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Some new results on parameter estimation of the exponential-Poisson distribution in ranked set sampling

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Abstract. The existence and uniqueness of the maximum likelihood estimator (MLE) of parameter for the exponential-Poisson distribution is discussed by Kuş [2007. A new lifetime distribution. Computational Statistics and Data Analysis 51(9): 4497-4509] in simple random sampling (SRS). As an alternative to the MLEs in SRS, Joukar et al. [2021. Parameter estimation for the exponential-poisson distribution based on ranked set samples. Communication in Statistics-Theory and Methods 50(3): 560-581] discussed the MLE of parameter for this distribution in ranked set sampling (RSS). However, they did not discuss the existence and uniqueness of the MLE in RSS and did not provide explicit expressions for the Fisher information in RSS. In this article, we discuss the existence and uniqueness of the MLE of parameter in RSS and give explicit expressions for the Fisher information in RSS. The MLEs will be compared in terms of asymptotic efficiencies. Numerical studies and a real data application show that these MLEs in RSS can be real competitors for those in SRS.

§1 Introduction

Lifetime distributions play an important role in many real practical situations. One of the popular life distributions is the exponential distribution which is the simplest one, as well. However, the exponential distribution suffers from a constant hazard rate function and may be inappropriate in many situations. Kuş (2007) introduced an extension of the exponential distribution, called the exponential-Poisson (EP) distribution with distribution function

$$F(x;\beta,\lambda)=\frac{e^{\lambda}-e^{\lambda e^{-\beta x}}}{e^{\lambda}-1}I(x>0), \tag{1}$$

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where $\beta > 0$ and $\lambda > 0$, and I denotes an indicator function. The probability density function (pdf) corresponding to the distribution function in (1) is then given by

$$f(x;\beta,\lambda) = \frac{\beta\lambda}{e^{\lambda}-1}\exp\left(-\beta x + \lambda e^{-\beta x}\right).$$

Barreto-Souza and Silva (2015) pointed out that the EP distribution is a good alternative to the gamma distribution for modelling lifetime, reliability and time intervals of successive natural disasters. For further details on the importance and applications of the EP distribution one may refer to Karlis (2009), Macera et al. (2015) and Xu et al. (2016).

The estimation problem for this distribution has not been discussed extensively in the literature yet. There are a few works regarding the inference problem for the EP distribution. For example, the existence and uniqueness of the maximum likelihood estimator (MLE) of parameter is discussed by Kuş (2007) in simple random sampling (SRS). As an alternative to the MLEs in SRS, Joukar et al. (2021) discussed the MLE of parameter in ranked set sampling (RSS). However, they did not discuss the existence and uniqueness of the MLE in RSS and did not provide explicit expressions for the Fisher information in RSS.

In this article, we discuss the existence and uniqueness of the MLE of parameter in RSS and give explicit expressions for the Fisher information in RSS. The remainder of the paper is organized as follows. In Sect. 2, RSS is introduced. In Sect. 3, we present some lemmas and notations, which are required for building our theorems. In Sect. 4, the existence and uniqueness of the MLE of β from the EP distribution in RSS are proved. In Sect. 5, the existence and uniqueness of the MLE of λ from the EP distribution in RSS are proved. In Sect. 6, explicit expressions for the Fisher information in RSS is computed. A comparison and conclusions will be respectively presented in Sects. 7 and 8.

§2 Ranked set sampling

RSS was introduced by McIntyre (1952) for estimating the pasture yields. It is appropriate for situations where quantification of sampling units is either costly or difficult, but ranking the units in a small set is easy and inexpensive. The procedure of RSS involves randomly drawing m^2 units from the population and then randomly partitioning them into m sets of size m. The units are then ranked within each set. Here ranking could be judgement, visual perception, covariates, or any other method that does not require actual measurement of the units. For each set, one unit is selected and measured. The basic version of RSS can be elucidated as follows. First, the experimenter draws m independent simple random samples, each of size m from the population. Then units within the i-th$(i = 1, 2, \ldots, m)$ sample are subjected to judgement ordering, with negligible cost, and the unit possessing i-th lowest rank is identified. Finally, the identified units are measured. For further introduction of RSS, refer to Al-Omari (2012), Chen et al. (2003), Dmbgen and Zamanzade (2020), Frey and Feeman (2017), He et al. (2021), Mahdizadeh and Arghami (2012), Mahdizadeh and Strzalkowska-Kominia (2017), Mahdizadeh and Zamanzade (2021), Samawi and Al-Sagheer (2001), Strzalkowska-Kominiak and Mahdizadeh (2014), Yang et al. (2020) and Zamanzade and Mahdizadeh (2017).

§3 Notations and preliminaries

In this section, we present some lemmas and notations, which are required for building our theorems.

3.1 Some lemmas

Lemma 1. For $\beta > 0$, $x > 0$ and $\lambda \in \left(0, e^2\right)$,

$$-\left(e^{\lambda} - e^{\lambda e^{-\beta x}}\right) + \beta x\left(e^{\lambda} - e^{\lambda e^{-\beta x}}\right) + \lambda e^{\lambda}\beta x e^{-\beta x} > 0. \tag{2}$$

Proof. Denote $-\left(e^{\lambda} - e^{\lambda e^{-\beta x}}\right) + \beta x\left(e^{\lambda} - e^{\lambda e^{-\beta x}}\right) + \lambda e^{\lambda}\beta x e^{-\beta x}$ as

$$g(t) = -\left(e^{\lambda} - e^{\lambda t}\right) - \ln t\left(e^{\lambda} - e^{\lambda t}\right) - \lambda e^{\lambda} t \ln t,$$

where $t = e^{-\beta x}$, $0 < t < 1$. Then

$$g'(t) = -\left(e^{\lambda} - e^{\lambda t}\right)\left(\lambda + \lambda \ln t + \frac{1}{t}\right). \tag{3}$$

Let

$$h(t) = \lambda + \lambda \ln t + \frac{1}{t},$$

then

$$h'(t) = \frac{1}{t^2}\left(\lambda t - 1\right).$$

Then $t = \frac{1}{\lambda}$ is the minimum point of $h(t)$. Thus

$$h(t) \geq \lambda\left(2 - \ln \lambda\right) > 0. \tag{4}$$

Note that

$$e^{\lambda} - e^{\lambda t} > 0. \tag{5}$$

Based on (3), (4), and (5), we observe that $g'(t) < 0$, indicating that $g(t)$ is strictly decreasing within the interval $0 < t < 1$. Considering the limit $\lim_{t\to 1^-} g(t) = 0$, it follows that $g(t) > 0$ for all t in this interval. This concludes the proof of Lemma 1. □

Lemma 2. For $\beta > 0$, $x > 0$ and $\lambda \in \left(0, e^2\right)$,

$$\left(e^{\lambda e^{-\beta x}} - 1\right) - \beta x\left(e^{\lambda e^{-\beta x}} - 1\right) + \lambda\beta x e^{-\beta x} > 0. \tag{6}$$

Proof. Denote $\left(e^{\lambda e^{-\beta x}} - 1\right) - \beta x\left(e^{\lambda e^{-\beta x}} - 1\right) + \lambda\beta x e^{-\beta x}$ as

$$p(t) = \left(e^{\lambda t} - 1\right) + \ln t\left(e^{\lambda t} - 1\right) - \lambda t \ln t,$$

where $t = e^{-\beta x}$, $0 < t < 1$. Then

$$p'(t) = \left(e^{\lambda t} - 1\right)\left(\lambda + \lambda \ln t + \frac{1}{t}\right), \tag{7}$$

Note that

$$e^{\lambda t} - 1 > 0. \tag{8}$$

From (4), (7) and (8), we can obtain $p'(t) > 0$, which means $p(t)$ is monotonically increasing within $0 < t < 1$. Taking into account the $\lim\limits_{t\to 0^+} p(t) = 0$. Thus $p(t) > 0$. This completes the proof of Lemma 2. □

Lemma 3. For $\lambda > 0$ and $m > 0$,

$$-\frac{m}{\lambda^2} + \frac{me^{\lambda}}{\left(e^{\lambda}-1\right)^2} < 0. \tag{9}$$

Proof. Let $q(\lambda) = e^{\frac{\lambda}{2}} - e^{-\frac{\lambda}{2}} - \lambda$, it is clear that $q(\lambda)$ is monotonically increasing for $\lambda > 0$ and $\lim\limits_{\lambda\to 0^+} q(\lambda) = 0$. It follows that

$$e^{\frac{\lambda}{2}} - e^{-\frac{\lambda}{2}} > \lambda.$$

Then

$$e^{\lambda} + e^{-\lambda} - 2 = \frac{\left(e^{\lambda}-1\right)^2}{e^{\lambda}} > \lambda^2 \tag{10}$$

From (10), we have

$$-\frac{m}{\lambda^2} + \frac{me^{\lambda}}{\left(e^{\lambda}-1\right)^2} < 0.$$

This completes the proof of Lemma 3. □

Lemma 4. For $\beta > 0$, $x > 0$ and $\lambda > 0$,

$$\left(1 - e^{-\beta x} - e^{-\frac{\lambda}{2}e^{-\beta x}}\right) e^{\lambda} + e^{\frac{\lambda}{2}e^{-\beta x}} - \left(1 - e^{-\beta x}\right) > 0. \tag{11}$$

Proof. Let $\mu(\lambda) = \left(1 - e^{-\beta x} - e^{-\frac{\lambda}{2}e^{-\beta x}}\right) e^{\lambda} + e^{\frac{\lambda}{2}e^{-\beta x}} - \left(1 - e^{-\beta x}\right)$, then

$$\begin{aligned}\mu'(\lambda) &= \left(1 - e^{-\beta x} - e^{-\frac{\lambda}{2}e^{-\beta x}}\right) e^{\lambda} + \frac{1}{2}e^{-\beta x}e^{-\frac{\lambda}{2}e^{-\beta x}}e^{\lambda} + \frac{1}{2}e^{-\beta x}e^{\frac{\lambda}{2}e^{-\beta x}} \\ &\geq \left(\left(1 - e^{-\beta x}\right) e^{\frac{\lambda}{2}} - e^{\frac{\lambda}{2}\left(1-e^{-\beta x}\right)} + e^{-\beta x}\right) e^{\frac{\lambda}{2}}.\end{aligned} \tag{12}$$

Denote $\nu(\lambda) = \left(1 - e^{-\beta x}\right) e^{\frac{\lambda}{2}} - e^{\frac{\lambda}{2}\left(1-e^{-\beta x}\right)} + e^{-\beta x}$, then

$$\nu'(\lambda) = \frac{1}{2}\left(1 - e^{-\beta x}\right)\left(e^{\frac{\lambda}{2}} - e^{\frac{\lambda}{2}\left(1-e^{-\beta x}\right)}\right) > 0.$$

Then $\nu(\lambda)$ is monotonically increasing for $\lambda > 0$ and $\lim\limits_{\lambda\to 0^+} \nu(\lambda) = 0$. Thus $\nu(\lambda) > 0$. Combining this with (12), we infer that $\mu'(\lambda) > 0$, indicating that $\mu(\lambda)$ is monotonically increasing in λ. Given that $\lim\limits_{\lambda\to 0^+} \mu(\lambda) = 0$, it follows that $\mu(\lambda) > 0$ for $\lambda > 0$. This concludes the proof of Lemma 4. □

Lemma 5. For $\beta > 0$, $x > 0$ and $\lambda > 0$,

$$e^{-\beta x}\left(e^{\frac{\lambda}{2}} - e^{-\frac{\lambda}{2}}\right) - \left(e^{\frac{\lambda}{2}e^{-\beta x}} - e^{-\frac{\lambda}{2}e^{-\beta x}}\right) > 0. \tag{13}$$

Proof. Let $\varphi(\lambda) = e^{-\beta x}\left(e^{\frac{\lambda}{2}} - e^{-\frac{\lambda}{2}}\right) - \left(e^{\frac{\lambda}{2}e^{-\beta x}} - e^{-\frac{\lambda}{2}e^{-\beta x}}\right)$, then

$$\varphi'(\lambda) = \frac{1}{2}e^{-\beta x}\left(\left(e^{\frac{\lambda}{2}} + e^{-\frac{\lambda}{2}}\right) - \left(e^{\frac{\lambda}{2}e^{-\beta x}} + e^{-\frac{\lambda}{2}e^{-\beta x}}\right)\right) \tag{14}$$

Denote $s(\lambda) = e^{\lambda} + e^{-\lambda}$, it is not difficult to see that $s(\lambda)$ is monotonically increasing for $\lambda > 0$ and $\frac{\lambda}{2} > \frac{\lambda}{2}e^{-\beta x}$. Then

$$s\left(\frac{\lambda}{2}\right) > s\left(\frac{\lambda}{2}e^{-\beta x}\right). \tag{15}$$

Based on (14) and (15), it follows that $\varphi'(\lambda) > 0$, indicating that $\varphi(\lambda)$ is monotonically increasing for $\lambda > 0$. Additionally, considering the limit $\lim\limits_{\lambda\to 0^+} \varphi(\lambda) = 0$, we conclude that $\varphi(\lambda) > 0$ for all $\lambda > 0$. This completes the proof of Lemma 5. □

3.2 Notations

For the sake of writing formulas conveniently, denote

$$\xi_j(\lambda) = \int_0^1 \frac{(\ln t)^j t^2 e^{3\lambda t}}{e^{\lambda t} - 1} dt \tag{16}$$

$$\varsigma_j(\lambda) = \int_0^1 \frac{(\ln t)^j t^2 e^{3\lambda t}}{e^{\lambda} - e^{\lambda t}} dt \tag{17}$$

$$\zeta_j(\lambda) = \int_0^1 \frac{(\ln t)^j t\, e^{2\lambda t}}{e^{\lambda} - e^{\lambda t}} dt \tag{18}$$

where $\lambda > 0$, $j = 1, 2$.

§4 The existence and uniqueness of the MLE of β in RSS

Joukar et al. (2021) discussed the MLE of β in RSS. However, they did not discuss the existence and uniqueness of the MLE in RSS. In this section, the existence and uniqueness of the MLE of β from the EP distribution in which λ is known in RSS are proved.

Let $\left\{x_{1(1)}, x_{2(2)}, x_{3(3)}, \cdots, x_{m(m)}\right\}$ be a ranked set sample of size m from the EP distribution in which λ is known. The pdf of $X_{i(i)}$ is

$$f_i(x;\beta,\lambda) = c(i,m)\frac{\beta\lambda}{(e^{\lambda}-1)^m}\left(e^{\lambda} - e^{\lambda e^{-\beta x}}\right)^{i-1}\left(e^{\lambda e^{-\beta x}} - 1\right)^{m-i} e^{-\beta x + \lambda e^{-\beta x}},$$

where $c(i,m) = \frac{m!}{(m-i)!(i-1)!}$. Then the log-likelihood function based on this sample is

$$lnL_{RSS} = d + mln\beta - \beta\sum_{i=1}^m x_{i(i)} + \lambda\sum_{i=1}^m e^{-\beta x_{i(i)}} + \sum_{i=1}^m (i-1)\ln\left(e^{\lambda} - e^{\lambda e^{-\beta x_{i(i)}}}\right)$$
$$+ \sum_{i=1}^m (m-i)\ln\left(e^{\lambda e^{-\beta x_{i(i)}}} - 1\right),$$

where d is a value which is free of β. By computing the first derivative of lnL_{RSS} with respect to β, we obtain

$$J(\beta) = \frac{m}{\beta} - \sum_{i=1}^m x_{i(i)} - \lambda\sum_{i=1}^m x_{i(i)}e^{-\beta x_{i(i)}} + \lambda\sum_{i=1}^m (i-1)x_{i(i)}\frac{e^{-\beta x_{i(i)} + \lambda e^{-\beta x_{i(i)}}}}{\left(e^{\lambda} - e^{\lambda e^{-\beta x_{i(i)}}}\right)}$$
$$- \lambda\sum_{i=1}^m (m-i)x_{i(i)}\frac{e^{-\beta x_{i(i)} + \lambda e^{-\beta x_{i(i)}}}}{\left(e^{\lambda e^{-\beta x_{i(i)}}} - 1\right)}.$$

Then the MLE of β is the solution of $J(\beta) = 0$, the solution is denoted as $\widehat{\beta}_{RSS,\ MLE}$. Building on the preceding analysis, we formally establish the following theorem and provide its rigorous mathematical justification.

Theorem 1. $\widehat{\beta}_{RSS,\ MLE}$ exists and is unique for a given $\lambda \in (0, e^2)$.

Proof. Since

$$\lim_{\beta\to 0} J(\beta) = +\infty \tag{19}$$

and

$$\lim_{\beta\to\infty} J(\beta) = -\sum_{i=1}^m (m-i+1)\, x_{i(i)} < 0, \tag{20}$$

the existence of $\widehat{\beta}_{RSS,\ MLE}$ is proved by combining (19) with (20). To prove the uniqueness of

$\widehat{\beta}_{RSS,\ MLE}$, we consider the second derivative of lnL_{RSS} with respect to β.

$$\frac{\partial^2 lnL_{RSS}}{\partial \beta^2} = -\frac{m}{\beta^2} + \lambda \sum_{i=1}^{m} x_{i(i)}^2 e^{-\beta x_{i(i)}} - \lambda^2 \sum_{i=1}^{m} (i-1) x_{i(i)}^2 \left(\frac{e^{-\beta x_{i(i)} + \lambda e^{-\beta x_{i(i)}}}}{e^{\lambda} - e^{\lambda e^{-\beta x_{i(i)}}}} \right)^2$$
$$- \lambda \sum_{i=1}^{m} (i-1) x_{i(i)}^2 \left(1 + \lambda e^{-\beta x_{i(i)}}\right) \left(\frac{e^{-\beta x_{i(i)} + \lambda e^{-\beta x_{i(i)}}}}{e^{\lambda} - e^{\lambda e^{-\beta x_{i(i)}}}} \right) - \lambda^2 \sum_{i=1}^{m} (m-i) x_{i(i)}^2 \left(\frac{e^{-\beta x_{i(i)} + \lambda e^{-\beta x_{i(i)}}}}{e^{\lambda e^{-\beta x_{i(i)}}} - 1} \right)^2$$
$$+ \lambda \sum_{i=1}^{m} (m-i) x_{i(i)}^2 \left(1 + \lambda e^{-\beta x_{i(i)}}\right) \left(\frac{e^{-\beta x_{i(i)} + \lambda e^{-\beta x_{i(i)}}}}{e^{\lambda e^{-\beta x_{i(i)}}} - 1} \right). \tag{21}$$

Denote β^* is a solution of $\dfrac{\partial^2 lnL_{RSS}}{\partial \beta^2} = 0$, then

$$\frac{m}{\beta^*} = \lambda\beta^* \sum_{i=1}^{m} x_{i(i)}^2 e^{-\beta^* x_{i(i)}} - \lambda^2\beta^* \sum_{i=1}^{m} (i-1) x_{i(i)}^2 \left(\frac{e^{-\beta^* x_{i(i)} + \lambda e^{-\beta^* x_{i(i)}}}}{e^{\lambda} - e^{\lambda e^{-\beta^* x_{i(i)}}}} \right)^2$$
$$- \lambda\beta^* \sum_{i=1}^{m} (i-1) x_{i(i)}^2 \left(1 + \lambda e^{-\beta^* x_{i(i)}}\right) \left(\frac{e^{-\beta^* x_{i(i)} + \lambda e^{-\beta^* x_{i(i)}}}}{e^{\lambda} - e^{\lambda e^{-\beta^* x_{i(i)}}}} \right)$$
$$- \lambda^2\beta^* \sum_{i=1}^{m} (m-i) x_{i(i)}^2 \left(\frac{e^{-\beta^* x_{i(i)} + \lambda e^{-\beta^* x_{i(i)}}}}{e^{\lambda e^{-\beta^* x_{i(i)}}} - 1} \right)^2$$
$$+ \lambda\beta^* \sum_{i=1}^{m} (m-i) x_{i(i)}^2 \left(1 + \lambda e^{-\beta^* x_{i(i)}}\right) \left(\frac{e^{-\beta^* x_{i(i)} + \lambda e^{-\beta^* x_{i(i)}}}}{e^{\lambda e^{-\beta^* x_{i(i)}}} - 1} \right) \tag{22}$$

and

$$J(\beta^*) = \lambda \sum_{i=1}^{m} x_{i(i)} e^{-\beta^* x_{i(i)}} \left(\beta^* x_{i(i)} - \frac{e^{\beta^* x_{i(i)}}}{\lambda} - 1 \right)$$
$$- \lambda \sum_{i=1}^{m} (i-1) x_{i(i)} \frac{e^{-\beta^* x_{i(i)} + \lambda e^{-\beta^* x_{i(i)}}}}{\left(e^{\lambda} - e^{\lambda e^{-\beta^* x_{i(i)}}}\right)^2} \left[-\left(e^{\lambda} - e^{\lambda e^{-\beta^* x_{i(i)}}}\right) + \beta^* x_{i(i)} \left(e^{\lambda} - e^{\lambda e^{-\beta^* x_{i(i)}}}\right) \right.$$
$$\left. + \lambda e^{\lambda} \beta^* x_{i(i)} e^{-\beta^* x_{i(i)}} \right] - \lambda \sum_{i=1}^{m} (m-i) x_{i(i)} \frac{e^{-\beta^* x_{i(i)} + \lambda e^{-\beta^* x_{i(i)}}}}{\left(e^{\lambda e^{-\beta^* x_{i(i)}}} - 1\right)^2} \left[\left(e^{\lambda e^{-\beta^* x_{i(i)}}} - 1\right) \right.$$
$$\left. - \beta^* x_{i(i)} \left(e^{\lambda e^{-\beta^* x_{i(i)}}} - 1\right) + \lambda \beta^* x_{i(i)} e^{-\beta^* x_{i(i)}} \right]. \tag{23}$$

Now we prove $J(\beta^*) < 0$ for $\lambda \in (0, e^2)$. Since $\beta x - \dfrac{e^{\beta x}}{\lambda} - 1 < 0$ for $\lambda \in (0, e^2)$ (see Kuş (2007)), the first term of (23) $\lambda \sum_{i=1}^{m} x_{i(i)} e^{-\beta^* x_{i(i)}} \left(\beta^* x_{i(i)} - \dfrac{e^{\beta^* x_{i(i)}}}{\lambda} - 1 \right) < 0$. We know that the second term and the third term of (23) are always negative from Lemma 1 and Lemma 2 for $\lambda \in (0, e^2)$. Hence $J(\beta^*) < 0$ for $\lambda \in (0, e^2)$. This implies that $J(\beta)$ is negative at its stationary points for $\lambda \in (0, e^2)$. Considering the limit $J(\beta) = -\sum_{i=1}^{m} (m-i+1)\, x_{i(i)} < 0$. Thus the uniqueness of $\widehat{\beta}_{RSS,\ MLE}$ is proved. This completes the proof of Theorem 1. □

§5 The existence and uniqueness of the MLE of λ in RSS

Joukar et al. (2021) discussed the MLE of λ in RSS. However, they did not discuss the existence and uniqueness of the MLE in RSS. In this section, the existence and uniqueness of the MLE of λ from the EP distribution in which β is known in RSS are proved.

Let $\left\{x_{1(1)}, x_{2(2)}, x_{3(3)}, \cdots, x_{m(m)}\right\}$ be a ranked set sample of size m from the EP distribution in which β is known, then the log-likelihood function based on this sample is

$$lnL_{RSS} = d_0 + mln\lambda - m^2ln\left(e^{\lambda}-1\right) + \lambda\sum_{i=1}^{m} e^{-\beta x_{i(i)}} + \sum_{i=1}^{m}(m-i)ln\left(e^{\lambda e^{-\beta x_{i(i)}}}-1\right)$$
$$+\sum_{i=1}^{m}(i-1)ln\left(e^{\lambda}-e^{\lambda e^{-\beta x_{i(i)}}}\right),$$

where d_0 is a value which is free of λ. Taking the first derivative for lnL_{RSS} with respect to λ, we have

$$\frac{\partial lnL_{RSS}}{\partial\lambda} = \frac{m}{\lambda} - \frac{m^2e^{\lambda}}{e^{\lambda}-1} + \sum_{i=1}^{m} e^{-\beta x_{i(i)}} + \sum_{i=1}^{m}(m-i)\frac{e^{-\beta x_{i(i)}+\lambda e^{-\beta x_{i(i)}}}}{\left(e^{\lambda e^{-\beta x_{i(i)}}}-1\right)}$$
$$+\sum_{i=1}^{m}(i-1)\frac{e^{\lambda}-e^{-\beta x_{i(i)}+\lambda e^{-\beta x_{i(i)}}}}{\left(e^{\lambda}-e^{\lambda e^{-\beta x_{i(i)}}}\right)}.$$

Then the MLE of λ is the solution of $\dfrac{\partial lnL_{RSS}}{\partial\lambda}=0$, the solution is denoted as $\widehat{\lambda}_{RSS,\ MLE}$. Building on the previous analysis, we formally establish the following theorem and present its detailed mathematical proof.

Theorem 2. $\widehat{\lambda}_{RSS,\ MLE}$ exists and is unique when $\sum\limits_{i=1}^{m} e^{-\beta x_{i(i)}} > \frac{m}{2}$.

Proof.

$$\lim_{\lambda\to 0}\frac{\partial lnL_{RSS}}{\partial\lambda} = -\frac{m\left(m+1\right)}{4} + \frac{\left(m+1\right)}{2}\sum_{i=1}^{m} e^{-\beta x_{i(i)}} \tag{24}$$

and

$$\lim_{\lambda\to\infty}\frac{\partial lnL_{RSS}}{\partial\lambda} = -\frac{m\left(m+1\right)}{2} + \sum_{i=1}^{m}\left(m-i+1\right)e^{-\beta x_{i(i)}}. \tag{25}$$

Since $-\frac{m(m+1)}{4} + \frac{(m+1)}{2}\sum\limits_{i=1}^{m} e^{-\beta x_{i(i)}} > -\frac{m(m+1)}{4} + \frac{m(m+1)}{4} = 0$ when $\sum\limits_{i=1}^{m} e^{-\beta x_{i(i)}} > \frac{m}{2}$, $\lim\limits_{\lambda\to 0}\frac{\partial lnL_{RSS}}{\partial\lambda} > 0$. Since $0 < e^{-\beta x} < 1$, $-\frac{m(m+1)}{2} + \sum\limits_{i=1}^{m}\left(m-i+1\right)e^{-\beta x_{i(i)}} < -\frac{m(m+1)}{2} + \frac{m(m+1)}{2} = 0$, $\lim\limits_{\lambda\to\infty}\frac{\partial lnL_{RSS}}{\partial\lambda} < 0$. Thus, the existence of $\widehat{\lambda}_{RSS,\ MLE}$ is proved when $\sum\limits_{i=1}^{m} e^{-\beta x_{i(i)}} > \frac{m}{2}$.

To prove the uniqueness of $\widehat{\lambda}_{RSS,\ MLE}$, the second derivative of lnL_{RSS} with respect to λ is computed as

$$\frac{\partial^2 lnL_{RSS}}{\partial\lambda^2} = -\frac{m}{\lambda^2} + \frac{m^2e^{\lambda}}{\left(e^{\lambda}-1\right)^2} - \sum_{i=1}^{m}(m-i)\frac{e^{-2\beta x_{i(i)}+\lambda e^{-\beta x_{i(i)}}}}{\left(e^{\lambda e^{-\beta x_{i(i)}}}-1\right)^2} - \sum_{i=1}^{m}(i-1)\frac{e^{\lambda}e^{\lambda e^{-\beta x_{i(i)}}}\left(1-e^{-\beta x_{i(i)}}\right)^2}{\left(e^{\lambda}-e^{\lambda e^{-\beta x_{i(i)}}}\right)^2}, \tag{26}$$

which is equivalent to the equation

$$\frac{\partial^2 lnL_{RSS}}{\partial\lambda^2}=-\frac{m}{\lambda^2}+\frac{me^{\lambda}}{\left(e^{\lambda}-1\right)^2}+\sum_{i=1}^{m}(m-i)\frac{e^{\lambda}}{\left(e^{\lambda}-1\right)^2}+\sum_{i=1}^{m}(i-1)\frac{e^{\lambda}}{\left(e^{\lambda}-1\right)^2}$$
$$-\sum_{i=1}^{m}(m-i)\frac{e^{-2\beta x_{i(i)}+\lambda e^{-\beta x_{i(i)}}}}{\left(e^{\lambda e^{-\beta x_{i(i)}}}-1\right)^2}-\sum_{i=1}^{m}(i-1)\frac{e^{\lambda}e^{\lambda e^{-\beta x_{i(i)}}}\left(1-e^{-\beta x_{i(i)}}\right)^2}{\left(e^{\lambda}-e^{\lambda e^{-\beta x_{i(i)}}}\right)^2}.$$

After some calculations and simplifications, we have that

$$\begin{aligned}&\frac{\partial^2 lnL_{RSS}}{\partial\lambda^2}=\left(-\frac{m}{\lambda^2}+\frac{me^{\lambda}}{(e^{\lambda}-1)^2}\right)\\&-\sum_{i=1}^{m}(i-1)\frac{e^{\lambda}e^{\frac{\lambda}{2}e^{-\beta x_{i(i)}}}}{\left(e^{\lambda}-e^{\lambda e^{-\beta x_{i(i)}}}\right)^2(e^{\lambda}-1)^2}\left\{\left[e^{\frac{\lambda}{2}e^{-\beta x_{i(i)}}}\left(1-e^{-\beta x_{i(i)}}\right)\left(e^{\lambda}-1\right)+\left(e^{\lambda}-e^{\lambda e^{-\beta x_{i(i)}}}\right)\right]\right.\\&\left.\left[\left(1-e^{-\beta x_{i(i)}}-e^{-\frac{\lambda}{2}e^{-\beta x_{i(i)}}}\right)e^{\lambda}+e^{\frac{\lambda}{2}e^{-\beta x_{i(i)}}}-\left(1-e^{-\beta x_{i(i)}}\right)\right]\right\}\\&-\sum_{i=1}^{m}(m-i)\frac{e^{\frac{\lambda}{2}}e^{\frac{\lambda}{2}e^{-\beta x_{i(i)}}}}{\left(e^{\lambda e^{-\beta x_{i(i)}}}-1\right)^2(e^{\lambda}-1)^2}\left\{\left[e^{-\beta x_{i(i)}}e^{\frac{\lambda}{2}e^{-\beta x_{i(i)}}}\left(e^{\lambda}-1\right)+e^{\frac{\lambda}{2}}\left(e^{\lambda e^{-\beta x_{i(i)}}}-1\right)\right]\right.\\&\left.\left[e^{-\beta x_{i(i)}}\left(e^{\frac{\lambda}{2}}-e^{-\frac{\lambda}{2}}\right)-\left(e^{\frac{\lambda}{2}e^{-\beta x_{i(i)}}}-e^{-\frac{\lambda}{2}e^{-\beta x_{i(i)}}}\right)\right]\right\}.\end{aligned}\tag{27}$$

Now we prove $\frac{\partial^2 lnL_{RSS}}{\partial\lambda^2}<0$. From Lemma 3, we know that the first term of (27) $-\frac{m}{\lambda^2}+\frac{me^{\lambda}}{\left(e^{\lambda}-1\right)^2}<0$. From Lemma 4, we know that the second term of (27) is always negative. From Lemma 5, we know that the third term of (27) is always negative. Thus the uniqueness of $\widehat{\lambda}_{RSS,\ MLE}$ is proved. This completes the proof of Theorem 2. □

§6 Explicit expressions for the Fisher information

Joukar et al. (2021) discussed Fisher information matrix from the EP distribution in RSS. However, they did not provide explicit expressions for the Fisher information. In this section, we will give explicit expressions for the Fisher information in RSS.

Let $\left\{x_{1(1)},x_{2(2)},x_{3(3)},\cdots,x_{m(m)}\right\}$ be a ranked set sample of size m from the EP distribution, then the log-likelihood function based on this sample is

$$lnL_{RSS}=d_1+mln\beta+mln\lambda-m^2ln\left(e^{\lambda}-1\right)-\beta\sum_{i=1}^{m}x_{i(i)}+\lambda\sum_{i=1}^{m}e^{-\beta x_{i(i)}}$$
$$+\sum_{i=1}^{m}(i-1)ln\left(e^{\lambda}-e^{\lambda e^{-\beta x_{i(i)}}}\right)+\sum_{i=1}^{m}(m-i)ln\left(e^{\lambda e^{-\beta x_{i(i)}}}-1\right),$$

where d_1 is a value which is free of β and λ. This function implies that

$$\frac{\partial lnL_{RSS}}{\partial\beta}=\frac{m}{\beta}-\sum_{i=1}^{m}x_{i(i)}-\lambda\sum_{i=1}^{m}x_{i(i)}e^{-\beta x_{i(i)}}+\lambda\sum_{i=1}^{m}(i-1)x_{i(i)}\frac{e^{-\beta x_{i(i)}+\lambda e^{-\beta x_{i(i)}}}}{\left(e^{\lambda}-e^{\lambda e^{-\beta x_{i(i)}}}\right)}$$

$$-\lambda\sum_{i=1}^{m}(m-i)x_{i(i)}\frac{e^{-\beta x_{i(i)}+\lambda e^{-\beta x_{i(i)}}}}{\left(e^{\lambda e^{-\beta x_{i(i)}}}-1\right)} \tag{28}$$

and

$$\begin{aligned}\frac{\partial lnL_{RSS}}{\partial\lambda}=&\frac{m}{\lambda}-\frac{m^2e^{\lambda}}{e^{\lambda}-1}+\sum_{i=1}^{m}e^{-\beta x_{i(i)}}+\sum_{i=1}^{m}(i-1)\frac{e^{\lambda}-e^{-\beta x_{i(i)}+\lambda e^{-\beta x_{i(i)}}}}{\left(e^{\lambda}-e^{\lambda e^{-\beta x_{i(i)}}}\right)}\\&+\sum_{i=1}^{m}(m-i)\frac{e^{-\beta x_{i(i)}+\lambda e^{-\beta x_{i(i)}}}}{\left(e^{\lambda e^{-\beta x_{i(i)}}}-1\right)}.\end{aligned} \tag{29}$$

The solutions of (28) and (29) are the MLEs of β and λ, the solutions are denoted by $(\widehat{\beta}_{RSS,\ MLE},\widehat{\lambda}_{RSS,\ MLE})$.

Explicit expressions for the Fisher information for β and λ in RSS of size m is given as follows.

Theorem 3. Under the standard regularity assumptions (see Azzalini, 1996, p.71),, the Fisher information matrix for β and λ in RSS of size m is given by

$$I_{RSS}(\beta,\lambda)=\begin{pmatrix} I_{11,\ RSS} & I_{12,\ RSS}\\ I_{12,\ RSS} & I_{22,\ RSS}\end{pmatrix}, \tag{30}$$

where

$$I_{11,\ RSS}=\frac{m}{\beta^2}-\frac{m}{\beta^2\left(e^{\lambda}-1\right)}\sum_{n=0}^{\infty}\frac{2\lambda^{n+2}}{n!(n+2)^3}+\frac{m\,(m-1)\,\lambda^3}{\beta^2(e^{\lambda}-1)^2}\xi_2\left(\lambda\right)+\frac{m\,(m-1)\,\lambda^3}{\beta^2(e^{\lambda}-1)^2}\varsigma_2\left(\lambda\right),$$

$$I_{12,\ RSS}=\frac{m}{\beta\left(e^{\lambda}-1\right)}\sum_{n=0}^{\infty}\frac{\lambda^{n+1}}{n!(n+2)^2}-\frac{m\,(m-1)\,\lambda^2}{\beta(e^{\lambda}-1)^2}\left(e^{\lambda}\zeta_1\left(\lambda\right)-\varsigma_1\left(\lambda\right)\right)+\frac{m\,(m-1)\,\lambda^2}{\beta(e^{\lambda}-1)^2}\xi_1\left(\lambda\right)$$

and

$$I_{22,\ RSS}=\frac{m}{\lambda^2}-\frac{m^2e^{\lambda}}{\left(e^{\lambda}-1\right)^2}+\sum_{i=1}^{m}(m-i)\,\frac{c\,(i,m)\,a_{i,\lambda}}{\lambda^2(e^{\lambda}-1)^m}+\sum_{i=1}^{m}(i-1)\,\frac{c\,(i,m)\,b_{i,\lambda}}{\lambda^2(e^{\lambda}-1)^m},$$

where

$a_{i,\lambda}=\sum\limits_{j=0}^{i-1}\binom{i-1}{j}\sum\limits_{k=0}^{m-i-2}\binom{m-i-2}{k}(-1)^{i-j+k-1}e^{\lambda j}\frac{\left[e^{\lambda(m-j-k-1)}\left(\lambda^2(m-j-k-1)^2-2\lambda(m-j-k-1)+2\right)-2\right]}{(m-j-k-1)^3}$

and

$b_{i,\lambda}=\sum\limits_{j=0}^{i-3}\binom{i-3}{j}\sum\limits_{k=0}^{m-i}\binom{m-i}{k}(-1)^{i-j+k-3}e^{\lambda(j+1)}\frac{\left[2e^{\lambda(m-j-k-1)}-\lambda^2(m-j-k-1)^2-2\lambda(m-j-k-1)-2\right]}{(m-j-k-1)^3}.$

Proof.

$$\begin{aligned}\frac{\partial^2 lnL_{RSS}}{\partial\beta\partial\lambda}=&-\sum_{i=1}^{m}x_{i(i)}e^{-\beta x_{i(i)}}-\sum_{i=1}^{m}(i-1)\,\lambda x_{i(i)}\left(e^{\lambda}-e^{-\beta x_{i(i)}+\lambda e^{-\beta x_{i(i)}}}\right)\frac{e^{-\beta x_{i(i)}+\lambda e^{-\beta x_{i(i)}}}}{\left(e^{\lambda}-e^{\lambda e^{-\beta x_{i(i)}}}\right)^2}\\&+\sum_{i=1}^{m}(i-1)x_{i(i)}\left(1+\lambda e^{-\beta x_{i(i)}}\right)\frac{e^{-\beta x_{i(i)}+\lambda e^{-\beta x_{i(i)}}}}{\left(e^{\lambda}-e^{\lambda e^{-\beta x_{i(i)}}}\right)}+\sum_{i=1}^{m}(m-i)\lambda x_{i(i)}\left(\frac{e^{-\beta x_{i(i)}+\lambda e^{-\beta x_{i(i)}}}}{e^{\lambda e^{-\beta x_{i(i)}}}-1}\right)^2\\&-\sum_{i=1}^{m}(m-i)x_{i(i)}\left(1+\lambda e^{-\beta x_{i(i)}}\right)\frac{e^{-\beta x_{i(i)}+\lambda e^{-\beta x_{i(i)}}}}{\left(e^{\lambda e^{-\beta x_{i(i)}}}-1\right)}.\end{aligned}$$

Under the assumed regularity conditions of Theorem 3.,

$$
\begin{aligned}
&I_{11,\ RSS} = -E\left(\frac{\partial^2 lnL_{RSS}}{\partial\beta^2}\right)\\
&= \frac{m}{\beta^2} - E\left(\lambda\sum_{i=1}^{m} x_{i(i)}^2 e^{-\beta x_{i(i)}}\right) + E\left[\lambda^2\sum_{i=1}^{m}(i-1)x_{i(i)}^2\left(\frac{e^{-\beta x_{i(i)}+\lambda e^{-\beta x_{i(i)}}}}{e^{\lambda}-e^{\lambda e^{-\beta x_{i(i)}}}}\right)^2\right]\\
&+E\left[\lambda\sum_{i=1}^{m}(i-1)x_{i(i)}^2\left(1+\lambda e^{-\beta x_{i(i)}}\right)\left(\frac{e^{-\beta x_{i(i)}+\lambda e^{-\beta x_{i(i)}}}}{e^{\lambda}-e^{\lambda e^{-\beta x_{i(i)}}}}\right)\right]\\
&+E\left[\lambda^2\sum_{i=1}^{m}(m-i)x_{i(i)}^2\left(\frac{e^{-\beta x_{i(i)}+\lambda e^{-\beta x_{i(i)}}}}{e^{\lambda e^{-\beta x_{i(i)}}}-1}\right)^2\right]\\
&-E\left[\lambda\sum_{i=1}^{m}(m-i)x_{i(i)}^2\left(1+\lambda e^{-\beta x_{i(i)}}\right)\left(\frac{e^{-\beta x_{i(i)}+\lambda e^{-\beta x_{i(i)}}}}{e^{\lambda e^{-\beta x_{i(i)}}}-1}\right)\right]\\
&= \frac{m}{\beta^2} - m\lambda E\left(x^2 e^{-\beta x}\right) + \frac{m\,(m-1)\,\lambda^2}{e^{\lambda}-1}E\left(x^2\frac{e^{-2\beta x+2\lambda e^{-\beta x}}}{e^{\lambda}-e^{\lambda e^{-\beta x}}}\right) + \frac{m\,(m-1)\,\lambda^2}{e^{\lambda}-1}E\left(x^2\frac{e^{-2\beta x+2\lambda e^{-\beta x}}}{e^{\lambda e^{-\beta x}}-1}\right)\\
&= \frac{m}{\beta^2} - \frac{m\lambda^2\beta}{(e^{\lambda}-1)}\int_0^{\infty} x^2 e^{-2\beta x}e^{\lambda e^{-\beta x}}dx + \frac{m\,(m-1)\,\lambda^3\beta}{(e^{\lambda}-1)^2}\int_0^{\infty} x^2\frac{e^{-3\beta x+3\lambda e^{-\beta x}}}{e^{\lambda}-e^{\lambda e^{-\beta x}}}dx\\
&+\frac{m\,(m-1)\,\lambda^3\beta}{(e^{\lambda}-1)^2}\int_0^{\infty} x^2\frac{e^{-3\beta x+3\lambda e^{-\beta x}}}{e^{\lambda e^{-\beta x}}-1}dx\\
&= \frac{m}{\beta^2} - \frac{m\lambda^2}{\beta^2\,(e^{\lambda}-1)}\int_0^1 t(lnt)^2\left(e^{\lambda t}\right)dt + \frac{m\,(m-1)\,\lambda^3}{\beta^2(e^{\lambda}-1)^2}\int_0^1\frac{t^2(lnt)^2\left(e^{3\lambda t}\right)}{e^{\lambda}-e^{\lambda t}}dt\\
&+\frac{m\,(m-1)\,\lambda^3}{\beta^2(e^{\lambda}-1)^2}\int_0^1\frac{t^2(lnt)^2\left(e^{3\lambda t}\right)}{e^{\lambda t}-1}dt\\
&= \frac{m}{\beta^2} - \frac{m}{\beta^2\,(e^{\lambda}-1)}\sum_{n=0}^{\infty}\frac{2\lambda^{n+2}}{n!(n+2)^3} + \frac{m\,(m-1)\,\lambda^3}{\beta^2(e^{\lambda}-1)^2}\varsigma_2(\lambda) + \frac{m\,(m-1)\,\lambda^3}{\beta^2(e^{\lambda}-1)^2}\xi_2(\lambda),
\end{aligned}
$$

$$
\begin{aligned}
&I_{12,\ RSS} = -E\left(\frac{\partial^2 lnL_{RSS}}{\partial\beta\partial\lambda}\right)\\
&= E\left(\sum_{i=1}^{m} x_{i(i)}e^{-\beta x_{i(i)}}\right) + E\left(\sum_{i=1}^{m}(i-1)\,\lambda x_{i(i)}\left(e^{\lambda}-e^{-\beta x_{i(i)}+\lambda e^{-\beta x_{i(i)}}}\right)\frac{e^{-\beta x_{i(i)}+\lambda e^{-\beta x_{i(i)}}}}{\left(e^{\lambda}-e^{\lambda e^{-\beta x_{i(i)}}}\right)^2}\right)\\
&-E\left(\sum_{i=1}^{m}(i-1)x_{i(i)}\left(1+\lambda e^{-\beta x_{i(i)}}\right)\frac{e^{-\beta x_{i(i)}+\lambda e^{-\beta x_{i(i)}}}}{\left(e^{\lambda}-e^{\lambda e^{-\beta x_{i(i)}}}\right)}\right)\\
&-E\left(\sum_{i=1}^{m}(m-i)\lambda x_{i(i)}\left(\frac{e^{-\beta x_{i(i)}+\lambda e^{-\beta x_{i(i)}}}}{e^{\lambda e^{-\beta x_{i(i)}}}-1}\right)^2\right)\\
&+E\left(\sum_{i=1}^{m}(m-i)x_{i(i)}\left(1+\lambda e^{-\beta x_{i(i)}}\right)\frac{e^{-\beta x_{i(i)}+\lambda e^{-\beta x_{i(i)}}}}{\left(e^{\lambda e^{-\beta x_{i(i)}}}-1\right)}\right)
\end{aligned}
$$

$$= mE\left(xe^{-\beta x}\right) + \frac{m(m-1)\lambda}{e^{\lambda}-1}E\left(x\left(e^{\lambda}-e^{-\beta x+\lambda e^{-\beta x}}\right)\frac{e^{-\beta x+\lambda e^{-\beta x}}}{e^{\lambda}-e^{\lambda e^{-\beta x}}}\right)$$
$$-\frac{m(m-1)\lambda}{e^{\lambda}-1}E\left(x\frac{e^{-2\beta x+2\lambda e^{-\beta x}}}{e^{\lambda e^{-\beta x}}-1}\right)$$
$$=\frac{m\lambda\beta}{(e^{\lambda}-1)}\int_0^{\infty}xe^{-2\beta x}e^{\lambda e^{-\beta x}}dx+\frac{m(m-1)\lambda^2\beta}{(e^{\lambda}-1)^2}\int_0^{\infty}x\left(e^{\lambda}-e^{-\beta x+\lambda e^{-\beta x}}\right)\frac{e^{-2\beta x+2\lambda e^{-\beta x}}}{e^{\lambda}-e^{\lambda e^{-\beta x}}}dx$$
$$-\frac{m(m-1)\lambda^2\beta}{(e^{\lambda}-1)^2}\int_0^{\infty}x\frac{e^{-3\beta x+3\lambda e^{-\beta x}}}{e^{\lambda e^{-\beta x}}-1}dx$$
$$=-\frac{m\lambda}{\beta(e^{\lambda}-1)}\int_0^1 t(lnt)\left(e^{\lambda t}\right)dt-\frac{m(m-1)\lambda^2}{\beta(e^{\lambda}-1)^2}\int_0^1\left(e^{\lambda}-te^{\lambda t}\right)\frac{t(lnt)\left(e^{2\lambda t}\right)}{e^{\lambda}-e^{\lambda t}}dt$$
$$+\frac{m(m-1)\lambda^2}{\beta(e^{\lambda}-1)^2}\int_0^1\frac{t^2(lnt)\left(e^{3\lambda t}\right)}{e^{\lambda t}-1}dt$$
$$=\frac{m}{\beta(e^{\lambda}-1)}\sum_{n=0}^{\infty}\frac{\lambda^{n+1}}{n!(n+2)^2}-\frac{m(m-1)\lambda^2}{\beta(e^{\lambda}-1)^2}\left(e^{\lambda}\zeta_1(\lambda)-\varsigma_1(\lambda)\right)+\frac{m(m-1)\lambda^2}{\beta(e^{\lambda}-1)^2}\xi_1(\lambda)$$

and

$$I_{22,\ RSS}=-E\left(\frac{\partial^2 lnL_{RSS}}{\partial\lambda^2}\right)$$
$$=\frac{m}{\lambda^2}-\frac{m^2e^{\lambda}}{(e^{\lambda}-1)^2}+E\left(\sum_{i=1}^m(m-i)\frac{e^{-2\beta x_{i(i)}+\lambda e^{-\beta x_{i(i)}}}}{\left(e^{\lambda e^{-\beta x_{i(i)}}}-1\right)^2}\right)+E\left(\sum_{i=1}^m(i-1)\frac{e^{\lambda}e^{\lambda e^{-\beta x_{i(i)}}}\left(1-e^{-\beta x_{i(i)}}\right)^2}{\left(e^{\lambda}-e^{\lambda e^{-\beta x_{i(i)}}}\right)^2}\right)$$
$$=\frac{m}{\lambda^2}-\frac{m^2e^{\lambda}}{(e^{\lambda}-1)^2}+\sum_{i=1}^m(m-i)\frac{c(i,m)\beta\lambda}{(e^{\lambda}-1)^m}\int_0^{\infty}\left(e^{\lambda}-e^{\lambda e^{-\beta x}}\right)^{i-1}\left(e^{\lambda e^{-\beta x}}-1\right)^{m-i-2}e^{-3\beta x}e^{2\lambda e^{-\beta x}}dx$$
$$+\sum_{i=1}^m(i-1)e^{\lambda}\frac{c(i,m)\beta\lambda}{(e^{\lambda}-1)^m}\int_0^{\infty}\left(e^{\lambda}-e^{\lambda e^{-\beta x}}\right)^{i-3}\left(e^{\lambda e^{-\beta x}}-1\right)^{m-i}\left(1-e^{-\beta x}\right)^2e^{-\beta x}e^{2\lambda e^{-\beta x}}dx$$
$$=\frac{m}{\lambda^2}-\frac{m^2e^{\lambda}}{(e^{\lambda}-1)^2}+\sum_{i=1}^m(m-i)\frac{c(i,m)\lambda}{(e^{\lambda}-1)^m}\int_0^1\left(e^{\lambda}-e^{\lambda t}\right)^{i-1}\left(e^{\lambda t}-1\right)^{m-i-2}t^2e^{2\lambda t}dt$$
$$+\sum_{i=1}^m(i-1)e^{\lambda}\frac{c(i,m)\lambda}{(e^{\lambda}-1)^m}\int_0^1\left(e^{\lambda}-e^{\lambda t}\right)^{i-3}\left(e^{\lambda t}-1\right)^{m-i}(1-t)^2e^{2\lambda t}dt$$
$$=\frac{m}{\lambda^2}-\frac{m^2e^{\lambda}}{(e^{\lambda}-1)^2}$$
$$+\sum_{i=1}^m(m-i)\frac{c(i,m)\lambda}{(e^{\lambda}-1)^m}\sum_{j=0}^{i-1}\binom{i-1}{j}\sum_{k=0}^{m-i-2}\binom{m-i-2}{k}(-1)^{i-j+k-1}e^{\lambda j}\int_0^1\left(e^{\lambda t}\right)^{m-j-k-1}t^2dt$$
$$+\sum_{i=1}^m(i-1)\frac{c(i,m)e^{\lambda}\lambda}{(e^{\lambda}-1)^m}\sum_{j=0}^{i-3}\binom{i-3}{j}\sum_{k=0}^{m-i}\binom{m-i}{k}(-1)^{i-j+k-3}e^{\lambda j}\int_0^1\left(e^{\lambda t}\right)^{m-j-k-1}(1-t)^2dt$$
$$=\frac{m}{\lambda^2}-\frac{m^2e^{\lambda}}{(e^{\lambda}-1)^2}+\sum_{i=1}^m(m-i)\frac{c(i,m)a_{i,\lambda}}{\lambda^2(e^{\lambda}-1)^m}+\sum_{i=1}^m(i-1)\frac{c(i,m)b_{i,\lambda}}{\lambda^2(e^{\lambda}-1)^m}.$$

This completes the proof of Theorem 3. □

§7 Numerical comparison

This section compares the maximum likelihood estimators obtained through RSS and SRS by assessing their asymptotic efficiencies.

7.1 Numerical studies

In this subsection, we compare the asymptotic efficiencies of the MLEs. Under certain regularity conditions, the asymptotic efficiency of the MLEs can be derived from the inverse of the Fisher information matrix (Barabesi and El-Sharaawi, 2001). Hence, the asymptotic efficiency of $\widehat{\beta}_{RSS,\ MLE}$ w.r.t. $\widehat{\beta}_{SRS,\ MLE}$ may be defined as $\mathrm{AE}^1=\dfrac{I_{11,\ RSS}}{I_{11,\ SRS}}$. The asymptotic efficiencies of $\widehat{\lambda}_{RSS,\ MLE}$ w.r.t. $\widehat{\lambda}_{SRS,\ MLE}$ and $(\widehat{\beta}_{RSS,\ MLE},\widehat{\lambda}_{RSS,\ MLE})$ w.r.t. $(\widehat{\beta}_{SRS,\ MLE},\widehat{\lambda}_{SRS,\ MLE})$ may be respectively defined as $\mathrm{AE}^2=\dfrac{I_{22,\ RSS}}{I_{22,\ SRS}}$ and $\mathrm{AE}^3=\dfrac{\det\{I_{RSS}(\beta,\ \lambda)\}}{\det\{I_{SRS}(\beta,\ \lambda)\}}$. $I_{11,\ SRS}$, $I_{22,\ SRS}$ and $\det\{I_{SRS}(\beta,\ \lambda)\}$ are from Kuş (2007). Since $\mathrm{AE}^i(i=1,2,3)$ are free of β, without loss of generality the numerical results is given for $\beta=1$. The numerical results are given in Table 1.

From Table 1, we conclude the following:

(1) $\mathrm{AE}^1>1$, which means $\widehat{\beta}_{RSS,\ MLE}$ is more efficient than $\widehat{\beta}_{SRS,\ MLE}$.

(2) $\mathrm{AE}^2>1$, which means $\widehat{\lambda}_{RSS,\ MLE}$ is more efficient than $\widehat{\lambda}_{SRS,\ MLE}$.

(3) $\mathrm{AE}^3>1$, which means $(\widehat{\beta}_{RSS,\ MLE},\widehat{\lambda}_{RSS,\ MLE})$ is more efficient than $(\widehat{\beta}_{SRS,\ MLE},\widehat{\lambda}_{SRS,\ MLE})$.

(4) In conclusion, the MLEs of β and λ in RSS are more efficient than that in SRS.

7.2 A real data application

Now, we focus on the analysis of earth quakes in the last century in North Anatolia fault zone between 39.00°-42.00° North latitude and 30.00°-40.00° East longitude. In Table 2, the dates of the successive earthquakes with magnitudes greater than or equal to 6Mw (moment magnitude), which are recorded with their exact locations, magnitudes and depths between the years 1900 and 2000. The data set given in Table 3 includes the time intervals (in days) of the successive earthquakes mentioned above. The data is taken from University of Bosphoros, Kandilli Observatory and Earthquake Research Institute-National Earthquake Monitoring Center (KOERI-NEMC, web address: http://www.koeri.boun.edu.tr).

Kuş (2007) pointed out that EP distribution fits the data very well. The MLE of λ, β and the p-value of the Kolmogorov-Smirnov test are 2.6377, 3.56×10^{-4} and 0.9772, respectively. The result of the analysis are presented in Table 4. It can be seen from the table that there is the same conclusions as numerical results of the previous sections. This agrees with the numerical results of the previous sections.

Table 1. Asymptotic efficiencies of MLEs of β and λ.

λ	m	AE^1	AE^2	AE^3
3	2	1.4494	1.4629	1.4779
	3	1.9015	1.9250	1.9350
	4	2.3528	2.3872	2.4361
	5	2.8061	2.8492	2.8620
	6	3.2584	3.3127	3.3879
4	2	1.4522	1.4506	1.0723
	3	1.9157	1.9011	2.4628
	4	2.3790	2.3517	4.4115
	5	2.8424	2.8028	6.9113
	6	3.3057	3.2533	9.7776
5	2	1.4699	1.4344	1.5818
	3	1.9342	1.8683	2.4338
	4	2.3985	2.3014	3.3015
	5	2.8629	2.7354	4.3380
	6	3.3273	3.1704	5.4182

Table 2. Earthquakes in North Anatolia fault zones.

Date	Longitude	Latitude	Magnitude(Mw)	Depth(km)
04.12.1905	39	39	6.8	0
09.02.1909	38	40	6.3	60
25.06.1910	34	41	6.2	0
24.01.1916	36.83	40.27	7.1	10
18.05.1929	37.9	40.2	6.1	10
19.04.1938	33.79	39.44	6.6	10
26.12.1939	39.51	39.8	7.9	20
30.07.1940	35.25	39.64	6.2	50
20.12.1940	39.2	39.11	6	0
08.11.1941	39.5	39.74	6	0
11.12.1942	34.83	40.76	6.1	40
20.12.1942	36.8	40.7	7	16
20.06.1943	30.51	40.85	6.5	10
26.11.1943	33.72	41.05	7.2	10
01.02.1944	32.69	41.41	7.2	10
26.10.1945	33.29	41.54	6	50
13.08.1951	32.87	40.88	6.9	10
07.09.1953	33.01	41.09	6.4	40
20.02.1956	30.49	39.89	6.4	40
26.05.1957	31	40.67	7.1	10
22.07.1967	30.69	40.67	7.2	33

Table 2. Continued.

Date	Longitude	Latitude	Magnitude(Mw)	Depth(km)
03.09.1968	32.39	41.81	6.5	5
13.03.1992	39.63	39.72	6.1	23
08.03.1997	35.44	40.78	6	5
12.11.1999	31.21	40.74	7.2	25

Table 3. Time intervals of the successive earthquakes (To be read down the columns).

1163	3258	323	159	756	409
501	616	398	67	896	8592
2039	217	9	633	461	1821
4863	143	182	2117	3709	979

Table 4. Asymptotic efficiencies of MLEs of β and λ.

$(\lambda,\ \beta)$	m	AE^1	AE^2	AE^3
$(2.6377, 3.56\times10^{-4})$	2	1.4252	1.4404	1.3006
	3	1.8875	1.8808	2.3231
	4	2.2339	2.3212	2.7801

§8 Conclusion

In this article, we proved the existence and uniqueness of the MLE of parameter in RSS and gave explicit expressions for the Fisher information in RSS. The MLEs were compared based on their asymptotic efficiencies. Numerical studies and a real data application demonstrate that the MLEs derived from RSS are comparable alternatives to those obtained from SRS. A further stage would be to extend the use of moving extremes RSS to the EP distribution.

Declarations

Conflict of interest The authors declare no conflict of interest.

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