

The transmuted epsilon distribution with applications in reliability engineering

Christophe Chesneau¹

Hassan S. Bakouch²

Idika E. Okorie^{3,*}

Bader Almohaimeed⁴

Abstract. Since its inception, the epsilon distribution has piqued the interest of statisticians. It has been successfully used to solve a variety of statistical problems. In this article, we propose to use the quadratic rank transmutation map mechanism to extend this distribution. This mechanism is not new; it was already used to improve the modeling capabilities of a number of existing distributions. For the original epsilon distribution, we expect the same benefits. As a result, we implement the transmuted epsilon distribution as a flexible three-parameter distribution with a bounded domain. We demonstrate its key features, focusing on the properties of its distributional mechanism and conducting quantile and moment analyses. Applications of the model are presented using two data sets. We also perform a regression analysis based on this distribution.

§1 Introduction

Bounded domain distributions naturally arise when the observed phenomena vary in a finite interval. The uniform, beta, arcsin, Bates, Kumaraswamy, noncentral beta, power, triangular, and Von Mises distributions are the most popular bounded domain distributions in the statistical literature. Dombi et al [1] recently proposed the epsilon distribution as an alternative to them. It is defined by the cumulative distribution function (CDF) as follows:

$$F_{\lambda,d}(x) = 1 - \epsilon_{-\lambda,d}(x), \quad x \in (0, d),$$

$F_{\lambda,d}(x) = 0$ if $x \leq 0$ and $F_{\lambda,d}(x) = 1$ if $x \geq d$, where $\lambda, d > 0$, and $\epsilon_{-\lambda,d}(x)$ denotes the following power-ratio function:

$$\epsilon_{-\lambda,d}(x) = \left(\frac{d+x}{d-x} \right)^{-\lambda d/2}, \quad x \in (0, d).$$

Hence, it is a two-parameter bounded domain distribution. It takes its source from the so-called first-order epsilon differential equation, and has the feature to satisfy the following limiting

Received: 2021-10-11. Revised: 2021-12-05.

MR Subject Classification: 60E05, 62E15.

Keywords: epsilon distribution, quadratic rank transmutation map, quantile, estimation, generalized linear modelling, regression.

Digital Object Identifier(DOI): <https://doi.org/10.1007/s11766-025-4580-7>.

*Corresponding author.

property: $\lim_{d \rightarrow +\infty} F_{\lambda,d}(x) = 1 - e^{-\lambda x}$, for $x > 0$, and $\lim_{d \rightarrow +\infty} F_{\lambda,d}(x) = 0$ otherwise, limit function which corresponds to the CDF of the exponential distribution with parameter $\lambda > 0$. It is shown in [1] that, in the context of reliability management, the epsilon probability distribution may be used to characterize the mortality and useful life cycle under the assumption of a common bathtub-shaped failure rate.

In view of its interesting modeling properties, recent works have proposed to modify or extend the epsilon distribution for diverse modelling aims. One can refer to Dombi et al [2], who proposed the omega probability distribution with the introduction of a new power parameter in the definition of $F_{\lambda,d}(x)$. A more pliant distribution family is derived and studied in Dombi and Jónás [3, 4] and Vasileva [5]. One can also mention the inverse epsilon distribution established by Jónás et al [6] corresponding to the distribution of $1/X$, where X is a random variable with the epsilon distribution, offering a new lower truncated distribution for novel modeling perspectives.

In this article, we explore a new mathematical horizon by applying an existing mechanism, called quadratic rank transmutation map (QRTM), to extend the functionality of the epsilon distribution. The QRTM was introduced in probability theory by Shaw and Buckley [7], and is formally defined by $T_\theta(u) = u(1 + \theta - \theta u)$, with $u \in (0, 1)$ and $\theta \in [-1, 1]$. It was rapidly used to add the new parameter θ to a baseline continuous distribution in order to improve its versatility. More precisely, if $G(x)$ denotes a baseline continuous CDF, the QRTM yields the following CDF:

$$H_\theta(x) = T_\theta(G(x)) = G(x)(1 + \theta - \theta G(x)), \quad x \in \mathbb{R},$$

where $\theta \in [-1, 1]$, defining an extended version of the baseline distribution. One can notice that, when $\theta = 0$, $H_\theta(x)$ is reduced to $G(x)$, when $\theta = -1$, it corresponds to the CDF of the maximum of two independent variables with CDF $G(x)$, and when $\theta = 1$, it becomes the CDF of the minimum of two independent variables with CDF $G(x)$. All the intermediary values of θ provide an easy switch between these specific distributions. This method was applied in numerous studies to extend a plethora of existing distributions. We may refer to the articles of Bourguignon et al [8] and Rahman et al [9], and the references therein.

In the light of this state of art, we introduce the transmuted epsilon (TE) distribution defined by the following CDF:

$$F_{\lambda,d,\theta}(x) = F_{\lambda,d}(x)(1 + \theta - \theta F_{\lambda,d}(x)), \quad x \in \mathbb{R}.$$

That is, after minor developments,

$$F_{\lambda,d,\theta}(x) = (1 - \epsilon_{-\lambda,d}(x))(1 + \theta \epsilon_{-\lambda,d}(x)), \quad x \in (0, d), \quad (1)$$

with $F_{\lambda,d,\theta}(x) = 0$ if $x \leq 0$ and $F_{\lambda,d,\theta}(x) = 1$ if $x \geq d$, where $\theta \in [-1, 1]$. By construction, when $d \rightarrow +\infty$, we get the CDF of the transmuted exponential distribution as proposed in [10]. Also, by the intrinsic nature of the QRTM approach, when $\theta = 0$, the TE distribution becomes the power epsilon distribution, when $\theta = -1$, it corresponds to the CDF of the maximum of two independent variables with the epsilon distribution, which is also identified as the exponentiated epsilon distribution with power parameter 2, and when $\theta = 1$, it becomes the CDF of the minimum of two independent variables with the epsilon distribution. Also, it can be noted that $F_{\lambda,d,\theta}(x)$ is an increasing function with respect to θ , implying the following first-order stochastic ordering properties involving the epsilon distribution: For $\theta \in [0, 1]$, we have $F_{\lambda,d,\theta}(x) \geq F_{\lambda,d}(x)$

and, for $\theta \in [-1, 0]$, $F_{\lambda,d,\theta}(x) \leq F_{\lambda,d}(x)$. We thus see that the parameter θ adds a certain dose of flexibility in the former epsilon distribution, which we aim to study and analyse from a statistical point of view. This aspect is detailed on several complementary plans in the article.

Concretely, the article is divided into seven sections. Other functions of interest of the TE distribution are discussed in Section 2. Quantile and moment analyses are performed in Sections 3 and 4, respectively. The estimation of the model parameters is investigated in Section 5. Applications in reliability engineering are presented with regression modeling in Section 6. Conclusion and final remarks are formulated in Section 7.

§2 More functions of the TE distribution

First, based on $F_{\lambda,d,\theta}(x)$ as given in Eq. (1), the survival function of the TE distribution is obtained as

$$\begin{aligned} S_{\lambda,d,\theta}(x) &= 1 - F_{\lambda,d,\theta}(x) \\ &= \epsilon_{-\lambda,d}(x) (1 + \theta - \theta \epsilon_{-\lambda,d}(x)), \quad x \in (0, d), \end{aligned} \quad (2)$$

with $S_{\lambda,d,\theta}(x) = 1$ if $x \leq 0$ and $S_{\lambda,d,\theta}(x) = 0$ if $x \geq d$.

By a simple differentiation of $F_{\lambda,d,\theta}(x)$ for $x \in (0, d)$, we get the corresponding probability density function (PDF) as

$$\begin{aligned} f_{\lambda,d,\theta}(x) &= F'_{\lambda,d,\theta}(x) \\ &= \lambda \frac{d^2}{d^2 - x^2} (1 - \theta + 2\theta \epsilon_{-\lambda,d}(x)) \epsilon_{-\lambda,d}(x), \quad x \in (0, d), \end{aligned}$$

and $f_{\lambda,d,\theta}(x) = 0$ for $x \notin (0, d)$. It can be noted that $f_{\lambda,d,\theta}(x)$ is a weighted version of the epsilon distribution [1] by the weight function $w(x) = 1 - \theta + 2\theta \epsilon_{-\lambda,d}(x)$. In some sense, this weight modulates some functionalities of the former epsilon distribution. Obviously, when $d \rightarrow +\infty$, we get the PDF of the transmuted exponential distribution.

Also, the following stochastic inequalities involving the PDF of the epsilon distribution hold:

$$(1 - \theta)f_{\lambda,d}(x) \leq f_{\lambda,d,\theta}(x) \leq (1 + \theta)f_{\lambda,d}(x).$$

One can also express $f_{\lambda,d,\theta}(x)$ as a general mixture of PDFs of the epsilon distribution by noting that

$$f_{\lambda,d,\theta}(x) = (1 - \theta)f_{\lambda,d}(x) + \theta f_{2\lambda,d}(x). \quad (3)$$

Therefore, selected properties of the epsilon distribution can be directly used to determine some properties of the TE distribution.

The precise analytical properties of $f_{\lambda,d,\theta}(x)$ are difficult to establish from a mathematical point of view. For this reason, we adopt a graphical approach. An exhaustive panel of curves for the PDF is presented in Figure 1.

On the other hand, the hazard rate function (HRF) of the TE distribution is obtained as

$$\begin{aligned} h_{\lambda,d,\theta}(x) &= \frac{f_{\lambda,d,\theta}(x)}{S_{\lambda,d,\theta}(x)} \\ &= \lambda \frac{d^2}{d^2 - x^2} \left(1 + \frac{\theta \epsilon_{-\lambda,d}(x)}{1 - \theta + \theta \epsilon_{-\lambda,d}(x)} \right), \quad x \in (0, d), \end{aligned}$$

and $h_{\lambda,d,\theta}(x) = 0$ for $x \notin (0, d)$. The following stochastic ordering inequalities can be estab-

lished: For $\theta \in [0, 1]$, we have $h_{\lambda,d}(x) \leq h_{\lambda,d,\theta}(x) \leq (1+\theta)h_{\lambda,d}(x)$ and for $\theta \in [-1, 0]$, we have $(1+\theta)h_{\lambda,d}(x) \leq h_{\lambda,d,\theta}(x) \leq h_{\lambda,d}(x)$, where $h_{\lambda,d}(x)$ refers to the HRF of the epsilon distribution.

The plots of the PDFs in Figure 1 indicate that the TE distribution is skewed to the right with unimodal and reversed-J shapes when θ is negative and positive, respectively, while the plots of the HRF indicate that the TE distribution has only J-shape regardless of the value of θ . From a practical point of view, this property of the HRF means that it can be used to model the second, quasi-constant, and the third, increasing phase of a bathtub-shaped failure rate curve (see, e.g. Lienig and Bruemmer [19]).

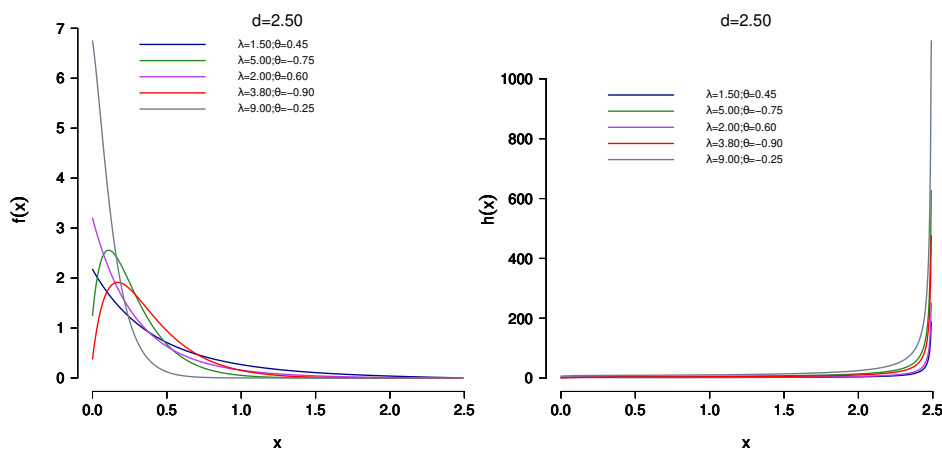


Figure 1. Plot of the PDF (left) and HRF (right) for some parameter values of the TE distribution.

§3 Quantile analysis

The quantile function (QF) of the TE distribution, say $Q_{\lambda,d,\theta}(u)$ with $u \in (0, 1)$, is characterized by the following functional equation: $F_{\lambda,d,\theta}[Q_{\lambda,d,\theta}(u)] = u$. After some developments, we establish the following sophisticated expression:

$$Q_{\lambda,d,\theta}(u) = \begin{cases} d \frac{\left[\theta - 1 + \sqrt{(1+\theta)^2 - 4\theta u} \right]^{-2/(\lambda d)} - (2\theta)^{-2/(\lambda d)}}{\left[\theta - 1 + \sqrt{(1+\theta)^2 - 4\theta u} \right]^{-2/(\lambda d)} + (2\theta)^{-2/(\lambda d)}} & \text{if } \theta \neq 0, \\ d \frac{(1-u)^{-2/(\lambda d)} - 1}{(1-u)^{-2/(\lambda d)} + 1} & \text{otherwise.} \end{cases}$$

The quantile analysis of the TE distribution is fully possible based on the expression of this QF. In particular, by taking $u = 1/2$, the median of the TE distribution is

$$M_{ed} = \begin{cases} d \frac{\left[\theta - 1 + \sqrt{1+\theta^2} \right]^{-2/(\lambda d)} - (2\theta)^{-2/(\lambda d)}}{\left[\theta - 1 + \sqrt{1+\theta^2} \right]^{-2/(\lambda d)} + (2\theta)^{-2/(\lambda d)}} & \text{if } \theta \neq 0, \\ d \frac{2^{2/(\lambda d)} - 1}{2^{2/(\lambda d)} + 1} & \text{otherwise.} \end{cases}$$

The first and third quartiles can be obtained in a similar manner. The inversion method, among other techniques, can be employed to generate values through the QF. Further general details on the quantile analysis of a distribution can be found in the book by Gilchrist [11].

The plots in Figure 2 show that the QF of the TE distribution could either be J-shaped or inverted-L-shaped.

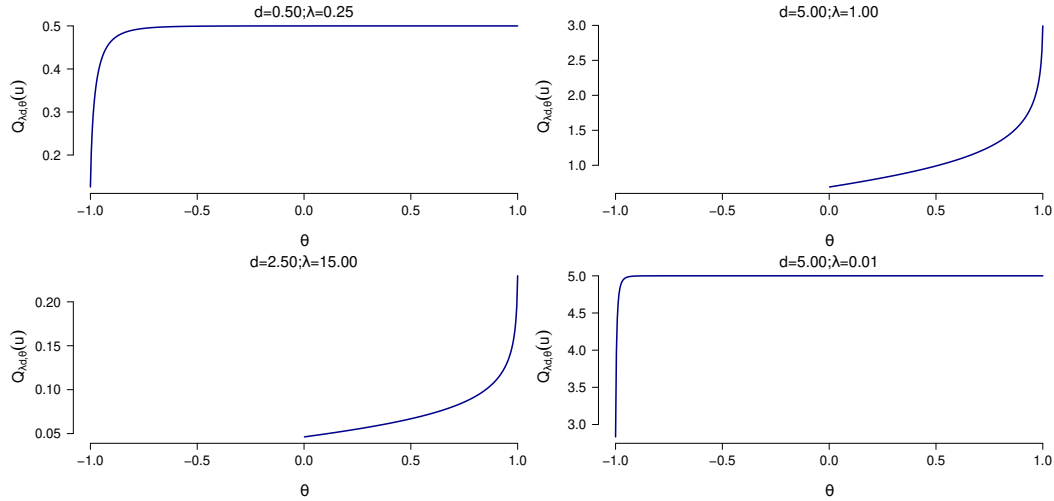


Figure 2. Plots of the QF for some parameter values of the TE distribution.

§4 Moment analysis

Let Y be a random variable with the TE distribution. The following result presents the r th moment of Y .

Proposition 1. *The r th moment of Y can be expressed as*

$$E(Y^r) = (1 - \theta)\lambda d^{1+r} B\left(r + 1, \frac{\lambda d}{2} + 1\right) {}_2F_1\left(\frac{\lambda d}{2} + 1, r + 1; r + \frac{\lambda d}{2} + 1; -1\right) \\ + 2\theta\lambda d^{1+r} B(r + 1, \lambda d + 1) {}_2F_1(\lambda d + 1, r + 1; r + \lambda d + 1; -1),$$

where $B(a, b) = \int_0^1 t^{a-1}(1-t)^{b-1}dt$, referring to the standard beta function, and ${}_2F_1(a, b; c; x)$ is the Gauss hypergeometric function defined by

$${}_2F_1(a, b; c; x) = \sum_{k=0}^{+\infty} \frac{(a)_k (b)_k}{(c)_k} \frac{x^k}{k!},$$

where $(e)_k = e(e+1)(e+2)\dots(e+k-1)$.

Proof. Let us introduce a random variable X_λ with the epsilon distribution, specified with the parameter λ . Then, by applying the approach of Section 3.1 in Okorie and Nadarajah [12] with $\beta = 1$, for any integer r , the r th moment of X_λ can be expressed as

$$E(X_\lambda^r) = \lambda d^{1+r} B\left(r + 1, \frac{\lambda d}{2} + 1\right) {}_2F_1\left(\frac{\lambda d}{2} + 1, r + 1; r + \frac{\lambda d}{2} + 1; -1\right). \quad (4)$$

Then, based on Eq. (3), the r th moment of Y is given as

$$\begin{aligned} E(Y^r) &= \int_{-\infty}^{+\infty} x^r f_{\lambda,d,\theta}(x) dx \\ &= (1-\theta) \int_{-\infty}^{+\infty} x^r f_{\lambda,d}(x) dx + \theta \int_{-\infty}^{+\infty} x^r f_{2\lambda,d}(x) dx \\ &= (1-\theta)E(X_\lambda^r) + \theta E(X_{2\lambda}^r), \end{aligned}$$

and, by virtue of Eq. (4), we get the desired result. $\square$

In particular, the mean and variance of Y are

$$\begin{aligned} E(Y) &= (1-\theta)\lambda d^2 B\left(2, \frac{\lambda d}{2} + 1\right) {}_2F_1\left(\frac{\lambda d}{2} + 1, 2; 2 + \frac{\lambda d}{2}; -1\right) \\ &\quad + 2\theta\lambda d^2 B(2, \lambda d + 1) {}_2F_1(\lambda d + 1, 2; 2 + \lambda d; -1) \end{aligned}$$

and

$$\begin{aligned} V(Y) &= (1-\theta)\lambda d^3 B\left(3, \frac{\lambda d}{2} + 1\right) {}_2F_1\left(\frac{\lambda d}{2} + 1, 3; 3 + \frac{\lambda d}{2}; -1\right) \\ &\quad + 2\theta\lambda d^3 B(3, \lambda d + 1) {}_2F_1(\lambda d + 1, 3; 3 + \lambda d; -1), \\ &\quad - \left[(1-\theta)\lambda d^2 B\left(2, \frac{\lambda d}{2} + 1\right) {}_2F_1\left(\frac{\lambda d}{2} + 1, 2; 2 + \frac{\lambda d}{2}; -1\right) \right. \\ &\quad \left. + 2\theta\lambda d^2 B(2, \lambda d + 1) {}_2F_1(\lambda d + 1, 2; 2 + \lambda d; -1) \right]^2, \end{aligned}$$

respectively. With similar expressions of $E(Y^2)$, $E(Y^3)$ and $E(Y^4)$, we can determine the skewness and kurtosis of Y , as follows:

$$S(Y) = \frac{E(Y^3) - 3E(Y)E(Y^2) + 2[E(Y)]^3}{[V(Y)]^{3/2}}$$

and

$$K(Y) = \frac{E(Y^4) - 4E(Y)E(Y^3) + 6[E(Y)]^2E(Y^2) - 3E(Y)^4}{[V(Y)]^2},$$

respectively.

The contour plot for the skewness in Figure 3 shows that, regardless of the value of θ , the TE distribution is always right skewed and the contour plot for the kurtosis still in Figure 3 shows that, whatever the value of θ is, the TE distribution is either a leptokurtic or a platykurtic distribution.

The moments are of relative complexity from the mathematical point of view. This complexity reduces considerably when r tends to infinity, as developed in the result below.

Proposition 2. *The asymptotic r th moment of Y satisfies*

$$\lim_{d \rightarrow \infty} E(Y^r) = \frac{r!}{\lambda^r} \left[1 - \theta + \frac{\theta}{2^r} \right].$$

Proof. It follows from the dominated convergence theorem that

$$\begin{aligned} \lim_{d \rightarrow \infty} E(Y^r) &= \int_{-\infty}^{\infty} x^r \left(\lim_{d \rightarrow \infty} f_{\lambda,d,\theta}(x) \right) dx \\ &= \lambda \int_0^{\infty} x^r (1 - \theta + 2\theta e^{-\lambda x}) e^{-\lambda x} dx = \frac{r!}{\lambda^r} \left[1 - \theta + \frac{\theta}{2^r} \right]. \end{aligned}$$

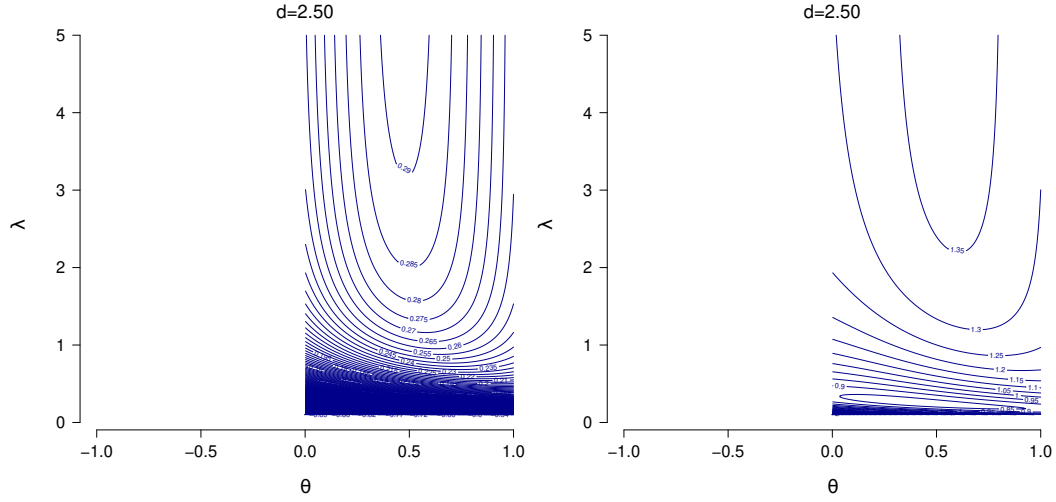


Figure 3. Contour plots for the skewness (left) and kurtosis (right) for some parameter values of the TE distribution.

□

We get the desired result.

§5 Maximum Likelihood Estimation

5.1 Method

Let X be a random variable with the TE distribution and $\boldsymbol{\Omega} = (d, \lambda, \theta)'$ be the parameter vector. The log-likelihood function $\mathcal{L} = \mathcal{L}(\boldsymbol{\Omega})$ based on a random sample of size n with the corresponding observations $x_1, x_2, \dots, x_n$ is given by

$$\begin{aligned} \mathcal{L}(\boldsymbol{\Omega}|x) = & - \sum_{i=1}^n \log(d^2 - x_i^2) + \sum_{i=1}^n \log\left(2\theta(d + x_i)^{-\frac{d\lambda}{2}}(d - x_i)^{\frac{d\lambda}{2}} - \theta + 1\right) \\ & - \frac{1}{2}d\lambda \sum_{i=1}^n \log\left(\frac{d + x_i}{d} - x_i\right) + 2n \log(d) + n \log(\lambda). \end{aligned} \quad (5)$$

Observe that we cannot obtain the estimate of d from Eq. (5) based on the sample data since d is free from x . However, from the order statistics, suppose $x_{(1)}, x_{(2)}, \dots, x_{(n)}$ such that $x_{(1)} < x_{(2)} < \dots < x_{(n)}$ denote the ordered sample corresponding to $x_1, x_2, \dots, x_n$; we can obtain the maximum likelihood estimate (MLE) for d as $\hat{d} = \max(x_1, x_2, \dots, x_n) + \epsilon$, i.e., $\hat{d} = x_{(n)} + \epsilon$ where $\epsilon > 0$ denotes an arbitrary constant. Throughout the remaining part of the article, we set $\epsilon = 0.1$ while estimating d . In fact, when we estimate a distribution parameter using the sample, we assume that all elements in the sample are in the domain of the random variable. This is because the sample should contain independent observations of the random variable in question. Since the value of parameter d determines the domain of the random

variable that has a TE distribution, in the estimation, it is a prerequisite that d is greater than the maximal element in the sample. This also means that the optimal value of d should be found under the condition that d is greater than the maximal element in the sample.

The partial derivative of $\mathcal{L}(\boldsymbol{\Omega}|x)$ given in Eq. (5) with respect to λ is

$$\begin{aligned} \frac{\partial \mathcal{L}(\boldsymbol{\Omega}|x)}{\partial \lambda} = & \sum_{i=1}^n \left[\frac{d\theta(d-x_i)^{\frac{d\lambda}{2}}(d+x_i)^{-\frac{d\lambda}{2}} \log(d-x_i)}{2\theta(d+x_i)^{-\frac{d\lambda}{2}}(d-x_i)^{\frac{d\lambda}{2}} - \theta + 1} \right. \\ & \left. - \frac{d\theta(d-x_i)^{\frac{d\lambda}{2}}(d+x_i)^{-\frac{d\lambda}{2}} \log(d+x_i)}{2\theta(d+x_i)^{-\frac{d\lambda}{2}}(d-x_i)^{\frac{d\lambda}{2}} - \theta + 1} \right] \\ & - \frac{1}{2}d \sum_{i=1}^n \log\left(\frac{d+x_i}{d} - x_i\right) + \frac{n}{\lambda}, \end{aligned} \quad (6)$$

and the partial derivative of $\mathcal{L}(\boldsymbol{\Omega}|x)$ with respect to θ is

$$\frac{\partial \mathcal{L}(\boldsymbol{\Omega}|x)}{\partial \theta} = \sum_{i=1}^n \frac{2(d-x_i)^{\frac{d\lambda}{2}}(d+x_i)^{-\frac{d\lambda}{2}} - 1}{2\theta(d+x_i)^{-\frac{d\lambda}{2}}(d-x_i)^{\frac{d\lambda}{2}} - \theta + 1}. \quad (7)$$

By substituting the MLE of d , which is $\hat{d}$ into Eq. (6) and Eq. (7), and setting the two equations to zero and solving them simultaneously, we can obtain the MLEs for λ and θ , denoted by $\hat{\lambda}$ and $\hat{\theta}$. However, the simultaneous equations can not be solved symbolically, but we can conveniently approach them numerically, for example, through the Newton-type algorithm by minimizing $\mathcal{L}(\boldsymbol{\Omega}|x)$. Several packages in some statistical and mathematical software like the `nlm` and the `optim` functions in R [13] can implement such Newton-type algorithm.

5.2 Simulation

Here, we use the MLE method to estimate the parameters of the TE distribution and we investigate the performance of this method through a Monte-Carlo simulation. The simulation involves different sample sizes, $n = 20, 50, 90, 180, 320$, and 700 , different parameter combinations, and 1000 replications in R software. Table 1 shows the values of the mean estimates, standard errors (SEs), biases, and mean square errors (MSEs) for λ and θ . We consider a wide range of initial guesses for the parameters and we did not notice any convergence issues, so any set of initial values will give a similar result in all scenarios. Generally, we note that the SEs, biases, and MSEs decrease as n increases.

The Monte-Carlo algorithm is detailed as follows.

- (i) For specific parameter values of $\boldsymbol{\Omega}$, simulate a random sample of size n from the TE distribution by using the inverse transformation method of random number generation.
- (ii) Estimate the parameters of the TE distribution by the MLE method.
- (iii) Perform 1000 replications of steps (i)-(ii).
- (iv) For each of the three parameters calculate the mean, SE, bias and MSE of the 1000 parameter estimates in (iii). The mean, SE, bias and MSE for the parameters are expressed as:

(1)

$$\bar{\bar{\Omega}} = \frac{1}{1000} \sum_{i=1}^{1000} \hat{\omega}_i,$$

(2)

$$SE_{\bar{\bar{\Omega}}} = \sqrt{\frac{1}{1000} \sum_{i=1}^{1000} (\hat{\omega}_i - \bar{\bar{\omega}})^2},$$

(3)

$$\text{Bias}_{\bar{\bar{\Omega}}} = \frac{1}{1000} \sum_{i=1}^{1000} (\hat{\omega}_i - \omega), \text{ and}$$

(4)

$$\text{MSE}_{\bar{\bar{\Omega}}} = \frac{1}{1000} \sum_{i=1}^{1000} (\hat{\omega}_i - \omega)^2,$$

respectively, where $\hat{\omega}_{i's}$ represents the MLEs of λ or θ for the i th iteration under specific sample size n , $\bar{\bar{\Omega}}$ corresponds to the mean of the parameter estimates, i.e., $\hat{\lambda}_{i's}$ and $\hat{\theta}_{i's}$ and ω denotes the actual values of the parameters.

Table 1. Simulation results for the MLEs.

Actual values			Sample size		MLEs		Measures					
d	λ	θ	n		$\hat{\lambda}$	$\hat{\theta}$	$SE_{\hat{\lambda}}$	$SE_{\hat{\theta}}$	$Bias_{\hat{\lambda}}$	$Bias_{\hat{\theta}}$	$MSE_{\hat{\lambda}}$	$MSE_{\hat{\theta}}$
1.70	0.30	1.00	20		0.3753	0.9982	0.0614	0.0328	0.0753	-0.0018	0.0094	0.0011
			50		0.3765	1.0000	0.0407	0.0000	0.0765	0.0000	0.0075	0.0000
			90		0.3746	1.0000	0.0291	0.0000	0.0746	0.0000	0.0064	0.0000
			180		0.3740	1.0000	0.0212	0.0000	0.0740	0.0000	0.0059	0.0000
			320		0.3743	1.0000	0.0162	0.0000	0.0743	0.0000	0.0058	0.0000
			700		0.3732	1.0000	0.0109	0.0000	0.0732	0.0000	0.0055	0.0000
6.00	4.80	0.70	20		5.0413	0.5862	2.1618	0.3440	0.2413	-0.1138	4.7271	0.1312
			50		4.6646	0.6638	1.2251	0.2311	-0.1354	-0.0362	1.5177	0.0547
			90		4.5820	0.7004	0.8981	0.1695	-0.2180	0.0004	0.8534	0.0287
			180		4.4891	0.7409	0.5573	0.1180	-0.3109	0.0409	0.4069	0.0156
			320		4.4783	0.7553	0.4267	0.0912	-0.3217	0.0553	0.2854	0.0114
			700		4.4848	0.7586	0.3105	0.0798	-0.3152	0.0586	0.1957	0.0098
3.00	1.00	-1.00	20		0.8994	-0.8521	0.1531	0.3027	-0.1006	0.1479	0.0335	0.1134
			50		0.9591	-0.9457	0.1047	0.1064	-0.0409	0.0543	0.0126	0.0142
			90		0.9751	-0.9626	0.0748	0.0710	-0.0249	0.0374	0.0062	0.0064
			180		0.9985	-0.9753	0.0559	0.0454	-0.0015	0.0247	0.0031	0.0027
			320		1.0018	-0.9838	0.0397	0.0307	0.0018	0.0162	0.0016	0.0012
			700		1.0104	-0.9907	0.0271	0.0176	0.0104	0.0093	0.0008	0.0004

§6 Applications in reliability theory

6.1 Basic data analysis

The first data that we consider are on the number of operating hours between successive failure times of air conditioning systems in Boeing airplanes (see Proschan [14]). The operating hours data and its descriptive statistics were given in Okorie and Nadarajah [12]. Here, we have reproduced only the descriptive statistics, which are listed in Table 2 for reference sake and from there, we can see that such data is right skewed and heavy tailed. Based on a KS p -value of 0.231 Okorie and Nadarajah [12] showed that the omega distribution, which is an extension of the epsilon distribution, fits the operating hours data better than many three-parameter distributions, namely the exponentiated Weibull, generalized power Weibull, generalized Weibull, modified Weibull, modified Weibull extension, odd Weibull, and reduced modified Weibull distributions in Dombi et al [2]. Here, we compare the TE distribution with some other well-known one-parameter and two-parameter distributions to justify the importance of the extra parameter θ . These competitors are defined with their CDFs as follows:

- (1) Epsilon distribution (see Dombi et al [1]):

$$F(x) = 1 - \left(\frac{d+x}{d-x} \right)^{-\frac{\lambda d}{2}}, \quad 0 < x < d, d, \lambda > 0,$$

- (2) Gamma distribution

$$F(x) = \frac{\gamma\left(\alpha, \frac{x}{\beta}\right)}{\Gamma(\alpha)}, \quad x > 0, \alpha, \beta > 0,$$

where $\gamma(\cdot, \cdot)$ denotes the standard incomplete gamma function,

- (3) Generalized exponential distribution (GED) (see Gupta and Kundu [15]):

$$F(x) = [1 - e^{-\lambda x}]^\alpha, \quad x > 0, \alpha, \lambda > 0,$$

- (4) Rayleigh geometric distribution (RGD) (see Okorie et al [16]):

$$F(x) = \frac{1 - e^{-\frac{x^2}{2\sigma^2}}}{1 - pe^{-\frac{x^2}{2\sigma^2}}}, \quad x > 0, \sigma > 0, p \in (0, 1),$$

- (5) Lindley distribution (see Lindley [17]):

$$F(x) = 1 - \left(1 + \frac{\lambda x}{\lambda + 1} \right) e^{-\lambda x}, \quad x > 0, \lambda > 0$$

- (6) Exponential distribution:

$$F(x) = 1 - e^{-\lambda x}, \quad x \geq 0, \lambda > 0, \text{ and}$$

- (7) Rayleigh distribution:

$$F(x) = 1 - e^{-\frac{x^2}{2\sigma^2}}, \quad x \geq 0, \sigma > 0.$$

The above CDFs can be completed as standard for the other values of X . We compare the fitting performance of all the fitted distributions by their Kolmogorov-Smirnov (KS) p -values (the KS is one of the most suitable goodness-of-fit measures for measuring the fit of distributions

with positive support) and AIC with a correction (AICc) due to Hurvich and Tsai [18] goodness-of-fit statistics. The distribution with the smallest goodness-of-fit statistics and largest KS p -value is the best. The goodness-of-fit measures considered can be expressed as follows:

(1) KS

$$KS = \max_{i=1,2,\dots,n} \left\{ \frac{i}{n} - \hat{F}(x_{(i)}), \hat{F}(x_{(i)}) - \frac{i-1}{n} \right\}.$$

(2) AICc

$$AICc = AIC + \frac{2k(k+1)}{n-k-1},$$

where, k , n , and $\hat{F}(x)$ correspond to the number of parameters, sample size of the data, and the estimated CDF of the distribution under the ordered data, respectively.

Table 2. Descriptive measures for the operating hours data.

Data	Min.	1st Qu.	Median	Mean	3rd Qu.	Max.	Std. Dev.	Skewness	Kurtosis
Operating hours	1	22	57	93.14	118	603	106.76	2.11	7.92

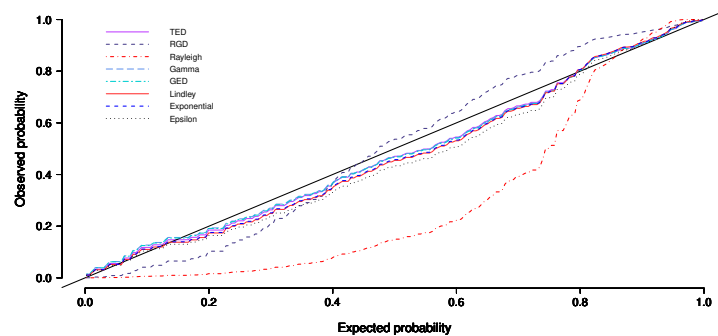


Figure 4. P-P plots of the fitted distributions for the number of operating hours data.

From Table 3, we can see that the TE distribution gives the smallest AICc and largest KS p -value. There is just one exceptional case where the exponential distribution though with a relatively larger log-likelihood value in comparison to the TE distribution, gave the smallest AICc value in Table 3. This is because the TE distribution has three parameters, whereas the exponential distribution has just one parameter and consequently the AICc penalizes the model based to the number of parameters in the model. In general, the TE distribution appears to fit the number of operating hours data better than the rest of the distributions and this evidence is conjectured in the probability-probability (P-P) plot in Figure 4.

Table 3. Parameter estimates, SEs (in bracket), log-likelihood, AICc and KS p -value for the fitted distributions for the number of operating hours data.

Distribution	MLEs ^a	$-\widehat{\mathcal{L}}(\boldsymbol{\Omega})$	Measures	
	Parameters [SEs]		AICc	KS p -value
TE distribution	$\hat{d} : 0.9885246$; $\hat{\lambda} : 4.2585507$; $\hat{\theta} : 0.6419139$ [−] [0.575417] [0.1667636]	-188.9262	-371.7376	0.4135
Epsilon	$\hat{d} : 0.9885246$; $\hat{\lambda} : 6.070133$ [−] [0.4159603]	-183.2891	-360.4634	0.0344
RGD	$\hat{p} : 0.9912079$; $\hat{\sigma} : 0.6563732$ [0.004901709] [0.18372903]	-164.7371	-325.4170	0.0034
Gamma	$\hat{\alpha} : 0.9217569$; $\hat{\beta} : 6.0367943$ [0.07807271] [0.6686258]	-187.7760	-371.4948	0.3768
GED	$\hat{\alpha} : 0.9269039$; $\hat{\lambda} : 6.2267767$ [0.08301857] [0.5767788]	-187.6644	-371.2717	0.3383
Lindley	$\hat{\lambda} : 7.334973$ [0.4539852]	-186.8464	-371.6738	0.1750
Exponential	$\hat{\lambda} : 6.549221$ [0.4487452]	-187.3007	-372.5825	0.2113
Rayleigh	$\hat{\sigma} : 0.1640222$ [0.005618628]	-22.09514	-42.17132	2.2×10^{-16}

^aFor computational stability with fitting of the distributions, the data was scaled by 610 as in Okorie and Nadarajah [12].

6.2 TE Distribution Regression Model

Classical regression modeling requires the response variable to follow the normal distribution. However, it is well known that this is not usually the case in practice as the response variable may follow a different distribution. In such situation, we can model the relationship between the response and predictor(s)/covariate(s) variable(s) by the generalized linear model (GLM). For instance, in simple case involving a single predictor variable, the GLM can relate the response variable, say X , with the predictor variable, say Y , in various ways.

Here, we describe the procedure for linking covariates with the TE distribution for regression analysis. There are many ways of doing this in the literature as we have noted; however, popular methods for incorporating the covariates into the model include: scale parameter, distribution mean or median, HRF and CDF. However, in the case of the TE distribution, it does not have any simple symbolic expression for the mean, so we can proceed by linking the covariates through any other convenient method. Here, the covariates are incorporated into the model through the proportional reversed hazard (PRH) model following Mulayath and Sankaran [20]. The PRH model can be expressed as

$$F(x|\mathbf{Y}) = [F_0(x|\mathbf{Y})]^{h(\boldsymbol{\beta};\mathbf{Y})},$$

where $F(x|\mathbf{Y})$ is the CDF of the response variable X given the covariates $\mathbf{Y}$, $F_0(x|\mathbf{Y})$ is the baseline CDF; in this case, the baseline CDF of the TE distribution, and $h(\boldsymbol{\beta};\mathbf{Y})$ denotes a non-negative function of the $n \times p$ covariate matrix $\mathbf{Y}$ and the $p \times 1$ vector of unknown regression

Table 4. Parameter estimates, SEs (in bracket), log-likelihood, AICc and KS p -value for the fitted distributions for the gradient of beach slope (X) data.

Distribution	MLEs	$-\widehat{\mathcal{L}}(\boldsymbol{\Omega})$	Measures	
	Parameters [SEs]		AICc	KS p -value
TE distribution	$\hat{d} : 11.4; \hat{\lambda} : 0.1442778; \hat{\theta} : 0.6227235$ [-] [0.05700563] [0.3868613]	18.78846	48.37691	0.12430
Normal	$\hat{\mu} : 3.186189; \hat{\sigma} : 3.572149$ [1.190717] [0.8419066]	24.22936	54.45872	0.00108

parameters $\boldsymbol{\beta}$. Here, we take $h(\boldsymbol{\beta}; \mathbf{Y}) = e^{\boldsymbol{\beta}^T \mathbf{Y}}$ hence, the CDF and PDF of our new regression model are given by

$$F(x|\mathbf{Y}) = \left\{ \left[1 - \left(\frac{d+x}{d-x} \right)^{-\frac{\lambda d}{2}} \right] \left[1 + \theta \left(\frac{d+x}{d-x} \right)^{-\frac{\lambda d}{2}} \right] \right\}^{e^{\boldsymbol{\beta}^T \mathbf{Y}}} \quad (8)$$

and

$$\begin{aligned} f(x|\mathbf{Y}) &= \frac{\lambda d^2}{d^2 - x^2} e^{\boldsymbol{\beta}^T \mathbf{Y}} \left\{ \left[1 - \left(\frac{d+x}{d-x} \right)^{-\frac{\lambda d}{2}} \right] \left[1 + \theta \left(\frac{d+x}{d-x} \right)^{-\frac{\lambda d}{2}} \right] \right\}^{e^{\boldsymbol{\beta}^T \mathbf{Y}} - 1} \\ &\times \left[1 - \theta + 2\theta \left(\frac{d+x}{d-x} \right)^{-\frac{\lambda d}{2}} \right] \left(\frac{d+x}{d-x} \right)^{-\frac{\lambda d}{2}}, \end{aligned} \quad (9)$$

respectively.

Given a pair of data (y, x) we can obtain the estimates for d , $\boldsymbol{\beta}$'s, λ and θ through Eq. (9) and using a similar estimation procedure in Section 5.

Next, we demonstrate the usefulness of the new regression model in construction engineering. We apply it to the pair of data on naturally occurring ocean beaches involving median diameter (mm) of granules of sand ($\mathbf{Y}$) and the gradient of beach slope in degrees (X), the data is available on https://college.cengage.com/mathematics/brase/understandable_statistics/7e/students/datasets/slr/frames/frame.html and are listed as follows:

$$\begin{aligned} Y &: 0.170, 0.190, 0.220, 0.235, 0.235, 0.300, 0.350, 0.420, 0.850 \\ X &: 0.630, 0.700, 0.820, 0.880, 1.150, 1.500, 4.400, 7.300, 11.30 \end{aligned}$$

First, a quick look at Table 4 shows that the normal distribution with a KS p -value < 0.05 is not a suitable candidate for the response variable, whereas the KS p -value > 0.05 of the TE distribution indicates that the distribution provides a good fit to the gradient of beach slope data. Both the negative log-likelihood and AICc values of the TE distribution are smaller than that of the normal distribution thus, it fits the data better than the normal distribution.

Since the TE distribution appears to provide a decent fit for the gradient of beach slope

Table 5. Parameter estimates, SEs (in bracket), log-likelihood, AICc and KS p -value for the fitted distributions for the gradient of beach slope (X) data.

Model I		Model II	
Parameters [SEs]	$-\widehat{\mathcal{L}}(\boldsymbol{\Omega})$	Parameters [SEs]	$-\widehat{\mathcal{L}}(\boldsymbol{\Omega})$
$\hat{d} : 11.4$ [—]	23.37715	$\hat{d} : 11.4$ [—]	18.23661
$\hat{\beta}_0 : -0.016885$ [1.0521478]		$\hat{\beta}_1 : 1.9815132$ [1.8567401]	
$\hat{\beta}_1 : -0.095625$ [4.1153665]		$\hat{\lambda} : 0.2012359$ [0.08622244]	
$\hat{\lambda} : 0.151875$ [0.06093317]		$\hat{\theta} : 0.7159927$ [0.2805690]	
$\hat{\theta} : -0.748125$ [0.4969280]			
AICc:76.75431		AICc:54.47322	
LR test:			
Δ : 0.4966559			
p -value: 0.4809732			

data, it suggests that we can describe the relationship between the gradient of beach slope in degrees and the median diameter (mm) of granules of sand via Eq. (9) for $p = 1$ and either with intercept β_0 (Model I) or without intercept (Model II). The MLEs for the parameters of the regression model for Model I and Model II are given in Table 5.

We compare Model I and Model II with the likelihood ratio test for testing, $H_0 : \beta_0 = 0$ against $H_1 : \beta_0 \neq 0$. Now, let $\hat{\mathcal{L}}_I$ and $\hat{\mathcal{L}}_{II}$ denote the estimated log-likelihoods under H_0 and H_1 , respectively. The test statistic for testing H_0 is $\Delta = -2[\hat{\mathcal{L}}_{II}/\hat{\mathcal{L}}_I]$ and it follows the chi-square distribution with 1 degree of freedom. For these models, the likelihood ratio test gave a p -value of > 0.05 hence, suggesting no statistically significant evidence against H_0 at 0.05 level of significance. In fact, from the scatter plot in Figure 5 (left) one can see that the gradient of beach slope in degrees tends to increase with the median diameter (mm) of granules of sand and this is in agreement with the positive value of $\hat{\beta}_1$ in Model II. To verify whether the fitted regression model is appropriate, we used the randomized quantile residual by Dunn and Smyth [21] which is defined as $\hat{r}_i = \Phi^{-1}(\hat{F}[x_i|y_i])$ for $i = 1, 2, \dots, n$, where $\Phi(x)$ denotes the CDF of the standard normal distribution and $\hat{F}(x_i|y_i)$ is the estimated CDF in Eq. (8). For valid models, the randomized quantile residuals are normal $N(0, 1)$ distributed. From Figure 5 (right), we can observe that nearly all the points fall between the 0.95 confidence band. The Jarque-Bera test for normality by Jarque and Bera [22] for testing H_0 indicates that the regression residuals are normally distributed against H_1 that the regression residuals are not normally distributed gave a p -value of 0.4614 (> 0.05) in favor of the null hypothesis. Hence, we can conclude that the fitted regression model is good for the data. These results show that median diameter (mm) of granules of sand is a significant predictor of gradient of beach slope

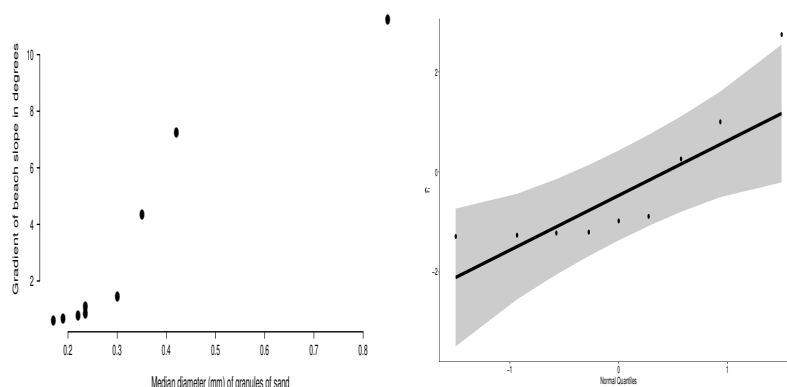


Figure 5. Scatter plot of median diameter (mm) of granules of sand versus gradient of beach slope in degrees (left) and the Q-Q plot of the quantile residuals (right) for the fitted regression model.

in degrees; furthermore, the gradient of beach slope in degrees increases with the increase in the median diameter (mm) of granules of sand.

§7 Conclusions

In this article, we introduced and studied the transmuted epsilon (TE) distribution. The main mathematical properties of the TE distribution have been derived, including linear representations, moments, quantile function and reliability functions. Statistical inference on the parameters of the TE distribution has been dealt with via the method of maximum likelihood estimation. Results from one of the applications in reliability engineering setting showed that the TE model can provide better results than the existing epsilon, gamma, generalized exponential, Rayleigh geometric, Lindley, exponential and Rayleigh models. Furthermore, we discussed the regression modeling using the TE distribution and found that the regression model based on the TE distribution offers a good fit to the applied data.

Acknowledgment

The authors would like to extend their appreciation to Dr. Tamás Jónás of the Institute of Business Economics, Eötvös Loránd University, Budapest, Hungary for his valuable discussion and comments which greatly improved the content of this article and we would also like to thank the three anonymous reviewers for their careful reading and comments which further improved the content of this article.

Declarations

Conflict of interest The authors declare no conflict of interest.

References

- [1] J Dombi, T Jónás, Z E Tóth. *The epsilon probability distribution and its application in reliability theory*, Acta Polytechnica Hungarica, 2018, 15(1): 197-216.
- [2] J Dombi, T Jónás, Z E Tóth, et al. *The omega probability distribution and its applications in reliability theory*, Quality and Reliability Engineering International, 2019, 35(2): 600-626.
- [3] J Dombi, T Jónás. *On an alternative to four notable distribution functions with applications in engineering and the business sciences*, Acta Polytech Hung, 2020, 17: 231-252.
- [4] J Dombi, T Jónás. *Advances in the theory of probabilistic and fuzzy data scientific methods with applications*, Springer, 2020.
- [5] M T Vasileva. *Some Notes on the Omega Distribution and the Pliant Probability Distribution Family*, Algorithms, 2020, 13(12): 324.
- [6] T Jónás, C Chesneau, J Dombi, et al. *The inverse epsilon distribution as an alternative to inverse exponential distribution with a survival times data example*, Acta Cybernetica, 2021.
- [7] W T Shaw, I R C Buckley. *The alchemy of probability distributions: beyond Gram-Charlier expansions, and a skew-kurtotic-normal distribution from a rank transmutation map*, 2009, arXiv:0901.0434.
- [8] M Bourguignon, I Ghosh, G M Cordeiro. *General results for the transmuted family of distributions and new models*, Journal of Probability and Statistics, 2016, 1: 2016.
- [9] M M Rahman, B Al-Zahrani, M Q Shahbaz. *A general transmuted family of distributions*, Pakistan Journal of Statistics and Operation Research, 2018, 1: 451-469.
- [10] E A Owoloko, P E Oguntunde, A O Adejumo. *Performance rating of the transmuted exponential distribution: an analytical approach*, Springerplus, 2015, 4: 818.
- [11] W Gilchrist. *Statistical modelling with quantile functions*, Chapman and Hall/CRC, 2000.
- [12] I E Okorie, S Nadarajah. *On the omega probability distribution*, Quality and Reliability Engineering International, 2019, 35(6): 2045-2050.
- [13] The R Core Team. *R: A Language and Environment for Statistical Computing*, R Foundation for Statistical Computing, Austria, 2021.
- [14] F Proschan. *Theoretical explanation of observed decreasing failure rate*, Technometrics, 1963, 5(3): 375-383.
- [15] R D Gupta, D Kundu. *Generalized exponential distribution: different method of estimations*, Journal of Statistical Computation and Simulation, 2001, 69(4): 315-337.

- [16] I E Okorie, A C Akpanta, J Ohakwe, et al. *On the Rayleigh-geometric distribution with applications*, Heliyon, 2019, 5(8): e02200.
- [17] D V Lindley. *Fiducial distributions and Bayes' theorem*, Journal of the Royal Statistical Society: Series B (Methodological), 1958, 1: 102-107.
- [18] C M Hurvich, C L Tsai. *Regression and time series model selection in small samples*, Biometrika, 1989, 76(2): 297-307.
- [19] J Lienig, H Bruemmer. *Fundamentals of electronic systems design*, Cham: Springer International Publishing, 2017.
- [20] A M Variyath, P G Sankaran. *Parametric regression models using reversed hazard rates*, Journal of Probability and Statistics, 2014, 2014: 645719.
- [21] P K Dunn, G K Smyth. *Randomized quantile residuals*, Journal of Computational and Graphical Statistics, 1996, 5(3): 236-244.
- [22] C M Jarque, A K Bera. *Efficient tests for normality, homoscedasticity and serial independence of regression residuals*, Economics letters, 1980, 6(3): 255-259.

¹Laboratoire de Mathématiques Nicolas Oresme, University of Caen Normandie, Caen, France.
Email: christophe.chesneau@gmail.com

²Department of Mathematics, College of Science, Qassim University, Buraydah, Saudi Arabia.
Email: h.bakouch@qu.edu.sa

³Department of Mathematics, Khalifa University, P.O. Box 127788, Abu Dhabi, United Arab Emirates.
Email: idika.okorie@ku.ac.ae

⁴Department of Mathematics, College of Science, Qassim University, Buraydah, Saudi Arabia.
Email: bsmhiemied@qu.edu.sa