

Existence for neutral fractional differential equations in composite relaxation process

HE Jia-wei^{1,2,*} YU Zi-cheng¹

Abstract. This paper is devoted to the existence results for a class of neutral abstract fractional differential equations involving the composite relaxation process. Based on the Laplace transform, the semigroup theory and the Wright functions, we first introduce a definition of mild solutions to the considered problem. By means of the noncompactness of measure and the fixed point technique, we establish existence criteria of solutions. Finally, an example is presented to illustrate our main result.

§1 Introduction

In this paper, we are concerned with the following neutral abstract fractional differential equations in composite relaxation process

$$\begin{cases} (u(t) - h(t, u_t))' = A {}^C D_{0+}^{1-\alpha} u(t) - u(t) + f(t, u_t), & t \in (0, a], \\ u(t) = \varphi(t), & t \in [-r, 0], \end{cases} \quad (1.1)$$

in a Banach space X , where ${}^C D_{0+}^{1-\alpha}$ is the Caputo fractional derivative of order $1 - \alpha$ for $\alpha \in (0, 1)$. The operator $A : D(A) \subset X \rightarrow X$ is the infinitesimal generator of an analytic semigroup of uniformly bounded linear operators $\{T(t)\}_{t \geq 0}$. We set u_t by $u_t(\theta) := u(t + \theta)$ for $\theta \in [-r, 0]$, and let $\mathcal{C} = C([-r, 0]; X)$ denote the Banach space of continuous functions from $[-r, 0]$ into X with the norm $\|u\|_{\mathcal{C}} = \sup_{\theta \in [-r, 0]} \|u(\theta)\|$. The functions $h, f : (0, a] \times \mathcal{C} \rightarrow X$ are given that will be specified later and $\varphi \in \mathcal{C}$.

We observe that, if $\alpha = 1/2$, $A = a > 0$ and $h = 0$ in the scalar case, the problem (1.1) is the so-called Basset problem, i.e., the equation

$$u'(t) + a {}^C D_{0+}^{1/2} u(t) + u(t) = f(t),$$

arises in fluid dynamics, modelling the unsteady motion of a particle accelerates in viscous fluid under the action of gravity. Ashyralyev [3] studied the well-posedness of this problem

Received: 2021-04-22. Revised: 2022-11-14.

MR Subject Classification: 26A33, 34K40.

Keywords: Caputo fractional derivative, composite relaxation process, noncompactness of measure.

Digital Object Identifier(DOI): <https://doi.org/10.1007/s11766-025-4428-1>.

Supported by the Guangxi Science and Technology base and Talent Project(AD23026249, AD22080047).

*Corresponding author.

in spaces of smooth functions associated with a strongly positive operator A , Gorenflo and Mainardi [16] found a fundamental solution by using the Laplace transform method. In [13,14], Lizama et al. considered the existence and qualitative properties of solutions by means of (a, k) -regularized resolvent. Mophou et al. [17] studied the existence and uniqueness of solutions for an initial/boundary value problem by using the eigenfunction expansions approach. Fan et al. [8] investigated the approximate controllability in Hilbert space with the compactness of C_0 -semigroup $T(t)$ generated by operator A . Wu and He [21] considered the maximal regularity for the initial value problem and the regularization of a backward problem based on a spectral problem. As we have already noted, the theory of fractional calculus has attracted widespread attention in recent years, with extensive contributions in various directions. For instance, the existence and controllability of solutions for fractional systems have been established in [6, 11, 12, 18-20, 24, 25]; the existence of solutions for fractional partial differential equations have been studied in [7, 9, 22, 26]; and the blow-up and decay estimates of solutions can be found in [1, 23], etc..

Neutral fractional differential equations have significant applications in the field of physics, particularly in describing complex physical phenomena such as anomalous relaxation and diffusion processes. These processes often involve memory effects and composite relaxation behaviours, which can be effectively captured by the fractional derivatives in neutral differential equations. Moreover these equations extend the classical integer order differential equations and pure fractional differential equations, introducing more flexibility and complexity in the description of dynamic systems. This motivates us to study neutral fractional differential equations with composite relaxation processes. As indicated in the provided references, despite extensive research on fractional evolution equations, there is a lack of work focusing on neutral fractional composite relaxation equations. In this paper, we will establish the existence of solution for this problem. We noted that, some classical analytical tools appearing in fractional evolution equations will be applied to derive the representation of solutions, like Laplace transform, the semigroup theory and the Wright function, by means of the operator theory and noncompactness of measure, the existence results of solutions for the current problem are obtained.

This paper is organized as follows. In section 2, we introduce some useful notations and preliminary results, and then we further introduce a suitable definition of the mild solutions by applying the Laplace transform. In section 3, the existence results of mild solutions are presented by means of noncompactness of measure. Finally, an example is presented to illustrate the main results.

§2 Preliminaries

Let $J = [0, a]$, $J' = (0, a]$ and let X be a Banach space with norm $\|\cdot\|$. We set $A : D(A) \subset X \rightarrow X$ by the infinitesimal generator of an analytic semigroup $\{T(t)\}_{t \geq 0}$ of uniformly bounded linear operators on X . Let $0 \in \rho(A)$, where $\rho(A)$ denotes the resolvent set of A . For any $0 < \eta \leq 1$, we denote the fractional power operator A^η as a closed linear operator on its

domain $D(A^\eta)$. We shall use the space $C(J, X)$ representing the Banach space of all continuous functions on J equipped with the norm

$$\|u\|_{C(J, X)} = \sup_{t \in J} e^{-Lt} \|u(t)\|,$$

where L is a positive constant.

Let us recall the following well-known definitions related to fractional integral and fractional derivative.

Definition 2.1. The Riemann-Liouville fractional integral of order $\alpha > 0$ for a function $x : [0, a] \rightarrow X$ is defined as

$$I_{0+}^\alpha x(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} x(s) ds, \quad t \in (0, a],$$

provided that the right-hand side is pointwise defined on $[0, a]$, where $\Gamma(\cdot)$ is the gamma function.

Definition 2.2. The Caputo fractional derivative of order $\alpha > 0$ for a function $x : [0, a] \rightarrow X$ is defined as

$${}^C D_{0+}^\alpha x(t) = \frac{d}{dt} I_{0+}^{1-\alpha} (x(t) - x(0)), \quad 0 < \alpha < 1.$$

In particular, if x is a smooth function, then ${}^C D_{0+}^\alpha x(t) = I_{0+}^{1-\alpha} x'(t)$.

Note that the Laplace transform for the Caputo fractional derivative of order $\alpha \in (0, 1)$ is given by

$$\mathcal{L}({}^C D_{0+}^\alpha x)(\lambda) = \lambda^\alpha \mathcal{L}(f)(\lambda) - \lambda^{\alpha-1} f(\lambda),$$

where the Laplace transform is defined by

$$\mathcal{L}(x)(\lambda) = \hat{x}(\lambda) = \int_0^\infty e^{-\lambda t} x(t) dt, \quad \operatorname{Re}(\lambda) > 0.$$

Next, we introduce the notations of measures of noncompactness and their properties. Denote by $\mathcal{P}(X)$ the collection of nonempty bounded subsets of X . For every $\Omega \in \mathcal{P}(X)$, there are two important examples of measure of noncompactness (MNC), the first one is the *Hausdorff* MNC defined by

$$\chi(\Omega) = \inf\{\varepsilon > 0 : \Omega \text{ has a finite } \varepsilon\text{-net}\},$$

and the second one is the *Kuratowski* MNC defined by

$$\tau(\Omega) = \inf\{d > 0 : \Omega \subset \bigcup_{j=1}^n M_j \text{ and } \operatorname{diam}(M_j) \leq d\},$$

where the diameter of M_j is given by $\operatorname{diam}(M_j) = \sup\{\|x - y\| : x, y \in M_j\}$, $j = 1, \dots, n$. The *Hausdorff* and *Kuratowski* MNCs are connected by the relations:

$$\chi(\Omega) \leq \tau(\Omega) \leq 2\chi(\Omega).$$

A MNC χ (or τ) is recalled: monotone if $\Omega_1, \Omega_2 \in \mathcal{P}(X)$ with $\Omega_1 \subseteq \Omega_2$ then $\chi(\Omega_1) \leq \chi(\Omega_2)$; nonsingular if $\chi(\{c\} \cup \Omega) = \chi(\Omega)$ for every $c \in X$, $\Omega \in \mathcal{P}(X)$; regular if $\chi(\Omega) = 0$, Ω is relatively compact.

We now introduce the MNC ν as follows: for a bounded set $D \subset C(J, X)$, the modulus of noncompactness of D is given by

$$\gamma_L(D) = \sup_{t \in J} e^{-Lt} \chi(D(t)),$$

where the modulus of equicontinuity of the set of functions D has the following form

$$\text{mod}_C(D) = \limsup_{\delta \rightarrow 0} \max_{x \in D} \max_{|t_2 - t_1| < \delta} \|x(t_2) - x(t_1)\|.$$

Let us fix a positive constant L and denote

$$\nu(D) = \max_{D \in \Theta(D)} (\gamma_L(D), \text{mod}_C(D)),$$

where $\Theta(D)$ is the collection of all denumerable subsets of D , the range for the MNC is a cone in $\mathbb{R}^2$ and the maximum is taken in the sense of the ordering induced by this cone. As remarked in [10], the MNC ν is known to be monotone, nonsingular, and regular. For more details of the definitions, properties and applications of the MNC, we refer to [4, 5, 24].

Lemma 2.1. [5] *Let X be a Banach space. If $W \subset X$ is bounded, then for each $\varepsilon > 0$, there is a sequence $\{u_n\}_{n=1}^\infty \subset W$, such that*

$$\chi(W) \leq 2\chi(\{u_n\}_{n=1}^\infty) + \varepsilon.$$

Lemma 2.2. [15] *Let χ_C be the Hausdorff MNC on $C(J, X)$, and let $W(t) = \{x(t) : x \in W\}$. If $W \subset C(J, X)$ is bounded, then for every $t \in J$,*

$$\chi(W(t)) \leq \chi_C(W).$$

Furthermore, if W is equicontinuous, then the map $t \mapsto \chi(W(t))$ is continuous on J and

$$\chi_C(W) = \sup_{t \in J} \chi(W(t)).$$

Lemma 2.3. [15]. *Let $\{x_n\}_{n=1}^\infty$ be a sequence of Bochner integrable functions from J into X . If there exists a function $\rho(\cdot) \in L^1(J, \mathbb{R}_+)$ satisfying $\|x_n(t)\| \leq \rho(t)$ for almost all $t \in J$ and for every $n \geq 1$, then the function $\psi(t) = \chi(\{x_n(t)\}_{n=1}^\infty) \in L^1(J, \mathbb{R}_+)$ satisfying*

$$\chi\left(\left\{\int_0^t x_n(s)ds : n \geq 1\right\}\right) \leq 2 \int_0^t \psi(s)ds.$$

Definition 2.3. Let D be a subset of Banach space X . A continuous function $G : D \rightarrow X$ is said to be ν -condensing with respect to MNC ν if $\nu(G(\Omega)) < \nu(\Omega)$ for every bounded and not relatively compact set $\Omega \subseteq D$.

Theorem 2.1. *Let D be a bounded convex closed subset of X and let $G : D \rightarrow D$ be a ν -condensing map. Then there exists at least one fixed point.*

Definition 2.4. A sequence $\{f_n\}_{n=1}^\infty$ is said to be semicompact if it satisfies the following conditions:

- (i) it is integrable bounded, i.e., there exists $q \in L^1(J, \mathbb{R}^+)$ such that $\|f_n(t)\| \leq q(t)$ for all n and for a.e. $t \in J$;
- (ii) the set $\{f_n(t)\}_{n=1}^\infty$ is relatively compact in X for a.e. $t \in J$.

Let us consider the following neutral fractional differential equations

$$\begin{cases} (u(t) - h(t, u_t))' = A {}^C D_{0+}^{1-\alpha} u(t) - u(t) + f(t), & t \in (0, a], \\ u(t) = \varphi(t), & t \in [-r, 0]. \end{cases} \quad (2.1)$$

Now, let $Re(\lambda) > 0$, $\lambda^\alpha \in \rho(A)$ and let us take Laplace transform to both sides of (2.1), we have

$$\lambda \hat{u}(\lambda) - \lambda \chi(\lambda) - u(0) + h(0, u_0) = A \lambda^{1-\alpha} \hat{u}(\lambda) - A \lambda^{-\alpha} u(0) - \hat{u}(\lambda) + \hat{f}(\lambda),$$

where

$$\chi(\lambda) = \int_0^\infty e^{-\lambda t} h(t, u_t) dt.$$

Therefore, we have

$$\begin{aligned} \hat{u}(\lambda) &= \lambda^{-1} u(0) - \lambda^{\alpha-1} (\lambda^\alpha I - A)^{-1} h(0, u_0) - \lambda^{\alpha-1} (\lambda^\alpha I - A)^{-1} \hat{u}(\lambda) \\ &\quad + \lambda^\alpha (\lambda^\alpha I - A)^{-1} \chi(\lambda) + \lambda^{\alpha-1} (\lambda^\alpha I - A)^{-1} \hat{f}(\lambda), \end{aligned} \quad (2.2)$$

where I is the identity operator. From the technique of [24], we have

$$\lambda^{\alpha-1} (\lambda^\alpha I - A)^{-1} v = \int_0^\infty e^{-\lambda t} \mathcal{Q}_\alpha(t) v dt, \quad (\lambda^\alpha I - A)^{-1} v = \int_0^\infty e^{-\lambda t} t^{\alpha-1} \mathcal{R}_\alpha(t) v dt,$$

for any $v \in X$ and

$$\mathcal{Q}_\alpha(t) = \int_0^\infty M_\alpha(\theta) T(t^\alpha \theta) d\theta, \quad \mathcal{R}_\alpha(t) = \int_0^\infty \alpha \theta M_\alpha(\theta) T(t^\alpha \theta) d\theta,$$

where $M_\alpha(\cdot)$ is the Mainardi Wright-type function defined by

$$M_\alpha(z) = \sum_{n=0}^\infty \frac{(-z)^n}{n! \Gamma(1 - \alpha(n+1))}, \quad \alpha \in (0, 1), \quad z \in \mathbb{C}.$$

Noting that $\lambda^\alpha (\lambda^\alpha I - A)^{-1} = I + A(\lambda^\alpha I - A)^{-1}$, hence, one has

$$\begin{aligned} \hat{u}(\lambda) &= \mathcal{L}(1(t)u(0)) - \mathcal{L}(\mathcal{Q}_\alpha(t)h(0, u_0)) - \mathcal{L}(\mathcal{Q}_\alpha(t)) \hat{u}(\lambda) \\ &\quad + \chi(\lambda) + \mathcal{L}(t^{\alpha-1} A \mathcal{R}_\alpha(t)) \chi(\lambda) + \mathcal{L}(\mathcal{Q}_\alpha(t)) \hat{f}(\lambda). \end{aligned}$$

By the uniqueness of inverse Laplace transform, we obtain

$$\begin{aligned} u(t) &= \varphi(0) - \mathcal{Q}_\alpha(t)h(0, u_0) - \int_0^t \mathcal{Q}_\alpha(t-s)u(s)ds + h(t, u_t) \\ &\quad + \int_0^t (t-s)^{\alpha-1} A \mathcal{R}_\alpha(t-s)h(s, u_s)ds + \int_0^t \mathcal{Q}_\alpha(t-s)f(s)ds. \end{aligned}$$

Lemma 2.4. [24] For any $t > 0$, the Mainardi Wright-type function has the properties

$$M_\alpha(t) \geq 0, \quad \int_0^\infty \theta^\delta M_\alpha(\theta) d\theta = \frac{\Gamma(1+\delta)}{\Gamma(1+\alpha\delta)}, \quad \text{for } -1 < \delta < \infty. \quad (2.3)$$

Throughout this paper, we suppose that A is the infinitesimal generator of an analytic semigroup $\{T(t)\}_{t \geq 0}$ on X . This means that there exists $M \geq 1$ such that

$$M = \sup_{t \in [0, \infty)} \|T(t)\|_{\mathcal{B}(X)} < \infty,$$

where $\mathcal{B}(X)$ is the space of all bounded linear operators from X to X with the norm $\|T\|_{\mathcal{B}(X)} = \sup\{\|T(x)\| : \|x\| = 1\}$. The following lemmas can be found in [24].

Lemma 2.5. For any fixed $t \geq 0$, $\mathcal{Q}_\alpha(t)$, $\mathcal{R}_\alpha(t)$ are linear and bounded operators, i.e., for any $x \in X$

$$\|\mathcal{Q}_\alpha(t)x\| \leq M\|x\|, \quad \|\mathcal{R}_\alpha(t)x\| \leq \frac{M}{\Gamma(\alpha)}\|x\|.$$

Lemma 2.6. $\{\mathcal{R}_\alpha(t)\}_{t \geq 0}, \{\mathcal{Q}_\alpha(t)\}_{t \geq 0}$ are strongly continuous, which means that, for any $x \in X$ and $t'' > t' \geq 0$, we have

$$\|\mathcal{Q}_\alpha(t'')x - \mathcal{Q}_\alpha(t')x\| \rightarrow 0, \quad \|\mathcal{R}_\alpha(t'')x - \mathcal{R}_\alpha(t')x\| \rightarrow 0 \quad \text{as } t'' \rightarrow t'.$$

Lemma 2.7. *If $T(t)$ is compact for every $t > 0$, then $\mathcal{Q}_\alpha(t)$ and $\mathcal{R}_\alpha(t)$ are compact for every $t > 0$.*

Let us introduce a fractional power space, see e.g. [19]. For $0 < \nu < 1$, we also introduce fractional power $A^{-\nu} \in \mathcal{B}(X)$ of A as

$$A^{-\nu} := \frac{1}{\Gamma(\nu)} \int_0^\infty s^{\nu-1} T(s) ds.$$

We set A^ν as the inverse of $A^{-\nu}$ and $D(A^\nu)$ as a Banach space with the graph norm $\|u\|_\nu = \|x\| + \|A^\nu u\|$ for any $u \in D(A^\nu)$.

Lemma 2.8. *For any $u \in D(A^\nu)$, $\nu \in (0, 1)$ and $\eta \in (0, 1]$, we have*

$$A\mathcal{R}_\alpha(t)u = A^{1-\nu}\mathcal{R}_\alpha(t)A^\nu u, \quad t \in J,$$

and there exists a constant $C_\eta > 0$ such that

$$\|A^\eta \mathcal{R}_\alpha(t)\|_{\mathcal{B}(X)} \leq \frac{C_\eta}{t^{\alpha\eta}}, \quad t \in J'.$$

In view of (2.2), we introduce the following definition of mild solutions to problem (1.1).

Definition 2.5. A function $u \in C([-r, a], X)$ is called a mild solution to composite fractional relaxation equation (1.1) if it satisfies

$$u(t) = \begin{cases} \varphi(0) - \mathcal{Q}_\alpha(t)h(0, u_0) + h(t, u_t) - \int_0^t \mathcal{Q}_\alpha(t-s)u(s)ds \\ \quad + \int_0^t (t-s)^{\alpha-1} A\mathcal{R}_\alpha(t-s)h(s, u_s)ds + \int_0^t \mathcal{Q}_\alpha(t-s)f(s, u_s)ds, & t \in [0, a], \\ \varphi(t), & t \in [-r, 0]. \end{cases}$$

§3 The existence results

In order to derive the main results, in this section, we first assume that the following conditions are satisfied:

- (Ht) The semigroup $T(t)$ is equicontinuous for $t > 0$, i.e., $T(t)$ is continuous in the uniform operator topology for $t > 0$;
- (Hf) The function f satisfies
 - (i) for almost all $t \in J$, the function $f(t, \cdot) : \mathcal{C} \rightarrow X$ is continuous and for each $u \in \mathcal{C}$, the function $f(\cdot, u) : J \rightarrow X$ is strongly measurable;
 - (ii) there exists a function $\sigma \in L^1(J, \mathbb{R}^+)$ such that $\|f(t, u)\| \leq \sigma(t)\|u\|_{\mathcal{C}}$ for all $u \in \mathcal{C}$ and almost all $t \in J$;
 - (iii) for every bounded set $D \subset \mathcal{C}_\varphi$, there exists a function $\rho \in L^1(J, \mathbb{R}^+)$ such that

$$\chi(f(t, D)) \leq \rho(t)\chi(D);$$

- (Hh) The function $h : J \times \mathcal{C} \rightarrow X$ is continuous and there exists a constant $\eta \in (0, 1)$ and two constants L_{h1}, L_{h2} with $2L_{hi}\|A^{-\eta}\|_{\mathcal{B}(X)} < 1$ ($i = 1, 2$) such that $h \in D(A^\eta)$ and for any $u \in \mathcal{C}_\varphi$, $t \in J$, the function $A^\eta h(\cdot, u)$ is strongly measurable and $A^\eta h(t, \cdot)$ satisfies the Lipschitz condition

$$\|A^\eta h(t, x) - A^\eta h(t, z)\| \leq L_{h1}\|x - z\|_{\mathcal{C}}$$

and $\|A^\eta h(t, u)\| \leq L_{h2}\|u\|_{\mathcal{C}}$. Moreover, for every bounded set $D' \subset \mathcal{C}_\varphi$ such that

$$\chi(A^\eta h(t, D')) \leq L_{h2}\chi(D').$$

Let

$$\mathcal{C}_\varphi := \{u \in C(J, X) : u(0) = \varphi(0)\},$$

and for $u \in \mathcal{C}_\varphi$, we set

$$u[\varphi](t) = \begin{cases} u(t), & \text{for } t \in [0, a]; \\ \varphi(t), & \text{for } t \in [-r, 0]. \end{cases}$$

Denote by $\mathbf{S}$ the operator acting on $\mathcal{C}_\varphi$ such that

$$\begin{aligned} \mathbf{S}(u)(t) = & \varphi(0) - \mathcal{Q}_\alpha(t)h(0, u[\varphi]_0) + h(t, u[\varphi]_t) - \int_0^t \mathcal{Q}_\alpha(t-s)u(s)ds \\ & + \int_0^t (t-s)^{\alpha-1} A\mathcal{R}_\alpha(t-s)h(s, u[\varphi]_s)ds + \int_0^t \mathcal{Q}_\alpha(t-s)f(s, u[\varphi]_s)ds. \end{aligned}$$

Define a mapping $\mathbf{Q}_\alpha : L^1(J, X) \rightarrow C(J, X)$ by

$$\mathbf{Q}_\alpha(g)(t) = \int_0^t \mathcal{Q}_\alpha(t-s)g(s)ds.$$

Additionally, we also consider an operator Φ_h as follows

$$\Phi_h(u)(t) = \varphi(0) - \mathcal{Q}_\alpha(t)h(0, u[\varphi]_0) + h(t, u[\varphi]_t) + \int_0^t (t-s)^{\alpha-1} A\mathcal{R}_\alpha(t-s)h(s, u[\varphi]_s)ds,$$

putting

$$N_f(u)(t) = f(t, u[\varphi]_t),$$

and

$$\Psi_f(u)(t) = \mathbf{Q}_\alpha(N_f(u) - u).$$

Hence, we have

$$\mathbf{S}(u) = \Psi_f(u) + \Phi_h(u).$$

It is evident that $u \in \mathcal{C}_\varphi$ is a fixed point of $\mathbf{S}$ if and only if $u[\varphi]$ is a mild solution of (1.1).

We shall consider an abstract operator $\mathcal{S} : L^1(J, X) \rightarrow C(J, X)$ satisfying

- (S1) there exists $\varpi \geq 0$ such that

$$\|\mathcal{S}f - \mathcal{S}g\|_{C(J, X)} \leq \varpi\|f - g\|_{L^1(J, X)}, \text{ for every } f, g \in L^1(J, X);$$

- (S2) for any compact $K \subset X$ and sequence $\{f_n(t)\}_{n=1}^\infty \subset L^1(J, X)$ such that $\{f_n\}_{n=1}^\infty \subset K$ for a.e. $t \in J$, the weak convergence $f_n \rightharpoonup f_0$ implies the strong convergence $\mathcal{S}f_n \rightarrow \mathcal{S}f_0$.

Lemma 3.1. *The operator $\mathbf{Q}_\alpha$ satisfies the properties (S1) and (S2).*

Proof. Since $T(t)$ is an uniformly bounded analytic semigroup, in view of Lemma 2.5, we have

$$\begin{aligned}\|\mathbf{Q}_\alpha f - \mathbf{Q}_\alpha g\|_{C(J,X)} &= \sup_{t \in J} e^{-Lt} \|\mathbf{Q}_\alpha(f - g)(t)\| \\ &\leq \sup_{t \in J} e^{-Lt} \int_0^t \|\mathcal{Q}_\alpha(t-s)(f(s) - g(s))\| ds \\ &\leq M \sup_{t \in J} e^{-Lt} \int_0^t \|f(s) - g(s)\| ds = M \|f - g\|_{L^1(J,X)}.\end{aligned}$$

The property (S1) follows. In order to check (S2), we first note that for every compact set $K \subset X$, in view of Lemma 2.6, the functions $s \mapsto \mathcal{Q}_\alpha(t-s)x$, $s \in [0, t]$, $x \in K$ are equicontinuous in s with respect to x on $[0, t]$, which means that the set

$$D = \bigcup_{s \in [0, t]} \mathcal{Q}_\alpha(t-s)K$$

is relatively compact.

Moreover, for every sequence $\{f_n(t)\}_{n=1}^\infty \subset L^1(J, X)$ such that $\{f_n(s)\}_{n=1}^\infty \subset K$ for a.e. $s \in J$, it yields

$$\{\mathbf{Q}_\alpha f_n(t)\}_{n=1}^\infty \subset tD,$$

which implies that the sequence $\{\mathbf{Q}_\alpha f_n(t)\}_{n=1}^\infty \subset X$ is relatively compact for every $t \in J$. Next, by means of Ascoli-Arzelà theorem, it remains to check the sequence $\{\mathbf{Q}_\alpha f_n\}_{n=1}^\infty \subset C(J, X)$ is equicontinuous.

Due to $f_n(t) \in K$ for $n \geq 1$ and a.e. $t \in J$, there is a constant $N > 0$ such that $\|f_n(t)\| \leq N$. For any $t'', t' \in J$, $0 < t'' - t' < a$, take $\varepsilon > 0$ enough small, we have

$$\begin{aligned}\|\mathbf{Q}_\alpha f_n(t'') - \mathbf{Q}_\alpha f_n(t')\| &\leq \int_{t'}^{t''} \|\mathcal{Q}_\alpha(t''-s)f_n(s)\| ds + \int_0^{t'} \|(\mathcal{Q}_\alpha(t''-s) - \mathcal{Q}_\alpha(t'-s))f_n(s)\| ds \\ &\leq \int_{t'}^{t''} \|\mathcal{Q}_\alpha(t''-s)f_n(s)\| ds \\ &\quad + \int_0^{t'-\varepsilon} \|(\mathcal{Q}_\alpha(t''-s) - \mathcal{Q}_\alpha(t'-s))f_n(s)\| ds \\ &\quad + \int_{t'-\varepsilon}^{t'} \|(\mathcal{Q}_\alpha(t''-s) - \mathcal{Q}_\alpha(t'-s))f_n(s)\| ds.\end{aligned}$$

In view of Lemma 2.5, Lemma 2.6, the functions $s \mapsto \mathcal{Q}_\alpha(t-s)x$, $s \in [0, t]$, $x \in K$ are equicontinuous in s with respect to x on $[0, t]$, we have

$$\begin{aligned}\|\mathbf{Q}_\alpha f_n(t'') - \mathbf{Q}_\alpha f_n(t')\| &\leq MN(t'' - t') + N(t' - \varepsilon) \sup_{s \in [0, t'-\varepsilon]} \|\mathcal{Q}_\alpha(t''-s) - \mathcal{Q}_\alpha(t'-s)\|_{\mathcal{B}(X)} \\ &\quad + 2MN\varepsilon \rightarrow 0, \quad \text{as } t'' \rightarrow t', \quad \varepsilon \rightarrow 0.\end{aligned}$$

Therefore, the sequence $\{\mathbf{Q}_\alpha f_n\}_{n=1}^\infty$ is relatively compact in $C(J, X)$. Finally, we know that (S1) implies that $\mathbf{Q}_\alpha$ is a bounded linear operator from $C(J, X)$ into $L^1(J, X)$. Hence, operator $\mathbf{Q}_\alpha$ is continuous if these spaces are endowed with the topology of weak sequential convergence and $f_n \rightharpoonup f$ implies $\mathbf{Q}_\alpha f_n \rightharpoonup \mathbf{Q}_\alpha f$, from the relative compactness of $\{\mathbf{Q}_\alpha f_n\}_{n=1}^\infty$, we have $\mathbf{Q}_\alpha f_n \rightarrow \mathbf{Q}_\alpha f$ in $C(J, X)$. The proof is completed. $\square$

It is readily seen from [10, Corollary 4.2.3] that the following lemma is true.

Lemma 3.2. *If the sequence $\{f_n\}_{n=1}^\infty \subset L^1(J, X)$ is semicompact, then there exists a compact set $K_\delta \subset X$ for every $\delta > 0$ and a δ -net of the sequence $\{f_n\}_{n=1}^\infty$ formed by the functions with values in K_δ .*

Lemma 3.3. *For every semicompact sequence $\{f_n\}_{n=1}^\infty \subset L^1(J, X)$, the sequence $\{\mathbf{Q}_\alpha f_n\}_{n=1}^\infty$ is relatively compact in $C(J, X)$.*

Proof. From Lemma 3.2, there exists a compact set $K_\delta \subset X$ for every $\delta > 0$ and an δ -net O_δ of the sequence $\{f_n\}_{n=1}^\infty$ formed by the functions with values in K_δ . In view of (S2), we get that $\mathbf{Q}_\alpha O_\delta$ is a relatively compact set in $C(J, X)$. From (S1), we obtain that $\mathbf{Q}_\alpha O_\delta$ is a δM -net of the sequence $\{\mathbf{Q}_\alpha f_n\}_{n=1}^\infty$ in $C(J, X)$. This ends the proof. $\square$

Lemma 3.4. *Let (Hf) holds. Then $\mathbf{S}(\{u_n\}_{n=1}^\infty)$ is relatively compact for every bounded sequence $\{u_n\}_{n=1}^\infty \subset \mathcal{C}_\varphi$ satisfying $\gamma_L(\{u_n\}_{n=1}^\infty) = 0$.*

Proof. Now let $\{u_n\}_{n=1}^\infty \subset \mathcal{C}_\varphi$ be a bounded sequence such that $\gamma_L(\{u_n\}_{n=1}^\infty) = 0$. Then it follows from (Hf) that, $f_n(t) = f(t, u_n[\varphi]_t)$ satisfies the estimate

$$\begin{aligned} \|f_n(t)\| &\leq \sigma(t) \|u_n[\varphi]_t\| \leq \sigma(t) \left(\sup_{s \in [0, t]} \|u_n(s)\| + \|\varphi\|_C \right) \\ &\leq \sigma(t) \overline{M} + \sigma(t) \|\varphi\|_C =: q(t), \end{aligned}$$

where $\overline{M}$ is an upper bound for $\{u_n\}_{n=1}^\infty$ in $C(J, X)$. In view of $\gamma_L(\{u_n\}_{n=1}^\infty) = 0$, one has $\chi(\{u_n(t)\}_{n=1}^\infty) = 0$ for all $t \in J$, i.e., $\{u_n(t)\}_{n=1}^\infty$ is relatively compact for each $t \in J$. Hence, $\{u_n[\varphi]_t\}$ is a relatively compact set in $C([-r, 0], X)$. Since $f(t, \cdot)$ is continuous, we get that $\{f(t, u_n[\varphi]_t)\}_{n=1}^\infty$ is relatively compact for $t \in J$. Thus $\{f_n\}_{n=1}^\infty$ is a semicompact sequence, and from Lemma 3.3 we obtain that

$$\Psi_f(\{u_n\}_{n=1}^\infty) = \mathbf{Q}_\alpha(\{f_n\}_{n=1}^\infty) - \mathbf{Q}_\alpha(\{u_n\}_{n=1}^\infty)$$

is relatively compact in $C(J, X)$. A similar argument in the previous proof shows that the set $\{h(t, u_n[\varphi]_t)\}_{n=1}^\infty$ is relatively compact for $t \in J$. Hence $\{h(\cdot, u_n)\}_{n=1}^\infty$ is also a semicompact sequence. Therefore, we get that $\{\Phi_h(u_n)\}_{n=1}^\infty$ is relatively compact in $C(J, X)$, which means that $\{\mathbf{S}(u_n)\}_{n=1}^\infty$ is relatively compact in $C(J, X)$. $\square$

Now we choose $L > 0$ in the definition of MNC γ_L such that

$$\begin{aligned} l := & 2M \int_0^t \sigma(s) e^{-L(t-s)} ds + 2M(1 - e^{-Lt})/L \\ & + L_{h2} \|A^{-\eta}\|_{\mathcal{B}(X)} + 2L_{h2} C_\eta \int_0^t (t-s)^{-\alpha(1-\eta)} e^{-L(t-s)} ds < 1. \end{aligned} \tag{3.1}$$

Lemma 3.5. *Let (Ht), (Hf) and (Hh) hold. Then $\mathbf{S}$ is ν -condensing.*

Proof. Let $\mathbb{B} \subseteq \mathcal{C}_\varphi$ be a bounded set such that

$$\nu(\mathbb{B}) \leq \nu(\mathbf{S}(\mathbb{B})). \tag{3.2}$$

We show below that $\mathbb{B}$ is a relatively compact set, that is, $\nu(\mathbb{B}) = 0$. By the definition of ν , there exists a sequence $\{u_n\}_{n=1}^\infty \subseteq \mathbb{B}$ such that

$$\nu(\mathbf{S}(\{u_n\}_{n=1}^\infty)) = \max \left(\gamma_L(\mathbf{S}(\{u_n\}_{n=1}^\infty)), \text{mod}_C(\mathbf{S}(\{u_n\}_{n=1}^\infty)) \right) \geq \gamma_L(\{u_n\}_{n=1}^\infty).$$

Claim I. $\gamma_L(\{\mathbf{S}(u_n)\}_{n=1}^\infty) = 0$.

For any $t \in J$, we have

$$\begin{aligned}\gamma_L(\{\Psi_f(u_n)\}_{n=1}^\infty) &= \sup_{t \in J} e^{-Lt} \chi(\{\Psi_f(u_n(t))\}_{n=1}^\infty) \\ &= \sup_{t \in J} e^{-Lt} \chi(\{\mathbf{Q}_\alpha(N_f(u_n(t)) - u_n(t))\}_{n=1}^\infty) \\ &\leq \sup_{t \in J} e^{-Lt} \chi(\{\mathbf{Q}_\alpha(N_f(u_n(t)))\}_{n=1}^\infty) + \sup_{t \in J} e^{-Lt} \chi(\{\mathbf{Q}_\alpha(u_n(t))\}_{n=1}^\infty).\end{aligned}$$

On the other hand, from Lemma 2.3 and Lemma 2.5 we get

$$\begin{aligned}\chi(\{\mathbf{Q}_\alpha(N_f(u_n(t)))\}_{n=1}^\infty) &\leq 2 \int_0^t \chi(\mathcal{Q}_\alpha(t-s)\{f(s, u_n[\varphi](s))\}_{n=1}^\infty) ds \\ &\leq 2M \int_0^t \sigma(s) \chi(\{u_n(s)\}_{n=1}^\infty) ds \\ &\leq 2M \gamma_L(\{u_n\}_{n=1}^\infty) \int_0^t \sigma(s) e^{Ls} ds.\end{aligned}$$

Similarly, we have

$$\chi(\{\mathbf{Q}_\alpha(u_n(t))\}_{n=1}^\infty) \leq 2M \gamma_L(\{u_n\}_{n=1}^\infty) (e^{Lt} - 1)/L.$$

Therefore, it follows that

$$\gamma_L(\{\Psi_f(u_n)\}_{n=1}^\infty) \leq \sup_{t \in J} \left[2M \int_0^t \sigma(s) e^{-L(t-s)} ds + 2M(1 - e^{-Lt})/L \right] \gamma_L(\{u_n\}_{n=1}^\infty),$$

which implies that $\gamma_L(\{u_n\}_{n=1}^\infty) = 0$ from $l < 1$. We thus obtain that $\gamma_L(\{\Psi_f(u_n)\}_{n=1}^\infty) = 0$.

Moreover, since the measure χ is monotone, from (Hh), for $t \in J$ we get

$$\begin{aligned}\gamma_L(\{h(t, u_n[\varphi]_t)\}_{n=1}^\infty) &= \sup_{t \in J} e^{-Lt} \chi(\{A^{-\eta} A^\eta h(t, u_n[\varphi]_t)\}_{n=1}^\infty) \\ &\leq L_{h2} \|A^{-\eta}\|_{\mathcal{B}(X)} \sup_{t \in J} e^{-Lt} \chi(\{u_n[\varphi]_t\}_{n=1}^\infty) \\ &\leq L_{h2} \|A^{-\eta}\|_{\mathcal{B}(X)} \sup_{t \in J} e^{-Lt} \left(\sup_{s \in [0, t]} \chi(\{u_n(s)\}_{n=1}^\infty) \right) \\ &\leq L_{h2} \|A^{-\eta}\|_{\mathcal{B}(X)} \gamma_L(\{u_n\}_{n=1}^\infty).\end{aligned}$$

On the other hand, from Lemma 2.8 we have

$$\begin{aligned}\chi(\{A\mathcal{R}_\alpha(t-s)h(s, u_n[\varphi]_s)\}_{n=1}^\infty) &= \chi(\{A^{1-\eta}\mathcal{R}_\alpha(t-s)A^\eta h(s, u_n[\varphi]_s)\}_{n=1}^\infty) \\ &\leq \|A^{1-\eta}\mathcal{R}_\alpha(t-s)\|_{\mathcal{B}(X)} \chi(\{A^\eta h(s, u_n[\varphi]_s)\}_{n=1}^\infty) \\ &\leq C_\eta L_{h2} (t-s)^{-\alpha(1-\eta)} \chi(\{u_n[\varphi]_s\}_{n=1}^\infty) \\ &\leq C_\eta L_{h2} (t-s)^{-\alpha(1-\eta)} e^{Ls} \gamma_L(\{u_n\}_{n=1}^\infty).\end{aligned}$$

By Lemma 2.4, one obtains

$$\begin{aligned}&\int_0^t (t-s)^{\alpha-1} \chi(\{A\mathcal{R}_\alpha(t-s)h(s, u_n[\varphi]_s)\}_{n=1}^\infty) ds \\ &\leq C_\eta L_{h2} \gamma_L(\{u_n\}_{n=1}^\infty) \int_0^t (t-s)^{-\alpha(1-\eta)} e^{Ls} ds.\end{aligned}$$

This turns out that $\gamma_L(\{u_n\}_{n=1}^\infty) \leq l \gamma_L(\{u_n\}_{n=1}^\infty)$. By the choice of $l \in (0, 1)$, it is easy to see that $\gamma_L(\{u_n\}_{n=1}^\infty) = 0$ and then $\gamma_L(\{\Phi_h(u_n)\}_{n=1}^\infty) = 0$. Together the above arguments, we

obtain the desired claim.

Claim II. $\text{mod}_C(\{u_n\}_{n=1}^\infty) = 0$, that is, the set $\mathbb{B}$ is equicontinuous. Firstly, by using the assumption (Hf) and the same arguments in Lemma 3.4, one can get that $\{f(t, u_n[\varphi](t))\}_{n=1}^\infty$ is a semicompact sequence. Then Definition 2.4 (ii) ensures that the set $\{\Psi_f(u_n)(t)\}_{n=1}^\infty$ is relatively compact in $C(J, X)$. Hence, $\text{mod}_C(\{\Psi_f(u_n)\}_{n=1}^\infty) = 0$. In fact, we just prove $\{\mathbf{Q}_\alpha(f_n)(t)\}_{n=1}^\infty$ is equicontinuous in $C(J, X)$, and the case of $\{\mathbf{Q}_\alpha(u_n)(t)\}_{n=1}^\infty$ follows.

For $0 \leq t_1 < t_2 \leq a$, by (Hf) we have

$$\begin{aligned} \|\mathbf{Q}_\alpha(f_n)(t_2) - \mathbf{Q}_\alpha(f_n)(t_1)\| &\leq \int_{t_1}^{t_2} \|\mathcal{Q}_\alpha(t_2 - s)f(s, u[\varphi]_s)\| ds \\ &\quad + \int_0^{t_1} \|(\mathcal{Q}_\alpha(t_2 - s) - \mathcal{Q}_\alpha(t_1 - s))f(s, u[\varphi]_s)\| ds \\ &\leq M \int_{t_1}^{t_2} \sigma(s) \|u[\varphi]_s\|_C ds \\ &\quad + \int_0^{t_1} \|(\mathcal{Q}_\alpha(t_2 - s) - \mathcal{Q}_\alpha(t_1 - s))\|_{\mathcal{B}(X)} \sigma(s) \|u[\varphi]_s\|_C ds. \end{aligned}$$

Since

$$\|u[\varphi]_t\|_C \leq \sup_{s \in [0, t]} \|u(s)\| + \|\varphi\|_C,$$

and $\sigma \in L^1(J, \mathbb{R}^+)$, it follows that

$$\int_{t_1}^{t_2} \sigma(s) \|u[\varphi]_s\|_C ds \rightarrow 0, \text{ as } t_2 \rightarrow t_1.$$

On the other hand, by (Ht), for any $\varepsilon > 0$

$$\begin{aligned} &\int_0^{t_1} \|(\mathcal{Q}_\alpha(t_2 - s) - \mathcal{Q}_\alpha(t_1 - s))\|_{\mathcal{B}(X)} \sigma(s) \|u[\varphi]_s\|_C ds \\ &\leq 2M \int_0^{t_1 - \varepsilon} \sigma(s) \|u[\varphi]_s\|_C ds \\ &\quad + \int_{t_1 - \varepsilon}^{t_1} \sigma(s) \|u[\varphi]_s\|_C ds \sup_{s \in [t_1 - \varepsilon, t_1]} \|\mathcal{Q}_\alpha(t_2 - s) - \mathcal{Q}_\alpha(t_1 - s)\|_{\mathcal{B}(X)} \\ &\rightarrow 0, \text{ as } t_2 \rightarrow t_1, \varepsilon \rightarrow 0. \end{aligned}$$

Therefore, it follows that

$$\|\mathbf{Q}_\alpha(f_n)(t_2) - \mathbf{Q}_\alpha(f_n)(t_1)\| \rightarrow 0, \text{ as } t_2 \rightarrow t_1,$$

which shows that $\{\mathbf{Q}_\alpha(f_n)(t)\}_{n=1}^\infty$ is equicontinuous.

Next we show that $\{\Phi_h(u_n)(t)\}_{n=1}^\infty$ is equicontinuous. For $0 \leq t_1 < t_2 \leq a$, one has

$$\begin{aligned} \|\Phi_h(u_n)(t_2) - \Phi_h(u_n)(t_1)\| &\leq \|(\mathcal{Q}_\alpha(t_2) - \mathcal{Q}_\alpha(t_1))h(0, u[\varphi]_0)\| \\ &\quad + \|h(t_2, u_n[\varphi]_{t_2}) - h(t_1, u_n[\varphi]_{t_1})\| \\ &\quad + \|\mathbf{R}_\alpha(u_n)(t_2) - \mathbf{R}_\alpha(u_n)(t_1)\|, \end{aligned}$$

in which

$$\mathbf{R}_\alpha(u_n)(t) := \int_0^t (t-s)^{\alpha-1} A \mathcal{R}_\alpha(t-s) h(s, u_n[\varphi]_s) ds.$$

In view of Lemma 2.6 and (Hh), we get

$$\|(\mathcal{Q}_\alpha(t_2) - \mathcal{Q}_\alpha(t_1))h(0, u[\varphi]_0)\| + \|h(t_2, u_n[\varphi]_{t_2}) - h(t_1, u_n[\varphi]_{t_1})\| \rightarrow 0, \text{ as } t_2 \rightarrow t_1.$$

For any $\{u_n\}_{n=1}^\infty \subset \mathbb{B}$, it follows from the previous proof that we obtain that $\{\mathbf{R}_\alpha(u_n)\}_{n=1}^\infty$ is equicontinuous in $C(J, X)$. It turns out that $\text{mod}_C(\{\mathbf{R}_\alpha(u_n)\}_{n=1}^\infty) = 0$. Thus, we obtain $\text{mod}_C(\{\Phi_h(u_n)\}_{n=1}^\infty) = 0$. The desired result is obtained. $\square$

Lemma 3.6. *Let (Hf) and (Hh) hold. Then $\mathbf{S}$ is continuous.*

Proof. Let B be a bounded subset of $\mathcal{C}_\varphi$ and let $\{u_n\}_{n=1}^\infty$ be a sequence of B such that $u_n \rightarrow u$ in B as $n \rightarrow \infty$. Due to the continuity of f and h , we have

$$f(s, u_n[\varphi]_s) \rightarrow f(s, u[\varphi]_s), \text{ and } h(s, u_n[\varphi]_s) \rightarrow h(s, u[\varphi]_s), \text{ as } n \rightarrow \infty.$$

Since from (Hh) and Lemma 2.6

$$\|(t-s)^{\alpha-1} A \mathcal{R}_\alpha(t-s)(h(s, u_n[\varphi]_s) - h(s, u[\varphi]_s))\| \leq 2L_{h1} C_\eta (t-s)^{\alpha\eta-1} \sup_{v \in B} \|v(s)\|,$$

$$\text{and } \|u_n[\varphi]_s - u[\varphi]_s\|_{\mathcal{C}} \leq 2(\sup_{v \in B} \|v(s)\| + \|\varphi\|_{\mathcal{C}}),$$

$$\|f(s, u_n[\varphi]_s) - f(s, u[\varphi]_s)\| \leq 2\sigma(s) \|u[\varphi]_s\|_{\mathcal{C}}, \text{ for } s \in [0, t].$$

In addition, function $s \mapsto \sigma(s)$ is L^1 -integral for a.e. $s \in [0, t]$, it follows from the Lebesgue dominated convergence theorem that

$$\begin{aligned} \|\mathbf{S}(u_n)(t) - \mathbf{S}(u)(t)\| &\leq \|\Psi_f(u_n)(t) - \Psi_f(u)(t)\| + \|\Phi_h(u_n)(t) - \Phi_h(u)(t)\| \\ &\leq \|h(s, u_n[\varphi]_s) - h(s, u[\varphi]_s)\| \\ &\quad + \left\| \int_0^t (t-s)^{\alpha-1} A \mathcal{R}_\alpha(t-s)(h(s, u_n[\varphi]_s) - h(s, u[\varphi]_s)) ds \right\| \\ &\quad + \left\| \int_0^t \mathcal{Q}_\alpha(t-s)(f(s, u_n[\varphi]_s) - f(s, u[\varphi]_s)) ds \right\| \\ &\quad + \left\| \int_0^t \mathcal{Q}_\alpha(t-s)(u_n[\varphi]_s - u[\varphi]_s) ds \right\| \\ &\rightarrow 0, \text{ as } n \rightarrow \infty. \end{aligned}$$

Thus, $\|\mathbf{S}(u_n) - \mathbf{S}(u)\| \rightarrow 0$ as $n \rightarrow \infty$. This means that $\mathbf{S}$ is continuous. $\square$

Based on the previous results, we get the existence of mild solutions.

Theorem 3.1. *Assume that (Ht), (Hf) and (Hh) are satisfied. Then the solution set of problem (1.1) is nonempty.*

Proof. By Lemma 3.5, the operator $\mathbf{S}$ is ν -condensing as well as it is continuous due to Lemma 3.6, hence it remains to check that $\mathbf{S}$ maps a bounded convex closed set into itself. To achieve this aim, let $\xi \in C(J, X)$ be the solution of the integral equation given by

$$\xi(t) = l_\varphi + \zeta(t), \tag{3.3}$$

where

$$l_\varphi := \frac{\|\varphi\|_{\mathcal{C}}}{1 - L_{h2} \|A^{-\eta}\|_{\mathcal{B}(X)}} (\|\varphi(0)\| + 2ML_{h2} \|A^{-\eta}\|_{\mathcal{B}(X)} + C_\eta L_{h2} a^{\alpha\eta} / (\alpha\eta) + M \|\sigma\|_{L^1(J, \mathbb{R}^+)}),$$

$$\zeta(t) := \frac{1}{1 - L_{h2}\|A^{-\eta}\|_{\mathcal{B}(X)}} \int_0^t (t-s)^{\alpha\eta-1} (C_\eta L_{h2} + M(\sigma(s) + 1)(t-s)^{1-\alpha\eta}) e^{-L(t-s)} \xi(s) ds.$$

In fact, we can find a solution of ξ satisfying (3.3) and belonging to $C(J, X)$, let

$$g(t, \xi(t)) := \frac{C_\eta L_{h2}}{1 - L_{h2}\|A^{-\eta}\|_{\mathcal{B}(X)}} e^{Lt} \xi(t), \quad k(t, \xi(t)) := \frac{M}{1 - L_{h2}\|A^{-\eta}\|_{\mathcal{B}(X)}} (\sigma(t) + 1) e^{Lt} \xi(t).$$

Obviously, $g, k \in L^1(J, \mathbb{R}^+)$ for any $\xi \in C(J, X)$, we set an operator

$$\mathcal{R}\xi(t) = l_\varphi + \zeta(t),$$

for any $\xi \in C(J, X)$, and then we rewrite the integral equation into the following equation

$$\mathcal{R}\xi(t) = l_\varphi + e^{-Lt} I_{0+}^{\alpha\eta} g(t, \xi(t)) + e^{-Lt} I_{0+}^1 k(t, \xi(t)).$$

Observe that $\mathcal{R}\xi \in C(J, X)$ for any $\xi \in C(J, X)$ from the continuity of the integral, and for any $\xi_1, \xi_2 \in C(J, X)$, by (3.1) we have

$$\begin{aligned} e^{-Lt} \|\mathcal{R}\xi_1(t) - \mathcal{R}\xi_2(t)\| &\leq e^{-2Lt} I_{0+}^{\alpha\eta} \|g(t, \xi_1(t)) - g(t, \xi_2(t))\| + e^{-2Lt} I_{0+}^1 \|k(t, \xi_1(t)) - k(t, \xi_2(t))\| \\ &\leq \frac{C_\eta L_{h2}}{1 - L_{h2}\|A^{-\eta}\|_{\mathcal{B}(X)}} \int_0^t (t-s)^{\alpha\eta-1} e^{-2L(t-s)} ds \|\xi_1 - \xi_2\|_{C(J, X)} \\ &\quad + \frac{M}{1 - L_{h2}\|A^{-\eta}\|_{\mathcal{B}(X)}} \int_0^t (\sigma(s) + 1) e^{-2L(t-s)} ds \|\xi_1 - \xi_2\|_{C(J, X)} \\ &\leq \frac{l - L_{h2}\|A^{-\eta}\|_{\mathcal{B}(X)}}{1 - L_{h2}\|A^{-\eta}\|_{\mathcal{B}(X)}} \|\xi_1 - \xi_2\|_{C(J, X)} \\ &< \|\xi_1 - \xi_2\|_{C(J, X)}. \end{aligned}$$

Thus, by using the Banach fixed point theorem, it is easy to see that the operator equation $\mathcal{R}\xi = \xi$ is satisfied and it belongs to $C(J, X)$. Set

$$D = \left\{ u \in \mathcal{C}_\varphi : \sup_{s \in [0, t]} e^{-Ls} \|u(s)\| \leq \xi(t), \quad t \in J \right\}.$$

It is clear that D is bounded, closed and convex set of $\mathcal{C}_\varphi$. In addition, for $u \in D$, in view of (Hf), (Hh) and Lemma 2.8, we have

$$\begin{aligned} \|\mathbf{S}(u)(t)\| &\leq \|\Psi_f(u)(t)\| + \|\Phi_h(u)(t)\| \\ &\leq \|\varphi(0)\| + \|\mathcal{Q}_\alpha(t)h(0, u[\varphi]_0)\| \\ &\quad + \|h(t, u[\varphi]_t)\| + \int_0^t (t-s)^{\alpha-1} \|A\mathcal{R}_\alpha(t-s)h(s, u[\varphi]_s)\| ds + \|\mathbf{Q}_\alpha(N_f(u) - u)\| \\ &\leq \|\varphi(0)\| + ML_{h2}\|A^{-\eta}\|_{\mathcal{B}(X)}\|u[\varphi]_0\|_C + L_{h2}\|A^{-\eta}\|_{\mathcal{B}(X)}\|u[\varphi]_t\|_C \\ &\quad + C_\eta L_{h2} \int_0^t (t-s)^{\alpha\eta-1} \|u[\varphi]_s\|_C ds + M \int_0^t \sigma(s) \|u[\varphi]_s\|_C ds + M \int_0^t \|u(s)\| ds. \end{aligned}$$

Therefore, one can see that

$$\begin{aligned} \|\mathbf{S}(u)(t)\| &\leq \|\varphi(0)\| + 2ML_{h2}\|A^{-\eta}\|_{\mathcal{B}(X)}\|\varphi\|_C + L_{h2}\|A^{-\eta}\|_{\mathcal{B}(X)} e^{Lt} \xi(t) \\ &\quad + C_\eta L_{h2} \int_0^t (t-s)^{\alpha\eta-1} e^{Ls} \xi(s) ds + C_\eta L_{h2} \|\varphi\|_C t^{\alpha\eta} / (\alpha\eta) \\ &\quad + M \int_0^t (\sigma(s) + 1) e^{Ls} \xi(s) ds + M \|\sigma\|_{L^1(J, \mathbb{R}^+)} \|\varphi\|_C, \end{aligned}$$

which implies that $e^{-Lt} \|\mathbf{S}(u)(t)\| \leq \xi(t)$. Therefore, we get $\mathbf{S}(D) \subseteq D$. The application of

Theorem 2.1 yields the conclusion of existence result. The proof is completed. $\square$

Note that if $T(t)$ is compact for every $t > 0$, then $\mathcal{Q}_\alpha(t)$ is compact for every $t > 0$ in view of Lemma 2.7, hence it is continuous in the operator topology for $t > 0$, the condition (Ht) holds immediately.

Theorem 3.2. Assume that $T(t)$ is compact for every $t > 0$, (Hf) and (Hh) are satisfied. Then the solution set of problem (1.1) is nonempty.

Example 3.1. Let $X = L^2([0, \pi], \mathbb{R})$ and let us consider the following delay fractional diffusion equations

$$\begin{cases} \partial_t(u(t, x) - h(t, x, u_t)) = \partial_t^{1-\alpha} \partial_{xx} u(t, x) - u(t, x) + g(t, x, u_t), & x \in [0, \pi], t \in [0, 1]; \\ u(t, 0) = u(t, \pi) = 0, & t \in [0, 1]; \\ u(t, x) = \varphi(t, x), & x \in [0, \pi], t \in [-1, 0]. \end{cases}$$

where ∂_t^α is the Caputo fractional partial derivative of order $0 < \alpha < 1$, g is a given function, $h : [0, 1] \times C([-1, 0], X) \rightarrow X$ is defined by

$$h(t, x, u_t) = c \int_0^\pi U(x, y) u_t(\theta, y) dy,$$

where $c > 0$ is a suitable constant, function U and $\partial_x U$ are measurable and $U(0, y) = U(\pi, y) = 0$,

$$\int_0^\pi \int_0^\pi U^2(x, y) dy dx + \int_0^\pi \int_0^\pi (\partial_x U(x, y))^2 dy dx < \infty.$$

Let A be the operator defined by $Av = v''$ with the domain

$$D(A) = \{v \in X : v, v' \text{ absolutely continuous, } v'' \in X, v(0) = v(\pi) = 0\}.$$

Then A generates an uniformly bounded compact analytic semigroup $\{T(t)\}_{t \geq 0}$. The problem can be reformulated as the following delay problem

$$\begin{cases} (w(t) - h(t, w_t))' = {}^C D_{0+}^{1-\alpha} A w(t) - w(t) + f(t, w_t(t)), & t \in [0, 1]; \\ w(t) = \varphi(t), & t \in [-1, 0]. \end{cases}$$

where $w(t) = u(t, \cdot)$, that is $w(t)(x) = u(t, x)$, $t \in [0, 1]$, $x \in [0, \pi]$. The function $f : J \times X \rightarrow X$ is given by

$$f(t, w_t)(x) = g(t, x, u_t(t, x)) = k e^{-t} \sin(u(t + \theta))(x), \quad \theta \in [-1, 0], \quad k > 0.$$

Then the assumptions (Hf), (Hh) are satisfied for some suitable $c, k > 0$. According to Theorem 3.1, the problem has a mild solution in $\mathcal{C}_\varphi$.

§4 Conclusion

In this paper, we are concerned with the existence of solutions for a class of neutral fractional differential equations of order $0 < \alpha < 1$. By using the techniques of fractional calculus, measure of noncompactness and the fixed point argument, some sufficient conditions are formulated to guarantee the existence of the solutions for such equations in composite relaxation process. Due to the abstract fractional differential equations in composite relaxation process can be used to

simulate the Basset problem, it is interesting to consider the existence and controllability results in the future works.

Acknowledgements

We thank the reviewers for their constructive comments that led to the improvement of the manuscript.

Declarations

Conflict of interest The authors declare no conflict of interest.

References

- [1] E Affili, E Valdinoci. *Decay estimates for evolution equations with classical and fractional time-derivatives*, J Differ Equ, 2019, 266(7): 4027-4060.
- [2] A Aparcana, C Cuevas, H Henríquez, et al. *Fractional evolution equations and applications*, Math Meth Appl Sci, 2018, 41: 1256-1280.
- [3] A Ashyralyev. *Well-posedness of the Basset problem in spaces of smooth functions*, Appl Math Letters, 2011, 24: 1176-1180.
- [4] J Banaś. *On measure of Noncompactness in Banach Spaces*, Comment Math Univ Carol, 1980, 21(1): 131-143.
- [5] D Bothe. *Multivalued perturbations of m -accretive differential inclusions*, Israel J Math, 1998, 108: 109-138.
- [6] C Dineshkumar, R Udhayakumar, V Vijayakumar, et al. *A discussion on the approximate controllability of Hilfer fractional neutral stochastic integro-differential systems*, Chaos Solitons Fractals, 2021, 142: 110472.
- [7] H Dong, D Kim. *L_p -estimates for time fractional parabolic equations with coefficients measurable in time*, Adv Math, 2019, 345: 289-345.
- [8] Z Fan, Q Dong, G Li. *Approximate controllability for semilinear composite fractional relaxation equations*, Fract Calc Appl Anal, 2016, 19(1): 267-284.
- [9] R Gorenflo, F Mainardi. *Fractional calculus: integral and differential equations of fractional order*, 2008, arXiv.0805.3823.
- [10] J W He, Y Zhou, L Peng, et al. *On well-posedness of semilinear Rayleigh-Stokes problem with fractional derivative on $\mathbb{R}^N$* , Adv Nonlinear Anal, 2022, 11: 580-597.
- [11] M Kamenskii, V Obukhovskii, P Zecca. *Condensing Multivalued Maps and Semilinear Differential Inclusions in Banach Spaces*, Walter de Gruyter, New York, 2001.
- [12] K Kavitha, V Vijayakumar, R Udhayakumar, et al. *Results on controllability of Hilfer fractional differential equations with infinite delay via measures of noncompactness*, Asian J Control, 2022, 24(3): 1406-1415.

- [13] A A Kilbas, H M Srivastava, J J Trujillo. *Theory and applications of fractional differential equations*, Elsevier, 2006.
- [14] C Lizama, G M N'Guérékata. *Mild solutions for abstract fractional differential equations*, Appl Anal, 2013, 92(8): 1731-1754.
- [15] C Lizama, H Prado. *On duality and spectral properties of (a, k) -regularized resolvents*, Proc R Soc Edinb Sect A, 2009, 139(3): 505-517.
- [16] H Monch. *Boundary value problems for nonlinear ordinary differential equations of second order in Banach spaces*, Nonlinear Anal, 1980, 49: 985-999.
- [17] G Mophou, S Tao, C Joseph. *Initial value/boundary value problem for composite fractional relaxation equation*, Appl Math Comput, 2015, 257: 134-144.
- [18] I Podlubny. *Fractional differential equations*, San Diego Academic Press, 1999.
- [19] R N Wang, D H Chen, T J Xiao. *Abstract fractional Cauchy problems with almost sectorial operators*, J Differ Equ, 2012, 252: 202-235.
- [20] W K Williams, V Vijayakumar, R Udhayakumar, et al. *A new study on existence and uniqueness of nonlocal fractional delay differential systems of order $1 < r < 2$ in Banach spaces*, Numer Methods Partial Differ Equ, 2021, 37: 949-961.
- [21] Y Q Wu, J W He. *Regularization of a backward problem for composite fractional relaxation equations*, Math Meth Appl Sci, 2022, 46(1): 180-196.
- [22] R Zacher. *A De Giorgi-Nash type theorem for time fractional diffusion equations*, Math Ann, 2013, 356(1): 99-146.
- [23] Q Zhang, Y Li. *Global well-posedness and blow-up solutions of the Cauchy problem for a time-fractional superdiffusion equation*, J Evol Equ, 2019, 19: 271-303.
- [24] Y Zhou. *Basic Theory of Fractional Differential Equations*, World Scientific, Singapore, 2014.
- [25] Y Zhou, J W He. *New results on controllability of fractional evolution systems with order $\alpha \in (1, 2)$* , Evol Equ Control Theory, 2021, 10(3): 491-509.
- [26] Y Zhou, J W He. *Well-posedness and regularity for fractional damped wave equations*, Monatsh Math, 2021, 194(2): 425-458.

¹School of Mathematics and Information Science, Guangxi University, Nanning 530004, China.

²Center for Applied Mathematics of Guangxi, Guangxi University, Nanning 530004, China.

Email: jwhe@gxu.edu.cn