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Optimal investment-consumption problem with discontinuous prices and random horizon

CHEN Tian[1] HUANG Zong-yuan[2,*] WU Zhen[2]

Abstract. This paper investigates an international optimal investmentCconsumption problem under a random time horizon. The investor may allocate wealth between a domestic bond and an international real project with production output, whose price may exhibit discontinuities. The model incorporates the effects of taxation and exchange rate dynamics, where the exchange rate follows a stochastic differential equation with jump-diffusion. The investor's objective is to maximize the utility of consumption and terminal wealth over an uncertain investment horizon. It is worth noting that, under our framework, the exit time is not assumed to be a stopping time. In particular, for the case of constant relative risk aversion (CRRA), we derive the optimal investment and consumption strategies by applying the separation method to solve the associated HamiltonCJacobiCBellman (HJB) equation. Moreover, several numerical examples are provided to illustrate the practical applicability of the proposed results.

§1 Introduction

Modern portfolio selection theory focuses on the optimal allocation of wealth among a variety of securities to achieve a desirable trade-off between return and risk. Under globalization, the domestic costs of human resources, raw materials, management, and other production factors have risen in many countries. As a result, an increasing number of enterprises are choosing to invest abroad. This paper investigates the investment consumption problem in international markets with discontinuous price processes. Moreover, we assume that the exit time is uncertain. It is worth noting that, under our assumption, the exit time is not modeled as a stopping time. We derive the optimal strategy under constant relative risk aversion (CRRA) preferences using the dynamic programming principle. The associated HamiltonCJacobiCBellman (HJB) equation is solved using the separation method. This yields the optimal solution. Finally,

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*Corresponding author.

several numerical examples are presented to illustrate the practical applicability of the proposed approach.

Portfolio selection aims to determine the optimal allocation of wealth among a basket of securities. Several useful theories and models have been developed, as discussed in Choi [4]. Karatzas et al. [13] and Karatzas [14] investigated optimal investment and consumption problems. Jeanblanc-Picqué and Pontier [12] considered optimal portfolio selection in a market model with discontinuous price processes. Duffie [6] initially introduced stochastic models for security markets, which laid the foundation for subsequent studies on optimal portfolio and consumption choices by many researchers. Bellalah and Wu [1] proposed a model for corporate international investment under incomplete information and taxation. Huang and Wu [10] studied an international corporate investment optimization problem. Lim [17] considered a meanCvariance hedging problem in an incomplete market, where the underlying asset prices follow jump-diffusion processes. Li et al. [16] addressed a linear-quadratic mean-field game problem for a class of stochastic large-population systems with jump-diffusion dynamics.

For most investors, maximizing profit is the primary objective; however, risk cannot be ignored. Their goal is to maximize overall utility, which includes both consumption and terminal wealth. Much of the existing literature assumes that investors know the exit time of their investments in advance. However, in reality, the timing of market exit is typically uncertain. Various unpredictable factorssuch as market behavior, social events, and exogenous shocks to wealthcan significantly influence the decision to exit. Therefore, it is essential to develop portfolio selection theory under a random time horizon. Yaari [23] was among the first to study an optimal consumption problem for individuals facing an uncertain lifetime. Subsequently, Hakansson [8, 9] and Richard [20] assumed that the random exit time is independent of other market uncertainties. In contrast, Karatzas and Wang [15] investigated an optimal investment problem in a complete market where the random exit time fully depends on the underlying asset prices. Bouchard and Pham [3], and Blanchet-Scalliet et al. [2] extended these two extreme cases to a general framework where the time horizon is a random variable. Assuming all market parameters and the conditional distribution of the exit time are deterministic, [2] derives an explicit optimal portfolio under CRRA utility, which coincides with that under a fixed horizon. Yu [25] and Lv et al. [18] studied meanCvariance portfolio selection with random horizons in complete and incomplete markets, respectively. Huang et al. [11] examined optimal investment problems under inflation and uncertain time horizons. Wang and Wu [21] considered meanCvariance portfolio selection with discontinuous prices and random horizons in an incomplete market. With the advancement of economic globalization, international investment has also grown substantially. Research on international investment is becoming increasingly important. In this paper, we investigate an international optimal investment-consumption problem under a random time horizon. Furthermore, we reformulate the investmentCconsumption problem as an equivalent optimal control problem.

The remainder of the paper is organized as follows. In Section 2, we formulate the expected utility maximization problem and present the basic model assumptions. In Section 3, we first establish the dynamic programming principle corresponding to the control problem. Then, under the assumption of a CRRA utility function, we solve the associated HJB equation using a separation method. In Section 4, we present several numerical examples to illustrate the main results derived above, and analyze the economic impact of the exchange rate and the random time horizon on the optimal portfolio and the maximum utility. Section 5 concludes the paper.

§2 Problem Formulation

Let $(\Omega, \mathcal{A}, \mathbf{P})$ be a complete probability space and $T > 0$ is a fixed time horizon. Suppose that $\{B(t), t \geq 0\}$ and $\{\bar{B}(t), t \geq 0\}$ are two 1-dimensional standard Brownian motions defined on this space. The two Brownian motions represent different sources of uncertainties in a market, which usually depend on each other with the correlation coefficient ρ , where $-1 < \rho < 1$. The filtration $\mathbb{F} = \{\mathcal{F}_t\}_{0\leq t\leq T}$ with $\mathcal{F}_T \subseteq \mathcal{A}$ is generated by $B(\cdot)$ and $\bar{B}(\cdot)$. The processes $N(\cdot)$ and $\bar{N}(\cdot)$ are independent Poisson processes defined on this space with $\mathbb{F}$-predictable non-negative intensity $\lambda(\cdot)$ and $\bar{\lambda}(\cdot)$ respectively and the filtration $\mathbb{D} = \{\mathcal{D}_t\}_{0\leq t\leq T}$ with $\mathcal{D}_T \subseteq \mathcal{A}$ is generated by $N(\cdot)$ and $\bar{N}(\cdot)$. The introduction of Poisson process indicates the impact of some emergencies on prices, for example, changes in the international situation and national policies. The filtration $\mathbb{G} = \{\mathcal{G}_t\}_{0\leq t\leq T}$ is defined as $\mathcal{G}_t = \mathcal{F}_t \vee \mathcal{D}_t \vee \mathcal{A}_0$, where $\mathcal{A}_0$ is generated by all $\mathbf{P}$-null sets of $\mathcal{A}$ so that $t \mapsto \mathcal{G}_t$ is continuous. Now we introduce some spaces of stochastic processes and random variables,

$$L^2_{\mathbb{G}}(0,T;\mathbb{R}^m) = \left\{ \varphi_t \text{ is a } \mathbb{R}^m\text{-valued, } \mathbb{G}\text{-predictable process s.t. } \mathbb{E}\int_0^T |\varphi_t|^2 \mathrm{d}t < \infty \right\};$$

$$L^{2,loc}_{\mathbb{G}}(0,T;\mathbb{R}^m) = \left\{ \varphi_t \text{ is a } \mathbb{R}^m\text{-valued, } \mathbb{G}\text{-predictable process s.t. } \int_0^T |\varphi_t|^2 \mathrm{d}t < \infty, \mathbf{P}\text{-a.s.} \right\};$$

$$L^\infty_{\mathbb{G}}(0,T;\mathbb{R}^m) = \left\{ \varphi_t \text{ is a } \mathbb{R}^m\text{-valued, } \mathbb{G}\text{-predictable uniformly bounded process } \right\};$$

$$S^2_{\mathbb{G}}(0,T;\mathbb{R}^m) = \left\{ \varphi_t \text{ is a } \mathbb{R}^m\text{-valued, } \mathbb{G}\text{-adapted process s.t. } \mathbb{E}\left[\sup_{t\in[0,T]} |\varphi_t|^2\right] < \infty \right\};$$

$$L^\infty_{\mathcal{G}_T}(\Omega,;\mathbb{R}^m) = \left\{ \varphi_t \text{ is a } \mathbb{R}^m\text{-valued, } \mathcal{G}_T\text{-measurable, uniformly bounded process } \right\}.$$

Suppose an investor can put her or his wealth into this market and benefit from it. Her or his initial capital is $x_0 > 0$ and there are two investment choices for her or him. One is a riskless bond, whose price process $\{P_0(t), 0 \leq t \leq T\}$ satisfies the ordinary differential equation (ODE):

$$\begin{cases} \mathrm{d}P_0(t) = r(t)P_0(t)\mathrm{d}t, \qquad 0 \leq t \leq T, \\ P_0(0) = p_0, \end{cases} \tag{1}$$

where $r(\cdot) > 0$ is the risk-free interest rate and $p_0 > 0$ is the current price of the bond. The other choice is a foreign real production project, which probably brings more profit but with risk. So the cost process $\{P(t), 0 \leq t \leq T\}$ and the selling price process $\{S(t), 0 \leq t \leq T\}$ per unit of the product can be described by the following two stochastic differential equations

(SDEs), respectively:

$$\begin{cases} \mathrm{d}P(t) = P(t)\mu_p(t)\mathrm{d}t + P(t)\sigma(t)\mathrm{d}B(t) + P(t-)\theta(t)\mathrm{d}N(t), \qquad 0 \le t \le T \\ P(0) = p, \end{cases}$$

$$\begin{cases} \mathrm{d}S(t) = S(t)\mu_s(t)\mathrm{d}t + S(t)\sigma(t)\mathrm{d}B(t) + S(t-)\theta(t)\mathrm{d}N(t), \qquad 0 \le t \le T \\ S(0) = s, \end{cases}$$

where $\mu_s(\cdot) > 0$ and $\mu_p(\cdot) > 0$ are the expected return and cost rate. $p > 0$ and $s > 0$ are the current cost and selling price, respectively. And $\sigma(\cdot) > 0$ is the common volatility rate of the cost and selling price processes of the product. $\theta(\cdot)$ represents the relative size of the jump to $P(\cdot)$ or $S(\cdot)$ given an arrival of $N(\cdot)$.

In economics, the production quantity $Q(\cdot)$ is usually supposed as $Q(t) = [S(t)]^{\beta}$, where β is generally a negative constant. So the cash flow from this project in foreign currency numeraire is

$$\widetilde{R}(t) = (1-\alpha)\left[S(t) - P(t)\right]Q(t),$$

where α is the tax rate in the foreign country market. Inspired by [1], we suppose $e(\cdot)$ to be the exchange rate from foreign currency to domestic currency, which satisfies the following stochastic differential equation (SDE):

$$\mathrm{d}e(t) = \mu_e(t)e(t)\mathrm{d}t + \sigma_e(t)e(t)\mathrm{d}\bar{B}(t) + \theta_e(t)e(t-)\mathrm{d}\bar{N}(t),$$

where $\mu_e(\cdot) > 0$ and $\sigma_e(\cdot) > 0$, and $\theta_e(\cdot)$ represents the relative size of the jump to $e(\cdot)$ given an arrival of $\bar{N}(\cdot)$.

Assumption 2.1. (i) *All market parameters $r(\cdot)$, $\mu_p(\cdot)$, $\mu_s(\cdot)$, $\mu_e(\cdot)$, $\sigma(\cdot)$, $\sigma_e(\cdot)$, $\theta(\cdot)$, $\theta_e(\cdot)$, $\lambda(\cdot)$ and $\bar{\lambda}(\cdot)$ are deterministic and bounded functions;*

(ii) *There exists a constant $\delta > 0$ such that $\sigma^2(t) \ge \delta$ and $\sigma_e^2(t) \ge \delta$ for any $t \in [0,T]$, $\mathbf{P}$-a.s.;*

(iii) *There exists a constant $\delta' > 0$ such that $\lambda^2(t) \ge \delta'$ and $\bar{\lambda}^2(t) \ge \delta'$ for any $t \in [0,T]$, $\mathbf{P}$-a.s..*

Assumption 2.2. *The relative size of the jump $\theta(\cdot)$ and $\theta_e(\cdot)$ satisfy the following conditions:*

$$\theta(t) > -1 \qquad \textit{and} \qquad \theta_e(t) > -1, \qquad \mathbf{P}\textit{-a.s.}.$$

Remark 2.3. Under Assumption 2.2, it is easy to verify that the price processes $P(t)$ and $S(t)$ and the exchange rate $e(t)$ are positive for any $t \in [0,T]$, $\mathbf{P}$-a.s..

The uniform bound on $\lambda(t)$ and $\bar{\lambda}(t)$ implies that $\int_0^t \lambda(s)\mathrm{d}s < \infty$ and $\int_0^t \bar{\lambda}(s)\mathrm{d}s < \infty$ for all $t \in [0,T]$ from which they follow the compensated Poisson processes $M(t) \triangleq N(t) - \int_0^t \lambda(s)\mathrm{d}s$ and $\bar{M}(t) \triangleq \bar{N}(t) - \int_0^t \bar{\lambda}(s)\mathrm{d}s$ are $\mathbb{G}$-martingales. To obtain independent Brownian motions, we can set

$$W(t) = B(t), \qquad \bar{W}(t) = \frac{\bar{B}(t) - \rho B(t)}{\sqrt{1-\rho^2}}.$$

Then the equation satisfied by cost price process $P(\cdot)$ can be rewritten as

$$\begin{cases} \mathrm{d}P(t) = P(t)\alpha_p(t)\mathrm{d}t + P(t)\sigma(t)\mathrm{d}W(t) + P(t-)\theta(t)\mathrm{d}M(t), \qquad 0 \le t \le T, \\ P(0) = p, \end{cases}$$

where $\alpha_p(\cdot) = \mu_p(\cdot) - \lambda(\cdot)\theta(\cdot)$, and the equation satisfied by selling price process $S(\cdot)$ can be rewritten as

$$\begin{cases} \mathrm{d}S(t) = S(t)\alpha_s(t)\mathrm{d}t + S(t)\sigma(t)\mathrm{d}W(t) + S(t-)\theta(t)\mathrm{d}M(t), & 0 \le t \le T, \\ S(0) = s, \end{cases}$$

where $\alpha_s(\cdot) = \mu_s(\cdot) - \lambda(\cdot)\theta(\cdot)$. Due to the above expressions, it is easy to derive

$$P(t) = pe^{\int_0^t \sigma(s)\mathrm{d}W(s) + \int_0^t \ln(1+\theta(s)\mathrm{d}N(s))} e^{\int_0^t \left(\alpha_p(s) - \frac{1}{2}\sigma^2(s) - \theta(s)\lambda(s)\right)\mathrm{d}s},$$

$$S(t) = se^{\int_0^t \sigma(s)\mathrm{d}W(s) + \int_0^t \ln(1+\theta(s)\mathrm{d}N(s))} e^{\int_0^t \left(\alpha_s(s) - \frac{1}{2}\sigma^2(s) - \theta(s)\lambda(s)\right)\mathrm{d}s}.$$

The equation satisfied by exchange rate $e(\cdot)$ can be rewritten as

$$\mathrm{d}e(t) = \alpha_e(t)e(t)\mathrm{d}t + \rho\sigma_e(t)e(t)\mathrm{d}W(t) + \sqrt{1-\rho^2}\sigma_e(t)e(t)\mathrm{d}\bar{W}(t) + \theta_e(t)e(t-)\mathrm{d}\bar{M}(t),$$

where $\alpha_e(\cdot) = \mu_e(\cdot) - \bar{\lambda}(\cdot)\theta_e(\cdot)$. Then the cash flow can be written as

$$\begin{aligned} \widetilde{R}(t) &= (1-\alpha)S^{\beta}(t)\left[S(t) - P(t)\right] \\ &= (1-\alpha)\left[S(0)\right]^{\beta} e^{(1+\beta)\left(\int_0^t \sigma(s)\mathrm{d}W(s) + \int_0^t \ln(1+\theta(s))\mathrm{d}N(s)\right)} H(t), \end{aligned}$$

where $H(t) = e^{\beta\int_0^t\left[\alpha_s(s) - \frac{1}{2}\sigma^2(s)\right]\mathrm{d}s}\left[se^{\int_0^t\left[\alpha_s(s) - \frac{1}{2}\sigma^2(s)\right]\mathrm{d}s} - pe^{\int_0^t\left[\alpha_p(s) - \frac{1}{2}\sigma^2(s)\right]\mathrm{d}s}\right]$.

Applying Itô′s formula to $\widetilde{R}(t)$, we can get

$$\mathrm{d}\widetilde{R}(t) = \widetilde{R}(t)h(t)\mathrm{d}t + \widetilde{R}(t)(1+\beta)\sigma(t)\mathrm{d}W(t) + \widetilde{R}(t-)l(t)\mathrm{d}M(t),$$

where $l(t) = \left((1+\beta)\ln(1+\theta(t)) + \frac{1}{2}(1+\beta)^2\ln^2(1+\theta(t))\right)$ and $h(t) = \frac{1}{2}(1+\beta)^2\sigma^2(t) + \lambda(t)l(t) + H'(t)/H(t)$, $H(t) \ne 0$. That is, the cost price is not equal to the selling price.

We translate the cash flow to domestic currency by exchange rate and let $R(t) = e(t)\widetilde{R}(t)$, then

$$\begin{aligned} \mathrm{d}R(t) &= g(t)R(t)\mathrm{d}t + \left[(1+\beta)\sigma(t) + \rho\sigma_e(t)\right]R(t)\mathrm{d}W(t) \\ &\quad + \sqrt{1-\rho^2}\sigma_e(t)R(t)\mathrm{d}\bar{W}(t) + \theta_e(t)R(t-)\mathrm{d}\bar{M}(t) + l(t)R(t-)\mathrm{d}M(t), \end{aligned} \tag{2}$$

where $g(t) = h(t) + \alpha_e(t) + (1+\beta)\rho\sigma(t)\sigma_e(t)$.

Now let us denote the total actual wealth of the investor and the actual amount of money invested into the foreign production project at time t by $X(t)$ and $\pi(t)$, respectively. Learning from Jeanblanc-Picque et al. [12] and Karatzas et al. [13], we give the following definition.

Definition 2.4. A portfolio process $\pi(t), 0 \le t \le T$ is a $\mathbb{G}$-predictable process on $(\Omega, \mathbb{G}, \mathbf{P})$. $k(t)$ denotes the proportion of the investment of total wealth, satisfying

$$\mathbb{E}\left[\int_0^T \pi^2(t)\mathrm{d}t\right] = \mathbb{E}\left[\int_0^T k^2(t)X^2(t)\mathrm{d}t\right] < \infty, \qquad \mathbf{P}\text{-a.s..} \tag{3}$$

A consumption rate process $c(t), 0 \le t \le T$ is a nonnegative, $\mathbb{G}$-predictable process satisfying

$$\int_0^T |c(t)X(t)|^2\mathrm{d}t < \infty, \qquad \mathbf{P}\text{-a.s..} \tag{4}$$

We also give the definition of admissible set $\mathcal{U}$.

$$\mathcal{U} = \Big\{(k(\cdot), c(\cdot))|k(\cdot) \text{ and } c(\cdot) \text{ are admissible such that } X_t \geq 0, \text{ a.s. for } t \geq 0\Big\}.$$

By the above definition, the process $\pi(t) = k(t)X(t)$ denotes the actual amount of money into the foreign production project at time t. Thus $X(t)(1 - k(t))$ is the actual amount of money invested into the risk-free asset at time t. If $C(t)$ stands for the consumption of the investor, then the change of the wealth would include three parts, the change of risk-free part $\mathrm{d}X(t)(1 - k(t))$, the risky part $\mathrm{d}k(t)X(t)$, and the consumption $\mathrm{d}C(t) = c(t)X(t)\mathrm{d}t$. So we have

$$\mathrm{d}X(t) = k(t)X(t)\frac{\mathrm{d}R(t)}{R(t)} + X(t)\left(1 - k(t)\right)\frac{\mathrm{d}P_0(t)}{P_0(t)} - \mathrm{d}C(t).$$

It follows from (1) and (2) that the actual wealth process $\{X(t), t \geq 0\}$ satisfies

$$\begin{cases} \mathrm{d}X(t) = X(t)\left[r(t) - c(t) + k(t)\left(g(t) - r(t)\right)\right]\mathrm{d}t \\ \qquad\qquad + X(t)\left[(1+\beta)\sigma(t) + \rho\sigma_e(t)\right]k(t)\mathrm{d}W(t) + X(t-)l(t)k(t)\mathrm{d}M(t) \\ \qquad\qquad + X(t)\sqrt{1-\rho^2}\sigma_e(t)k(t)\mathrm{d}\bar{W}(t) + X(t-)\theta_e(t)k(t)\mathrm{d}\bar{M}(t), \\ X(0) = x_0. \end{cases} \tag{5}$$

Remark 2.5. Under Assumption 2.2, we can know that the process $l(\cdot)$ satisfies the condition $l(t) > -1$ by the properties of elementary function.

Definition 2.6. The investment-consumption proportion $(k(\cdot), c(\cdot))$ satisfying (3) and (4) is admissible for the initial wealth $x_0 > 0$ if the corresponding wealth process $X(t) > 0$ for any $t \in [0, T]$, $\mathbf{P}$-a.s..

To make sure that the total wealth $X(t)$ is positive for any $t \in [0, T]$, we give the following assumption similar with [7].

Assumption 2.7. *The investment-consumption proportion* $(k(t), c(t)), 0 \leq t \leq T$ *satisfy that* $(k(t), c(t)) \in [0, K] \times [0, \infty)$ *for all* $t > 0$ *and* $K > 1$ *is chosen so that*

$$l(t)k > -1 \quad \text{and} \quad \theta_e(t)k > -1, \tag{6}$$

for all $k \in [0, K]$.

Remark 2.8. It is easy to see that $(k(\cdot), c(\cdot)) \equiv (0, 0)$ is an admissible investment-consumption pair for any $x_0 > 0$. So the admissible set $\mathcal{U}$ is not empty.

In reality, the investor cannot know when the investment should be terminated certainly. In this paper, we assume that an agent's exit time of the investment is $\tau \wedge T$, where τ is a $\mathcal{A}$-measurable positive random variable.

Remark 2.9. τ is not a stopping time of the filtration $\mathbb{G}$ because $\mathcal{A}$ is strictly bigger than $\mathcal{G}_T$. It means that the exit time relies on not only the uncertainties of the cash flow of the foreign production project and exchange rate level, but also other uncertain factors.

Following Blanchet-Scalliet et al. [2] (see also [21], [25]), we introduce the condition distribution function of exit time by $F(t) = \mathbf{P}(\tau \leq t|\mathcal{G}_t)$. It is easy to verify that $F(t)$ is a

$\mathbb{G}$-submartingale and the function $t \to \mathbb{E}[F(t)]$ is right continuous. Then $F(t)$ has a right-continuous modification. From the Doob-Meyer decomposition theorem, we have $F(t) = \Lambda(t) + \Psi(t)$, where Ψ is a martingale and Λ is an increasing process. Then we give the following assumptions.

Assumption 2.10. *The process $\Lambda(\cdot)$ is absolutely continuous process with respect to Lebesgue's measure, with a $\mathbb{G}$-predictable nonnegative bounded density $f(\cdot)$, i.e., $\Lambda(t) = \int_0^t f(s)\mathrm{d}s, t \in [0,T]$.*

Under Assumption 2.10, we get at once the boundedness of Λ, and for the martingale Ψ. From the martingale representation theorem, there exist unique processes $\phi(\cdot), \varphi(\cdot) \in L^2_{\mathbb{G}}(0,T;\mathbb{R}^2)$ such that

$$\Psi(t) = \int_0^t \psi(s)'\mathrm{d}\tilde{W}(s) + \int_0^t \varphi(s)'\mathrm{d}\tilde{M}(s), \quad t \in [0,T], \ a.s.$$

where $\tilde{W}(t) = \big(W(t), \bar{W}(t)\big)'$, $\tilde{M}(t) = \big(M(t), \bar{M}(t)\big)'$ and $\tilde{N}(t) = (N(t), \bar{N}(t))'$.

Assumption 2.11. *There exists a constant C such that $\int_0^T |\psi(t)|^2\mathrm{d}t \le C$, $\mathbf{P}$-a.s. and $\int_0^T |\varphi(t)|^2 \mathrm{d}\tilde{N}(t) \le C$, $\mathbf{P}$-a.s..*

Assumption 2.12. *There exists a constant $\varepsilon > 0$ such that $F(T) \le 1 - \varepsilon$, $\mathbf{P}$-a.s..*

The investors' preference for consumption and investment can be measured by the utility function $U_1(\cdot)$ and $U_2(\cdot)$, which are strictly increasing, strictly concave and continuously differentiable function defined on $(0,\infty) \to \mathbb{R}$, satisfying the following two conditions: $\lim_{x\to\infty} U'(x) = 0$ and $\lim_{x\to 0} U'(x) = +\infty$. For a rational investor, the problem is to find an admissible investment-consumption strategy $(k(\cdot), c(\cdot))$ which maximizes the expected utility of consumption process and terminal wealth

$$J(k,c) = \mathbb{E}\left[\int_0^{\tau\wedge T} e^{-\gamma t} U_1(c(t)X(t))\mathrm{d}t + e^{-\gamma(\tau\wedge T)}U_2(X(\tau\wedge T))\right], \tag{7}$$

where $\gamma > 0$ is the discount rate.

Finally, we can present our problem as follows:

Problem (P). If the wealth process $X(\cdot)$ satisfies (5), under the above assumptions, we want to find an optimal investment-consumption strategy $(k^*(\cdot), c^*(\cdot))$ such that

$$J(k^*, c^*) = \sup_{(k,c)\in\mathcal{U}} J(k,c).$$

From the definition of $F(t) = \mathbf{P}(\tau \le t|\mathcal{F}_t)$, a result of Dellacherie [5], we have

$$\begin{aligned} J(k,c) &= \mathbb{E}\bigg[\mathbb{1}_{\{\tau\le T\}}\int_0^\tau e^{-\gamma t}U_1(c(t)X(t))\mathrm{d}t + \mathbb{1}_{\{\tau\le T\}}e^{-\gamma\tau}U_2(X(\tau)) \\ &\qquad + \mathbb{1}_{\{\tau> T\}}\int_0^T e^{-\gamma t}U_1(c(t)X(t))\mathrm{d}t + \mathbb{1}_{\{\tau> T\}}e^{-\gamma T}U_2(X(T))\bigg] \\ &= \mathbb{E}\bigg[\int_0^T\int_0^t e^{-\gamma s}U_1(c(s)X(s))\mathrm{d}s\mathrm{d}F(t) + \int_0^T e^{-\gamma t}U_2(X(t))\mathrm{d}F(t) \\ &\qquad + \int_T^\infty\int_0^T e^{-\gamma s}U_1(c(s)X(s))\mathrm{d}s\mathrm{d}F(t) + \int_T^\infty e^{-\gamma T}U_2(X(T))\mathrm{d}F(t)\bigg]. \end{aligned} \tag{8}$$

Thus the above formulate is one kind of standard stochastic optimal control problems. Now we only need to solve the Problem (P).

In order to further formulate our problem, let us state a consequence of the Burkholder-Davis-Gundy inequality without proof.

Proposition 2.13. (i) *Let* $\xi \in L^{2,loc}_{\mathbb{G}}(0,T;\mathbb{R}^2)$ *such that* $\mathbb{E}\left[\left(\int_0^T |\xi(t)|^2 \mathrm{d}t\right)^{\frac{1}{2}}\right] < \infty$. *Then* $\left\{\int_0^t \xi(s)' \mathrm{d}\tilde{W}(s), 0 \le t \le T\right\}$ *is a uniform integrable martingale. In particular, for each* $t \in [0,T]$, *the random variable* $\int_0^t \xi(s)'\mathrm{d}\tilde{W}(s)$ *is integrable, and* $\mathbb{E}\big[\int_0^t \xi(s)'\mathrm{d}\tilde{W}(s)\big] = 0$.

(ii) *Let* $\eta \in L^{2,loc}_{\mathbb{G}}(0,T;\mathbb{R}^2)$ *such that* $\mathbb{E}\left[\left(\int_0^T |\eta(t)|^2 \mathrm{d}\tilde{N}(t)\right)^{\frac{1}{2}}\right] < \infty$. *Then* $\big\{\int_0^t \eta(s)'\mathrm{d}\tilde{M}(s), 0 \le t \le T\big\}$ *is a uniform integrable martingale. In particular, for each* $t \in [0,T]$, *the random variable* $\int_0^t \eta(s)'\mathrm{d}\tilde{M}(s)$ *is integrable, and* $\mathbb{E}\big[\int_0^t \eta(s)'\mathrm{d}\tilde{M}(s)\big] = 0$.

Lemma 2.14. *Let Assumption 2.11 hold. Let* $x(\cdot) \in S^2_{\mathbb{G}}(0,T;\mathbb{R})$, *then we have*

(i) *The stochastic integral* $\int_0^T \int_0^t e^{-\gamma s}\frac{(c(s)x(s))^{1-R}}{1-R}\mathrm{d}s\psi(t)'\mathrm{d}\tilde{W}(t)$ *and* $\int_0^T e^{-\gamma t}\frac{x(t)^{1-R}}{1-R}\psi(t)'\mathrm{d}\tilde{W}(t)$ *is integrable. Moreover,*

$$\mathbb{E}\int_0^T \int_0^t e^{-\gamma s}\frac{(c(s)x(s))^{1-R}}{1-R}\mathrm{d}s\psi(t)'\mathrm{d}\tilde{W}(t) = 0, \qquad \mathbb{E}\int_0^T e^{-\gamma t}\frac{x(t)^{1-R}}{1-R}\psi(t)'\mathrm{d}\tilde{W}(t) = 0.$$

(ii) *The stochastic integral* $\int_0^T \int_0^t e^{-\gamma s}\frac{(c(s)x(s-))^{1-R}}{1-R}\mathrm{d}s\varphi(t)'\mathrm{d}\tilde{M}(t)$ *and* $\int_0^T e^{-\gamma t}\frac{x(t-)^{1-R}}{1-R}\varphi(t)'\mathrm{d}\tilde{M}(t)$ *are integrable. Moreover,*

$$\mathbb{E}\int_0^T \int_0^t e^{-\gamma s}\frac{(c(s)x(s-))^{1-R}}{1-R}\mathrm{d}s\varphi(t)'\mathrm{d}\tilde{M}(t) = 0, \qquad \mathbb{E}\int_0^T e^{-\gamma t}\frac{x(t-)^{1-R}}{1-R}\varphi(t)'\mathrm{d}\tilde{M}(t) = 0.$$

Proof. (i) By Proposition 2.13, we only need to verify the following bounded condition.

$$\begin{aligned}
\mathbb{E}\left[\left(\int_0^T \left|e^{-\gamma t}\frac{x(t)^{1-R}}{1-R}\psi(t)\right|^2 \mathrm{d}t\right)^{\frac{1}{2}}\right] &\le C\mathbb{E}\left[\left(\sup_{t\in[0,T]}|x(t)|^2 \cdot \int_0^T |\psi(t)|^2\mathrm{d}t\right)^{\frac{1}{2}}\right] \\
&\le C\left(\mathbb{E}\left[\sup_{t\in[0,T]}|x(t)|^2\right]\right)^{\frac{1}{2}}\left(\mathbb{E}\int_0^T |\psi(t)|^2\mathrm{d}t\right)^{\frac{1}{2}} < \infty.
\end{aligned}$$

$$\begin{aligned}
&\mathbb{E}\left[\left(\int_0^T \left|\int_0^t e^{-\gamma s}\frac{(c(s)x(s))^{1-R}}{1-R}\mathrm{d}s\psi(t)\right|^2 \mathrm{d}t\right)^{\frac{1}{2}}\right] \\
&\le \mathbb{E}\left[\left(\left|\int_0^T e^{-\gamma t}\frac{(c(t)x(t))^{1-R}}{1-R}\mathrm{d}t\right|^2 \cdot \int_0^T |\psi(t)|^2\mathrm{d}t\right)^{\frac{1}{2}}\right] \\
&\le C\left(\mathbb{E}\left[\sup_{t\in[0,T]}|x(t)|^2\right]\right)^{\frac{1}{2}} \cdot \left(\mathbb{E}\int_0^T |c(t)|^2\mathrm{d}t\right)^{\frac{1}{2}} \cdot \left(\mathbb{E}\int_0^T |\psi(t)|^2\mathrm{d}t\right)^{\frac{1}{2}} < \infty.
\end{aligned}$$

(ii) According to the same argument and Proposition 2.13, we have the results. This finishes the proof of the lemma. □

§3 Dynamic Programming Principle

In this section, we consider the case where the process $(F_t, t \geq 0)$ is a deterministic function, hence equals the cumulative function of τ, with a derivative f. Note that a necessary and sufficient condition for the distribution F_t to be a deterministic function of time is to have τ independent of $\mathcal{G}$. Then we will show that the dynamic programming principle still holds for the optimization Problem (P) and we can use it to give the optimal investment-consumption choice of the CRRA case whose utility function is in the form of

$$U_1(x) = U_2(x) = \frac{x^{1-R}}{1-R},$$

where $R \in (0,1)$ is a constant.

In the CRRA case, noticing Lemma 2.14, the total expected utility (8) becomes

$$J(k,c) = \mathbb{E}\bigg[\int_0^T e^{-\gamma t}(1-F(t))a\frac{(c(t)X(t))^{1-R}}{1-R}\mathrm{d}t \\ + \int_0^T e^{-\gamma t} f(t)\frac{X(t)^{1-R}}{1-R}\mathrm{d}t + e^{-\gamma T}(1-F(T))\frac{X(T)^{1-R}}{1-R}\bigg].$$

Let $(s,x) \in [0,T) \times \mathbb{R}^+$, and consider the following control system over $[s,T]$,

$$\begin{cases} \mathrm{d}X(t) = X(t)\left[r(t) - c(t) + k(t)\left(g(t) - r(t)\right)\right]\mathrm{d}t \\ \qquad + X(t)\left[(1+\beta)\sigma(t) + \rho\sigma_e(t)\right]k(t)\mathrm{d}W(t) + X(t-)l(t)k(t)\mathrm{d}M(t) \\ \qquad + X(t)\sqrt{1-\rho^2}\sigma_e(t)k(t)\mathrm{d}\bar{W}(t) + X(t-)\theta_e(t)k(t)\mathrm{d}\bar{M}(t), \\ X(s) = x. \end{cases}$$

Then we define the value function

$$\begin{cases} V(s,x) = \max_{(k,c)\in\mathcal{U}} \mathbb{E}\bigg\{\int_s^T e^{-\gamma(t-s)}\bigg[(1-F(t))\frac{(c(t)X(t))^{1-R}}{1-R} + f(t)\frac{X(t)^{1-R}}{1-R}\bigg]\mathrm{d}t \\ \qquad\qquad + e^{-\gamma(T-s)}(1-F(T))\frac{X(T)^{1-R}}{1-R}\bigg\}, \\ V(T,x) = (1-F(T))\frac{x^{1-R}}{1-R}. \end{cases}$$

Now the problem is to get the optimal value function $V(s,x)$ by choosing the investment-consumption proportion. From Proposition 3.5 of chapter 4 in Yong and Zhou [24], we can obtain the following conclusion.

Proposition 3.1. *For all t in $[0,T]$, the function $V(s,\cdot)$ is increasing and strictly concave.*

Proposition 3.2. *If $V(s,x) \in C^{1,2}([0,T] \times \mathbb{R}^+)$, then it is the solution of the following HJB equation:*

$$\begin{cases} -\gamma V(s,x)+\dfrac{\partial V(s,x)}{\partial s}+\max\limits_{(k,c)}\left\{\mathcal{T}V(s,x)+(1-F(s))\dfrac{(cx)^{1-R}}{1-R}+f(s)\dfrac{x^{1-R}}{1-R}\right\}=0, \\ V(T,x)=(1-F(T))\dfrac{x^{1-R}}{1-R}, \end{cases} \tag{9}$$

where $\mathcal{T}$ is the operator defined by

$$\begin{aligned}
&\mathcal{T}V(s,x)\\
&=\frac{\partial V(s,x)}{\partial x}x\left[r(s)-c+k\left(g(s)-r(s)-l(s)\lambda(s)-\theta_e(s)\bar{\lambda}(s)\right)\right]\\
&+\frac{1}{2}\frac{\partial^2 V(s,x)}{\partial x^2}x^2k^2\left[\left((1+\beta)\sigma(s)+\rho\sigma_e(s)\right)^2+(1-\rho^2)\sigma_e^2(s)\right]\\
&+\lambda(s)\left[V(s,x+xkl(s))-V(s,x)\right]+\bar{\lambda}(s)\left[V(s,x+xk\theta_e(s))-V(s,x)\right].
\end{aligned}$$

Proof. For fixed $(t,x)\in[0,T)\times\mathbb{R}^+$, applying Itô's formula to $V(t,x)$ and then sending t to s, we can go through the same argument as that of Proposition 4.3.5 in Yong and Zhou [24] to get the desired conclusion. □

We want to have the sufficient of optimality condition, then we present the verification theorem before we give the explicit solution to the optimization problem.

Theorem 3.3. *Let $v(t,x)\in C^{1,2}([0,T]\times\mathbb{R}^+)$ be a solution to the HJB equation (9) with boundary condition $v(T,x)=(1-F(T))\frac{x^{1-R}}{1-R}$. Suppose the above assumptions hold and for all $(s,x)\in[0,T]\times\mathbb{R}^+$ and all admissible controls such that*

$$\mathbb{E}\left(\sup_{t\in[s,T]}|v(t,X(t))|\right)<\infty,$$

then we have

(i) *$v(t,x)\geq J(t,x;k(\cdot),c(\cdot))$;*

(ii) *if there exists an admissible strategy $(k^*(\cdot),c^*(\cdot))$ that is a maximizer of (9), then $v(t,x)=V(t,x)$ for any $t\in[0,T]$. Furthermore $(k^*(\cdot),c^*(\cdot))$ is an optimal strategy.*

Proof. (i) Applying Itô's formula to $e^{\gamma(T-t)}v(t,x)$, we obtain

$$\begin{aligned}
&(1-F(T))\frac{X(T)^{1-R}}{1-R}=v(T,X(T))\\
&=e^{\gamma(T-s)}v(s,x)+\int_s^T e^{\gamma(T-t)}\left(-\gamma v(t,X(t))+v_s(t,X(t))+\mathcal{T}v(t,X(t))\right)\mathrm{d}t\\
&+\int_s^T e^{\gamma(T-t)}v_x(t,X(t))\left\{\left[(1+\beta)\sigma(t)+\rho\sigma_e(t)\right]k(t)\mathrm{d}W(t)+\sqrt{1-\rho^2}\sigma_e(t)k(t)\mathrm{d}\bar{W}(t)\right\}\\
&+\int_s^T e^{\gamma(T-t)}\left[v(t,X(t-)+l(t)k(t))-v(t,X(t-))\right]\mathrm{d}M(t)\\
&+\int_s^T e^{\gamma(T-t)}\left[v(t,X(t-)+\theta_e(t)k(t))-v(t,X(t-))\right]\mathrm{d}\bar{M}(t)
\end{aligned}$$

Thus, we have

$$\begin{aligned} v(s,x) =& J(s,x;k(\cdot),c(\cdot)) - \mathbb{E}\left[\int_s^T e^{-\gamma(t-s)}\left((1-F(t))\frac{(c(t)X(t))^{1-R}}{1-R} + f(t)\frac{X(t)^{1-R}}{1-R}\right)\mathrm{d}t\right] \\ &- \mathbb{E}\int_s^T e^{-\gamma(t-s)}\left(-\gamma v(t,X(t)) + v_s(t,X(t)) + \mathcal{T}v(t,X(t))\right)\mathrm{d}t, \end{aligned}$$

which immediately implies that $v(t,x) \geq J(s,x;k(\cdot),c(\cdot))$. (ii) When taking the strategy $\{(k^*(t),c^*(t)); 0 \leq t \leq T\}$, the inequalities become equalities. Hence, conclusion (ii) holds. □

To obtain an analytical solution of the HJB equation, we try to use a separation method like in [7] (see also [19]) and set

$$V(s,x) = A(s)\frac{x^{1-R}}{1-R}.$$

So we obtain:

$$\max_{c\geq 0}\{f_1(c)\} + \max_{k\geq 0}\{f_2(k)\} = 0$$

with

$$f_1(c) = -cA(t)x^{1-R} + (1-F(t))\frac{(c(t)x)^{1-R}}{1-R} + \{-\gamma A(t) + A'(t) + f(t)\}\frac{x^{1-R}}{1-R},$$

and

$$\begin{aligned} f_2(k) = A(t)x^{1-R}\Big\{& r(t) + k\left[g(t) - r(t) - l(t)\lambda(t) - \theta_e(t)\bar{\lambda}(t)\right] \\ &- \frac{1}{2}Rk^2\left[((1+\beta)\sigma(t) + \rho\sigma_e(t))^2 + (1-\rho^2)\sigma_e^2(t)\right] \\ &+ \frac{\lambda(t)}{1-R}\left[(1+kl(t))^{1-R} - 1\right] + \frac{\bar{\lambda}(t)}{1-R}\left[(1+k\theta_e(t))^{1-R} - 1\right]\Big\}. \end{aligned}$$

Let $f_1'(c) = 0$, we can get the explicit optimal consumption proportion solution

$$c^*(t) = \left(\frac{A(t)}{(1-F(t))}\right)^{-\frac{1}{R}}.$$

But unfortunately, it is impossible to obtain the optimal investment proportion k^* by the basic property and calculation. However, it is easy to show that $f_2(k)$ is concave with respect to k, so we can obtain a unique k^* such that $f_2'(k^*) = 0$ by simple calculation. Then we obtain the equation that $A(t)$ satisfies

$$A'(t) + D(t)A(t) + [1-F(t)]^{\frac{1}{R}}[A(t)]^{1-\frac{1}{R}} + f(t) = 0, \tag{10}$$

where $D(t) = -\gamma + (1-R)r(t) + (1-R)k^*\left(g(t) - r(t) - l(t)\lambda(t) - \theta_e(t)\bar{\lambda}(t)\right) - \frac{1}{2}R(1-R)k^{*2}\left[((1+\beta)\sigma(t) + \rho\sigma_e(t))^2 + (1-\rho^2)\sigma_e^2(t)\right] + \lambda(t)\left[(1+k^*l(t))^{1-R} - 1\right] + \bar{\lambda}(t)\left[(1+\theta_e(t)k^*)^{1-R} - 1\right]$.

Next we shall show that $A(t) > 0$.

Lemma 3.4. *If $A(t)$ solves* (10), *then*

(i) $A(t) > 0$*; moreover $A(t)$ is bounded from below; that is, there exists a constant $\tilde{C} > 0$ such that $A(t) \geq \tilde{C}$;*

(ii) *$A(t)$ is the unique continuous solution of* (10), *and $A(t)$ has an upper bound in* $[0,T]$.

Proof. (i) Denote $K(s,t)=\exp(\int_s^t D(u)\mathrm{d}u)$, then in view of (10), we have

$$\mathrm{d}\left[K(s,t)A(t)\right]=K_s(s,t)A(t)+K(s,t)\mathrm{d}A(t)=-K(s,t)\left((1-F(t))^{\frac{1}{R}}A(t)^{1-\frac{1}{R}}+f(t)\right)\mathrm{d}t.$$

Then we have

$$A(t)=K(t,T)(1-F(T))+\int_t^T K(t,s)\left((1-F(s))^{\frac{1}{R}}A(s)^{1-\frac{1}{R}}+f(s)\right)\mathrm{d}s$$

To prove $A(t)>0$, We construct a Picard iterative sequence $\{A^{(i)}(t), i=0,1,2,\dots\}$ as follows:

$$A^{(0)}(t)=1,$$
$$A^{(i+1)}(t)=K(t,T)(1-F(T))+\int_t^T K(t,s)\left((1-F(s))^{\frac{1}{R}}A^{(i)}(s)^{1-\frac{1}{R}}+f(s)\right)\mathrm{d}s.$$

Noting that $K(t,s)>0$, $1-F(s)>0$ and $f(s)>0$, we have

$$A^{(i)}(t)\geq K(t,T)(1-F(T))>0,\quad i=1,2,\dots. \tag{11}$$

Since all the coefficients in our paper are bounded, (11) indicates that $A^{(i)}(t)>\tilde{C}>0$ for $i=1,2,\dots$. At the same time, it is well known that $A(t)$ is the limit of the sequence $\{A^{(i)}(t), i=1,2,\dots\}$ as $i\to\infty$. Thus, $A(t)\geq\tilde{C}>0,\ t\in[0,T]$.

(ii) Denote $h(t,A(t))=-\left(D(t)A(t)+(1-F(t))^{\frac{1}{R}}A(t)^{1-\frac{1}{R}}+f(t)\right)$, we have

$$A'(t)=h(t,A(t)).$$

Then h satisfies the following inequality for $\forall a,b\geq\tilde{C}$,

$$\begin{aligned}|h(t,a)-h(t,b)|&=\left|D(t)a+(1-F(t))^{\frac{1}{R}}a^{1-\frac{1}{R}}-D(t)b-(1-F(t))^{\frac{1}{R}}b^{1-\frac{1}{R}}\right|\\&\leq|D(t)||a-b|+|1-F(t)|^{\frac{1}{R}}\left|a^{1-\frac{1}{R}}-b^{1-\frac{1}{R}}\right|.\end{aligned}$$

Because $D(t)$, $(1-F(t))$ are bounded, there exists a constant C_1 such that $|D(t)|+|1-F(t)|^{\frac{1}{R}}\leq C_1$. Moreover

$$\left|\frac{\partial a^{1-\frac{1}{R}}}{\partial a}\right|=\left|\frac{1-R}{R}\right|\left|\frac{1}{a}\right|^{\frac{1}{R}}\leq\left|\frac{1-R}{R}\right|\left|\frac{1}{\tilde{C}}\right|^{\frac{1}{R}}.$$

So

$$\left|a^{1-\frac{1}{R}}-b^{1-\frac{1}{R}}\right|\leq\left|\frac{1-R}{R}\right|\left|\frac{1}{\tilde{C}}\right|^{\frac{1}{R}}|a-b|.$$

Then we have

$$|h(t,a)-h(t,b)|\leq C\left|a-b\right|.$$

Now it obvious that h satisfies Lipschitz condition. Consequently, (10) has a unique continuous solution denoted by $A(t)$ in $[0,T]$. A continuous function $A(t)$ defined in a close interval $[0,T]$ must have an upper bound $\bar{C}$. Then we complete the proof. □

It is easy to know that $(k^*(\cdot),c^*(\cdot))$ is admissible strategy by Lemma 3.4.

Theorem 3.5. *The optimal investment proportion and the optimal consumption for the CRRA case are, respectively,*

$$c^*(t)=\left(\frac{A(t)}{1-F(t)}\right)^{-\frac{1}{R}},$$

$$k^*(t) = \begin{cases} k^* & \text{if } \ \mathrm{k}^* \in [0,\mathrm{K}], \\ K & \text{if } \ \mathrm{k}^* \in (\mathrm{K},+\infty), \\ 0 & \text{otherwise}, \end{cases}$$

where K is defined in Assumption 2.7 and the maximum utility is

$$V(0,x_0) = A(0)\frac{x_0^{1-R}}{1-R}.$$

§4 Numerical examples

In this section, we present some numerical examples to show the application of the results obtained above. We further explain the influence of exchange rate and random exit time on the optimal investment proportion and the maximum utility in these examples from the economic viewpoint. For simplicity, all parameters except the density of conditional function of exit time are given. We assume that $r = 0.04$, $\mu_s = 0.12$, $\mu_p = 0.1$, $\mu_e = 0.2$, $\alpha = 0.17$, $\gamma = 0.04$, $\theta = 0.4$, $\theta_e = 0.5$, $\beta = -0.6$, $p = 3$, $s = 4$, $p_0 = 0.3$, $x_0 = 10$, $\rho = 0.2$, $R = 0.5$, $\lambda = 0.2$, $\bar{\lambda} = 0.24$, $\psi = 0$, $\varphi = 0$ for an investor. Below we present several examples.

Example 4.1. *Suppose that the fixed time horizon $T = 1$, the volatility of the exchange rate $\sigma_e = 0.15$, and the volatility of the price process σ varies from 0 to 1, we can obtain the optimal investment proportion by the above results. The Figure 1 shows the result with respect to each $\sigma \in [0,1]$:*

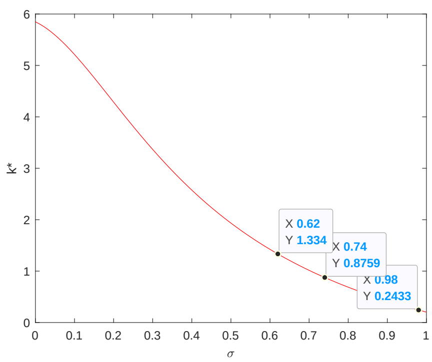

Figure 1. The optimal investment proportion with different σ.

We can see from Figure 1 that the optimal investment proportion k^* decreases as the volatility of price σ increases. It is quite reasonable since the greater the price volatility, the greater the risk. So when the risk increases, the optimal investment proportion will decrease. Next we will present the relationship of optimal investment proportion, the volatility of price and the volatility of exchange rate by $3D$ figure.

Example 4.2. *Suppose that the fixed time horizon* $T = 1$*, the volatility of price* σ *and the volatility of exchange rate* σ_e *vary from* 0.2 *to* 1*, we can obtain the optimal investment proportion by the above results. The Figure* 2 *shows the relationship between them.*

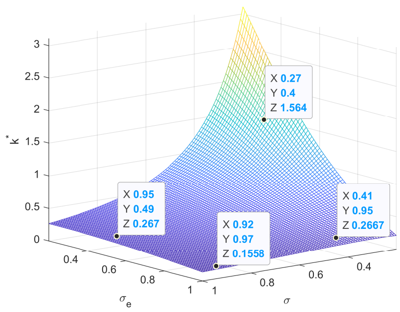

Figure 2. The optimal investment proportion with different σ and σ_e.

We can see from Figure 2 that the optimal investment proportion k^* decreases as the volatility of price σ and the volatility of exchange rate σ_e increase.

Example 4.3. *Suppose that the volatility of price and exchange rate* $\sigma = 0.4$*,* $\sigma_e = 0.5$*, respectively, and the density of conditional distribution of exit time* f *varies from* 0.1 *to* 0.9*. Then the maximum utility can be given in the Figure* 3*:*

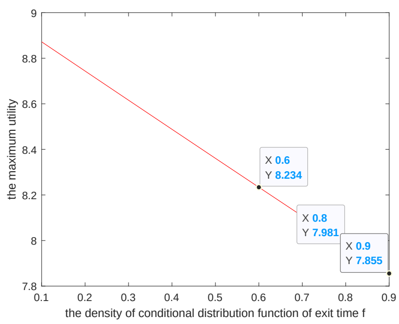

Figure 3. The maximum utility with different density f.

It's clear that the maximum utility becomes smaller as the density of conditional distribution function of exit time grows from Figure 3. As is well known, the randomness of exit time has no influence on the optimal proportion in the deterministic market parameters. However it may advance the exit time before T, resulting in a less accumulation of wealth by time. So it will decrease the utility of consumption and wealth. The shorter the expected duration of the investment and consumption are, the smaller the maximum utility is.

§5 Conclusion

In this paper, we have considered one kind of investment and consumption choices problem for an investor who can invest her or his wealth either in a domestic bond or in a foreign real project. From the contribution of predecessors, we can easily know that the Dynamic Programming Principle still holds for this optimal problem. The optimal investment and consumption strategies have also been achieved by solving the corresponding HJB equation. For a classical CRRA utility function case, we get the optimal investment-consumption strategy by solving the corresponding HJB equation and a separation method. Then we give some numerical examples of this problem and explain their economic significance. It shows that the research of this problem is of great significance to the market in the real world.

This paper uses Poisson process to simulate the market price, which makes the model more suitable for the market. So the problem is more meaningful. The following are the research directions of interest in the future. One of the future tasks is to extend our model to a more general case. We will study the optimal investment-consumption problem with discontinuous price and random time horizon under the general utility function. We will try to give an explicit solution to the investment and consumption strategy. In addition, we will combine this problem with partial information problem. We want to get some useful conclusions and apply them to financial markets.

Declarations

Conflict of interest The authors declare no conflict of interest.

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[1]Suzhou Research Institute, Shandong University, Suzhou 215123, China.
Email: chentian43@sdu.edu.cn

[2]School of Mathematics, Shandong University, Jinan 250100, China.
Emails: huangzy@sdu.edu.cn, wuzhen@sdu.edu.cn