|1 - \chi|(u+v)^3 t^3}{|(u+v)^2 t^2 \Omega_1 - [2 + (\beta - 1)(1 + \gamma)]^2 ((u^2 + v^2)(u+v)t^2 + uva)|}}, & |\chi - 1| \leq A_1 \\ |\chi - 1| \geq A_1, \end{cases}$$

where

$$A_1 = \frac{1}{|3 + (\beta - 1)(1 + \gamma + \gamma^2)|} \left| \Omega_1 - [2 + (\beta - 1)(1 + \gamma)]^2 \left(\frac{u^2 + v^2}{u + v} + \frac{uva}{(u + v)^2 t^2} \right) \right|$$

$$\Omega_1 = (\beta - 1) \left[\frac{(\beta - 2)(1 + \gamma)^2}{2} + \gamma^2 + 3\gamma + 3 \right] + 3,$$

$$G_4 = \left[(\beta - 1) \left[\frac{(\beta - 2)(1 + \gamma)^2}{2} + 3(1 + \gamma) + \gamma^2 \right] + 3 \right] (u + v)^2 t^2 \\ - [2 + (\beta - 1)(1 + \gamma)]^2 ((u^2 + v^2)(u + v)t^2 + uva).$$

Corollary 2. For $\beta = 0$, let $\mu \in \mathfrak{J}_{\sigma,\gamma}^{\mathfrak{B},0}(u, v)$. Then

$$|\tau_2| \leq \frac{(u+v)t\sqrt{(u+v)t}}{\sqrt{\left| (\gamma - \sigma)(1 - 2\sigma)(u + v)^2 t^2 - [2\sigma - (1 + \gamma)]^2 ((u^2 + v^2)(u + v)t^2 + uva) \right|}}, \quad (41)$$

$$|\tau_3| \leq (u+v)t \left[\frac{(u+v)t}{|[2\sigma - (1 + \gamma)]^2|} + \frac{1}{|3\sigma - (1 + \gamma + \gamma^2)|} \right], \quad (42)$$

and for any real number χ ,

$$|\tau_3 - \chi\tau_2^2| \leq \begin{cases} \frac{\frac{(u+v)t}{|3\sigma - (1 + \gamma + \gamma^2)|},}{\frac{|1 - \chi|(u+v)^3 t^3}{|(u+v)^2 t^2 \Omega_2 - [2\sigma - (1 + \gamma)]^2 ((u^2 + v^2)(u + v)t^2 + uva)|}}, & |\chi - 1| \leq A_2 \\ |\chi - 1| \geq A_2, \end{cases}$$

where

$$A_2 = \frac{1}{|3\sigma - (1 + \gamma + \gamma^2)|} \left| \Omega_2 - [2\sigma - (1 + \gamma)]^2 \left(\frac{u^2 + v^2}{u + v} + \frac{uva}{(u + v)^2 t^2} \right) \right|$$

$$\Omega_2 = (\gamma - \sigma)(1 - 2\sigma).$$

Corollary 3. For $\sigma = 1$, $\beta = 0$, let $\mu \in \mathfrak{J}_{1,\gamma}^{\mathfrak{B},0}(u, v)$. Then

$$|\tau_2| \leq \frac{(u+v)t\sqrt{(u+v)t}}{\sqrt{\left| (1-\gamma)(u+v)^2t^2 - [1-\gamma]^2((u^2+v^2)(u+v)t^2 + uva) \right|}}, \quad (43)$$

$$|\tau_3| \leq (u+v)t \left[\frac{(u+v)t}{|[1-\gamma]^2|} + \frac{1}{|2-(\gamma+\gamma^2)|} \right], \quad (44)$$

and for any real number χ ,

$$|\tau_3 - \chi\tau_2^2| \leq \begin{cases} \frac{(u+v)t}{|2-(\gamma+\gamma^2)|}, & |\chi - 1| \leq A_3 \\ \frac{|1-\chi|(u+v)^3t^3}{|(u+v)^2t^2\Omega_3 - [1-\gamma]^2((u^2+v^2)(u+v)t^2 + uva)|}, & |\chi - 1| \geq A_3, \end{cases}$$

where

$$A_3 = \frac{1}{|2-(\gamma+\gamma^2)|} \left| \Omega_3 - [1-\gamma]^2 \left(\frac{u^2+v^2}{u+v} + \frac{uva}{(u+v)^2t^2} \right) \right|$$

$$\Omega_3 = (1-\gamma).$$

Corollary 4. For $\beta = 0$ and $\gamma = 0$, let $\mu \in \mathfrak{J}_{\sigma,0}^{\mathfrak{B},0}(u,v)$. Then

$$|\tau_2| \leq \frac{(u+v)t\sqrt{(u+v)t}}{\sqrt{\left| \sigma(2\sigma-1)(u+v)^2t^2 - [2\sigma-1]^2((u^2+v^2)(u+v)t^2 + uva) \right|}}, \quad (45)$$

$$|\tau_3| \leq (u+v)t \left[\frac{(u+v)t}{|[2\sigma-1]^2|} + \frac{1}{|3\sigma-1|} \right], \quad (46)$$

and for any real number χ ,

$$|\tau_3 - \chi\tau_2^2| \leq \begin{cases} \frac{(u+v)t}{|3\sigma-1|}, & |\chi - 1| \leq A_4 \\ \frac{|1-\chi|(u+v)^3t^3}{|(u+v)^2t^2\Omega_4 - [2\sigma-1]^2((u^2+v^2)(u+v)t^2 + uva)|}, & |\chi - 1| \geq A_4, \end{cases}$$

where

$$A_4 = \frac{1}{|3\sigma-1|} \left| \Omega_4 - [2\sigma-1]^2 \left(\frac{u^2+v^2}{u+v} + \frac{uva}{(u+v)^2t^2} \right) \right|$$

$$\Omega_4 = \sigma(2\sigma-1).$$

Corollary 5. For $\sigma = 2$, let $\mu \in \mathfrak{J}_{2,0}^{\mathfrak{B},0}(u,v)$. Then

$$|\tau_2| \leq \frac{(u+v)t\sqrt{(u+v)t}}{\sqrt{\left| 6(u+v)^2t^2 - 3((u^2+v^2)(u+v)t^2 + uva) \right|}}, \quad (47)$$

$$|\tau_3| \leq (u+v)t \left[\frac{(u+v)t}{9} + \frac{1}{5} \right], \quad (48)$$

and for any real number χ ,

$$|\tau_3 - \chi\tau_2^2| \leq \begin{cases} \frac{(u+v)t}{5}, & |\chi - 1| \leq A_5 \\ \frac{|1-\chi|(u+v)^3t^3}{|6(u+v)^2t^2 - 9((u^2+v^2)(u+v)t^2 + uva)|}, & |\chi - 1| \geq A_5, \end{cases}$$

where

$$A_5 = \frac{1}{5} \left| 6 - 9 \left(\frac{u^2 + v^2}{u + v} + \frac{uva}{(u + v)^2t^2} \right) \right|.$$

Corollary 6. For $\sigma = 1$, let $\mu \in \mathfrak{J}_{1,0}^{\mathfrak{B},0}(u, v)$. Then

$$|\tau_2| \leq \frac{(u + v)t\sqrt{(u + v)t}}{\sqrt{\left| (u + v)^2t^2 - ((u^2 + v^2)(u + v)t^2 + uva) \right|}}, \quad (49)$$

$$|\tau_3| \leq (u + v)t \left[(u + v)t + \frac{1}{2} \right], \quad (50)$$

and for any real number χ ,

$$|\tau_3 - \chi\tau_2^2| \leq \begin{cases} \frac{(u+v)t}{2}, & |\chi - 1| \leq A_6 \\ \frac{|1-\chi|(u+v)^3t^3}{|(u+v)^2t^2 - ((u^2+v^2)(u+v)t^2 + uva)|}, & |\chi - 1| \geq A_6, \end{cases}$$

where

$$A_6 = \frac{1}{2} \left| 1 - \left(\frac{u^2 + v^2}{u + v} + \frac{uva}{(u + v)^2t^2} \right) \right|.$$

Corollary 7. For $\beta = 1$, let $\mu \in \mathfrak{J}_{\sigma}^{\mathfrak{B},1}(u, v)$. Then

$$|\tau_2| \leq \frac{(u + v)t\sqrt{(u + v)t}}{\sqrt{\left| \sigma(2\sigma + 1)(u + v)^2t^2 - 4\sigma^2((u^2 + v^2)(u + v)t^2 + uva) \right|}}, \quad (51)$$

$$|\tau_3| \leq (u + v)t \left[\frac{(u + v)t}{|4\sigma^2|} + \frac{1}{|3\sigma|} \right], \quad (52)$$

and for any real number χ ,

$$|\tau_3 - \chi\tau_2^2| \leq \begin{cases} \frac{(u+v)t}{|3\sigma|}, & |\chi - 1| \leq A_7 \\ \frac{|1-\chi|(u+v)^3t^3}{|(u+v)^2t^2\Omega_7 - 4\sigma^2((u^2+v^2)(u+v)t^2 + uva)|}, & |\chi - 1| \geq A_7, \end{cases}$$

where

$$A_7 = \frac{1}{3\sigma} \left| \Omega_7 - 4\sigma^2 \left(\frac{u^2 + v^2}{u + v} + \frac{uva}{(u + v)^2t^2} \right) \right|$$

$$\Omega_7 = \sigma(2\sigma + 1).$$

Corollary 8. For $\sigma = 1$, $\beta = 1$, let $\mu \in \mathfrak{J}_1^{\mathfrak{B},1}(u, v)$. Then

$$|\tau_2| \leq \frac{(u+v)t\sqrt{(u+v)t}}{\sqrt{\left|3(u+v)^2t^2 - 4((u^2+v^2)(u+v)t^2 + uva)\right|}}, \quad (53)$$

$$|\tau_3| \leq (u+v)t \left[\frac{(u+v)t}{4} + \frac{1}{3} \right], \quad (54)$$

and for any real number χ ,

$$|\tau_3 - \chi\tau_2^2| \leq \begin{cases} \frac{(u+v)t}{|3\sigma|}, & |\chi - 1| \leq A_8 \\ \frac{|1-\chi|(u+v)^3t^3}{|3(u+v)^2t^2 - 4((u^2+v^2)(u+v)t^2 + uva)|}, & |\chi - 1| \geq A_8, \end{cases}$$

where

$$A_8 = \frac{1}{3} \left| 3 - 4 \left(\frac{u^2 + v^2}{u + v} + \frac{uva}{(u + v)^2t^2} \right) \right|.$$

Declarations

Conflict of interest The authors declare no conflict of interest.

References

- [1] A Amourah. *Initial bounds for analytic and bi-univalent functions by means of (p,q) -Chebyshev polynomials defined by differential operator*, General Letters in Mathematics, 2019, 7(2): 45-51.
- [2] A Amourah. *Faber polynomial coefficient estimates for a class of analytic bi-univalent functions*, 2018, DOI: 10.48550/arXiv.1810.07018.
- [3] A Amourah, T Al-Hawary, B A Frasin. *Application of Chebyshev polynomials to certain class of bi-Bazilevic functions of order $\alpha + i\beta$* , Afr Math, 2021, 32(5-6): 1059-1066.
- [4] A Amourah, B A Frasin, G Murugusundaramoorthy, et al. *Bi-Bazilevic functions of order $\vartheta + i\delta$ associated with (p, q) -Lucas polynomials*, AIMS Math, 2021, 6(5): 4296-4305.
- [5] A Amourah. *A comprehensive subclass of analytic and biunivalent functions associated with subordination*, Palest J Math, 2020, 9(1): 187-193.
- [6] A Akgul. *(P, Q) -Lucas polynomial coefficient inequalities of the bi-univalent function class*, Turkish J Math, 2019, 43: 2170-2176.
- [7] A Akgul, F M Sakar. *A certain subclass of bi-univalent analytic functions introduced by means of the q -analogue of Noor integral operator and Horadam polynomials*, Turkish J Math, 2019, 43: 2275-2286.

- [8] R M Ali, S K Lee, V Ravichandran, et al. *Coefficient estimates for bi-univalent Ma-Minda starlike and convex functions*, Appl Math Lett, 2012, 25(3): 344-351.
- [9] Ş Altınkaya, S Yalçın. *Estimates on coefficients of a general subclass of bi-univalent functions associated with symmetric q -derivative operator by means of the Chebyshev polynomials*, Asia Pac J Math, 2017, 4(2): 90-99.
- [10] Ş Altınkaya, S Yalçın. *On the Chebyshev coefficients for a general subclass of univalent functions*, Turkish J Math, 2018, 42(6): 2885-2890.
- [11] Ş Altınkaya, S Yalçın. *The (p, q) -Chebyshev polynomial bounds of a general bi-univalent function class*, Bol Soc Mat Mexicana, 2019, 26(2): 341-348.
- [12] Ş Altınkaya, S Yalçın. *On the (p, q) -Lucas polynomial coefficient bounds of the bi-univalent function class*, Bol Soc Mat Mexicana, 2019, 25(3): 1015-1022.
- [13] H Bateman. *Higher Transcendental Functions*, McGraw-Hill, 1953.
- [14] S Bulut, N Magesh, C Abirami. *A comprehensive class of analytic bi-univalent functions by means of Chebyshev polynomials*, J Fract Calc Appl, 2017, 8(2): 32-39.
- [15] S Bulut, N Magesh, V K Balaaji. *Initial bounds for analytic and bi-univalent functions by means of chebyshev polynomials*, J Class Anal, 2017, 11(1): 83-89.
- [16] B Doman. *The Classical Orthogonal Polynomials*, World Scientific, 2015.
- [17] P L Duren. *Univalent Functions*, Grundlehren der Mathematischen Wissenschaften. Springer, New York, NY, 1983.
- [18] B A Frasin. *Coefficient bounds for certain classes of bi-univalent functions*, Hacet J Math Stat, 2014, 43(3): 383-389.
- [19] B A Frasin, M K Aouf. *New subclasses of bi-univalent functions*, Appl Math Lett, 2011, 24(9): 1569-1573.
- [20] C Kizilates, N Tuglu, B Çekim. *On the (p, q) -Chebyshev polynomials and related polynomials*, Mathematics, 2019, 7(2): 136.
- [21] G Y Lee, M Asci. *Some properties of the (p, q) -Fibonacci and (p, q) -Lucas polynomials*, J Appl Math, 2012, 2012: 264842.
- [22] A Legendre. *Recherches sur l'attraction des sphéroïdes homogènes*, Mém Acad R Sci Paris, Paris, 1785.
- [23] M Lewin. *On a coefficient problem for bi-univalent functions*, Proc Amer Math Soc, 1967, 18(1): 63-68.
- [24] A Lupas. *A guide of Fibonacci and Lucas polynomials*, Octagon Math Mag, 1999, 7: 2-12.

- [25] A Ozkoc, A Porsuk. *A note for the (p, q) -Fibonacci and Lucas quaternion polynomials*, Konuralp J Math, 2017, 5(2): 36-46.
- [26] A B Patil, T G Shaba. *On sharp Chebyshev polynomial bounds for general subclass of bi-univalent functions*, Appl Sci, 2021, 23: 109-117.
- [27] T G Shaba. *Subclass of bi-univalent functions satisfying subordinate conditions defined by Frasin differential operator*, Turkish J Ineq, 2020, 4(2): 50-58.
- [28] H M Srivastava, Ş Altınkaya, S Yalçın. *Certain subclasses of bi-univalent functions associated with the Horadam polynomials*, Iran J Sci Technol Trans A Sci, 2019, 43(4): 1873-1879.
- [29] H M Srivastava, S Bulut, M Caglar, et al. *Coefficient estimates for a general subclass of analytic and bi-univalent functions*, Filomat, 2013, 27(5): 831-842.
- [30] H M Srivastava, S S Eker, S G Hamidi, et al. *Faber polynomial coefficient estimates for bi-univalent functions defined by the Tremblay fractional derivative operator*, Bull Iranian Math Soc, 2018, 44(1): 149-157.
- [31] H M Srivastava, S Hussain, I Ahmad, et al. *Coefficient bounds for analytic and bi-univalent functions associated with some conic domains*, J Nonlinear Convex Anal, 2022, 23(4): 741-753.
- [32] H M Srivastava, S Hussain, A Raziq, et al. *The Fekete-Szegő functional for a subclass of analytic functions associated with quasi-subordination*, Carpathian J Math, 2018, 34(1): 103-113.
- [33] H M Srivastava, M Kamali, A Urdaletova. *A study of the Fekete-Szegő functional and coefficient estimates for subclasses of analytic functions satisfying a certain subordination condition and associated with the Gegenbauer polynomials*, AIMS Math, 2022, 7(2): 2568-2584.
- [34] H M Srivastava, S Khan, Q Z Ahmad, et al. *The Faber polynomial expansion method and its application to the general coefficient problem for some subclasses of bi-univalent functions associated with a certain q -integral operator*, Stud Univ Babes-Bolyai Math, 2018, 63(4): 419-436.
- [35] H M Srivastava, A K Mishra, P Gochhayat. *Certain subclasses of analytic and bi-univalent functions*, Appl Math Lett, 2010, 23(10): 1188-1192.
- [36] H M Srivastava, G Murugusundaramoorthy, T Bulboaca. *The second Hankel determinant for subclasses of bi-univalent functions associated with a nephroid domain*, Rev R Acad Cienc Exactas Fis Nat Ser A Mat RACSAM, 2022, 116(4): 145.
- [37] H M Srivastava, A K Wanas, H O Guney. *New families of bi-univalent functions associated with the Bazilevic functions and the λ -pseudo-starlike functions*, Iran J Sci Technol Trans A Sci, 2021, 45(5): 1799-1804.

- [38] H M Srivastava, A K Wanas, R Srivastava. *Applications of the q -Srivastava-Attiya operator involving a certain family of bi-univalent functions associated with the Horadam polynomials*, Symmetry, 2021, 13(7): 1230.
- [39] T Wang, W Zhang. *Some identities involving Fibonacci, Lucas polynomials and their applications*, Bull Math Soc Sci Math Roumanie (NS), 2012, 55(1): 95-103.
- [40] F Yousef, S Alroud, M Illafe. *New subclasses of analytic and bi-univalent functions endowed with coefficient estimate problems*, Anal Math Phys, 2021, 11(2): 58.

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