Appl. Math. J. Chinese Univ.
2025, 40(2): 327-342

Pricing power option under NIG model using fast Fourier transform

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Abstract. The aim of this paper is to price power option with its underlying asset price following exponential normal inverse gaussian (NIG) process. We first find the risk neutral equivalent martingale measure Q by Esscher transform. Then, using the Fourier transform and its inverse, we derive the analytical pricing formulas of power options which are expressed in the form of Fourier integral. In addition, the fast Fourier transform (FFT) algorithm is applied to calculate these pricing formulas. Finally, Shangzheng 50ETF options are chosen to test our results. Estimating the parameters in NIG process by maximum likelihood method, we show that the NIG prices are much closer to market prices than the Black-Scholes-Merton (BSM) ones.

§1 Introduction

Nowadays, options are very popular financial derivatives and have been traded all over the world. There is a great variety of exotic options designed to meet the particular demands of financial market participants (see [26]). The power option is a type of exotic options whose payoff depends on some power of the underlying asset price at maturity. Compared with a plain vanilla option whose payoff is piecewise-linear, the payoff of a power option is a nonlinear function and can afford great flexibility and a substantial amount of leverage. Consequently, it has been extensively used in financial markets. For example, Bankers Trust in Germany issued capped symmetric FX power options on US dollars, Swiss Francs and Japanese Yen with a power of order 2; Polynomial options on Nikkei were issued in Tokyo. We refer to [9,16,24,25] for many examples of the instruments with power option payoffs.

It is well known that the distribution form of the underlying asset price plays a key role in valuing derivative securities. Black and Scholes [3] and Merton [18] obtained the pricing formula of plain vanilla option under assumption that the price process of underlying asset follows geometric Brownian motion, with constant risk-free interest rate r and constant volatility σ.

Received: 2020-09-01. Revised: 2021-09-01.
MR Subject Classification: 60A10, 65T50, 91B24, 91G20.
Keywords: power option, NIG process, Esscher transform, Fourier transform, FFT algorithm.
Digital Object Identifier(DOI): https://doi.org/10.1007/s11766-025-4249-2.
Supported by National Natural Science Foundation of China(11571089, 11501164), Natural Science Foundation of Hebei Province(A2019205299), the Foundation of Hebei Education Department(ZD2018065, ZD2019053) and Hebei Normal University(L2019Z01).

This classic model is called the Black-Scholes-Merton model (BSM henceforth) and has been widely used among practitioners for pricing options. However, Bakshi et al. [1] and Rubinstein [21] showed that the logarithmic return of the underlying asset does not follow the normal distribution, but has the characteristic of peak and fat tail. In spite of the great success, BSM model fails to reflect these characteristics. During the last four decades, there have been numerous efforts to improve BSM model. Kou [14], Merton [19] and Zhou [27] introduced jumps into BSM model, while Heston [8] and Hull and White [11] suggested a stochastic volatility model. In addition, various Lévy process models have also been applied to option pricing such as variance gamma model [6,15,17] and tempered stable model [4].

Barndorff-Nielsen [2] introduced the *Normal inverse Gaussian* (NIG) distribution which is a variance-mean mixture of a normal distribution with the inverse Gaussian as the mixing distribution. This distribution has lots of good qualities, such as half heavy tail and infinite divisibility. It provides a very good fit to the distributions of logarithmic asset returns. The NIG distribution determines a homogeneous Lévy process, that is, NIG process, and this process can also be obtained by replacing the fixed time in Brownian motion by the inverse Gaussian process. It is therefore natural to model the logarithmic stock price processes as NIG process.

Heynen and Kat [9] and Zhang [26] obtained the pricing formula of power option under BSM model. Kim et al. [13] valued the power option under Heston's volatility model and derived a semi-analytic pricing formula. Macovschi and Quittard-Pinon [16] discussed the power option pricing in BSM model, Heston's model and a pure jump Lévy process respectively. To the best of our knowledge, there is no literature studying the power option pricing problem under NIG model. This is the motivation of this study.

One of the most important approaches to value derivative securities is the martingale pricing. Under the risk neutral equivalent martingale measure, the price of option is the discounted expectation of the exercise payoff, namely, the integral of the product of the exercise payoff and the density function. As we know, the characteristic function of NIG process is simpler than its density function, and the Fourier transform of density function is exactly the characteristic function. So we can express the pricing formula of power option into the Fourier integral of its characteristic function by Fourier transform and its inverse. Subsequently, we can use the *fast Fourier transform* (FFT) algorithm to obtain the power option price across the whole spectrum of exercise price. Ibrahim et al. [12] have shown that the FFT can be used to price power options under BSM model.

The rest of this paper is organized as follows. In section 2, we give the financial market setting and find the risk neutral equivalent martingale measure by Esscher transform. Section 3 derives power option pricing formulas by Fourier transform and inverse Fourier transform and introduces the FFT algorithm to calculate Fourier integral. Section 4 gives a numerical example.

§2 Preliminary

In this section, we begin by introducing the financial market setting. Then we find the risk neutral equivalent martingale measure. Given any $0 < T < +\infty$, let $(\Omega, \mathcal{F}, P, \{\mathcal{F}_t\}_{0\leq t\leq T})$ be a probability space, where Ω is the sample space, P is the physical measure representing the real world probability or the historical probability and $\{\mathcal{F}_t\}_{0\le t\le T}$ is the natural filtration with $\mathcal{F} = \sigma\left(\bigcup_{0\le t\le T}\mathcal{F}_t\right)$. Write $\mathbb{E}_P$ as the expectation under the probability measure P.

2.1 Financial Market Setting

The NIG distribution with parameters α, β, δ, μ $(NIG(\alpha,\beta,\delta,\mu))$ is continuously distributed with density function defined as (see [2])

$$f(x) = \alpha\delta\exp\left(\delta\sqrt{\alpha^2-\beta^2}+\beta(x-\mu)\right)\frac{K_1(\alpha\sqrt{\delta^2+(x-\mu)^2})}{\pi\sqrt{\delta^2+(x-\mu)^2}}, \quad x\in\mathbf{R}, \tag{1}$$

where $K_1(\cdot)$ denotes the modified Bessel function of the third kind with index 1, and parameters α (shape), β (skewness), δ (scale), μ (location) satisfy $\alpha>0$, $0\le|\beta|<\alpha$, $\delta>0$ and $\mu\in\mathbf{R}$. The characteristic function $\varphi(z)$ of the $NIG(\alpha,\beta,\delta,\mu)$ has the following explicit expression

$$\varphi(z) = \exp\left(\delta\left(\sqrt{\alpha^2-\beta^2}-\sqrt{\alpha^2-(\beta+\mathrm{i}z)^2}\right)+\mathrm{i}\mu z\right), \quad z\in\mathbf{R}. \tag{2}$$

where i is the imaginary unit.

The NIG process $\{L(t)\}_{0\le t\le T}$ with parameters α, β, δ, μ is a Markov process satisfying the following conditions (see [2,22]):

(i) $L(0)=0$;

(ii) $\{L(t)\}_{0\le t\le T}$ has independent and stationary increments;

(iii) $L(t)$ has $NIG(\alpha,\beta,t\delta,t\mu)$ distribution.

By the definition of $\{L(t)\}_{0\le t\le T}$, we know that it is a Lévy process.

In this paper, we consider a financial market with two primary assets: a risk-free asset and a risky asset. The risk-free asset is a money market account with earning continuous compound interest rate r which is assumed to be constant. The risky asset (underlying asset), such as a stock or a stock index, provides fixed continuous compound yield q, and its price process $\{S(t)\}_{0\le t\le T}$ is an exponential of NIG process $\{L(t)\}_{0\le t\le T}$ with parameters α, β, δ, μ, i.e.

$$S(t) = S(0)e^{L(t)}, \quad 0\le t\le T. \tag{3}$$

2.2 Risk Neutral Equivalent Martingale Measure

A probability measure Q defined on $(\Omega,\mathcal{F})$ is a *risk neutral equivalent martingale measure* if

- Q is equivalent to P, i.e. they have the same null sets.
- The discounted price process $\{e^{-(r-q)t}S(t)\}_{0\le t\le T}$ is a martingale under Q.

If there is a risk neutral equivalent martingale measure Q in financial market, we can obtain the price of a derivative security by calculating the expectation of the discounted payoff according to the fundamental theorem of asset pricing. There are several methods to find the risk neutral equivalent martingale measure, such as Esscher transform [7], the minimal entropy martingale measure [5] and the mean-correcting martingale measure [6,22]. The method of Esscher transform is efficient in our case because the risk neutral Esscher measure preserves the Lévy structure of the process.

Lemma 2.1 *([23]) Let $\{\Lambda(t)\}_{0\le t\le T}$ be a positive P-martingale such that $\mathbb{E}_P[\Lambda(T)]=1$. Define the new probability measure $\widetilde{P}$ by the relation $\frac{\mathrm{d}\widetilde{P}}{\mathrm{d}P}=\Lambda(T)$. Then $\widetilde{P}$ is equivalent to P, and for any $\mathcal{F}_t$-measurable random variable Y, $0\le t\le T$,*

$$\mathbb{E}_{\widetilde{P}}[Y]=\mathbb{E}_P[Y\Lambda(t)].$$

From (2), the moment-generating function $M(u,t)$ of the $L(t)$ exists if and only if $u\in[-\alpha-\beta,\alpha-\beta]$ and

$$M(u,t)=\mathbb{E}_P[e^{uL(t)}]=\exp\left(t\delta\left(\sqrt{\alpha^2-\beta^2}-\sqrt{\alpha^2-(\beta+u)^2}\right)+\mu ut\right). \tag{4}$$

Lemma 2.2 *Let $u\in[-\alpha-\beta,\alpha-\beta]$ and*

$$\Lambda(u,t)=\frac{e^{uL(t)}}{\mathbb{E}_P[e^{uL(t)}]}=\frac{e^{uL(t)}}{M(u,t)}, \tag{5}$$

then $\{\Lambda(u,t)\}_{0\le t\le T}$ is a P-martingale.

Proof. Because of the independent and stationary increments of the NIG process $\{L(t)\}_{0\le t\le T}$, we have, for $0\le s\le t\le T$,

$$\begin{aligned}\mathbb{E}_P[\Lambda(u,t)|\mathcal{F}_s]&=\frac{1}{M(u,t)}\mathbb{E}_P[e^{uL(t)}|\mathcal{F}_s]\\
&=\frac{e^{uL(s)}}{M(u,t)}\mathbb{E}_P[e^{u(L(t)-L(s))}|\mathcal{F}_s]\\
&=\frac{e^{uL(s)}}{M(u,t)}\mathbb{E}_P[e^{u(L(t)-L(s))}]\\
&=\frac{e^{uL(s)}}{M(u,t)}\mathbb{E}_P[e^{uL(t-s)}]\\
&=\frac{e^{uL(s)}M(u,t-s)}{M(u,t)}\\
&=\frac{e^{uL(s)}}{M(u,s)}=\Lambda(u,s).\end{aligned}$$

Thus, Lemma 2.2 has been proved. □

From Lemma 2.1 and Lemma 2.2, for every $u\in[-\alpha-\beta,\alpha-\beta]$, we can define a new probability measure P_u by the relation $\frac{\mathrm{d}P_u}{\mathrm{d}P}=\Lambda(u,T)$. In order to find the risk neutral equivalent martingale measure, we need to seek θ such that the discounted process $\{e^{-(r-q)t}S(t)\}_{0\le t\le T}$ is a martingale under P_θ. Therefore, θ is a solution to the equation

$$S(0)=\mathbb{E}_{P_\theta}[e^{-(r-q)t}S(t)]. \tag{6}$$

Noticing that $L(t)$ is $\mathcal{F}_t$-measurable, by Lemma 2.1, (3), (4) and (5), one has

$$\begin{aligned}&\mathbb{E}_{P_\theta}[e^{-(r-q)t}S(t)]\\
=\;&e^{-(r-q)t}S(0)\mathbb{E}_{P_\theta}[e^{L(t)}]\\
=\;&e^{-(r-q)t}S(0)\mathbb{E}_P[e^{L(t)}\Lambda(\theta,t)]\\
=\;&e^{-(r-q)t}S(0)M(\theta+1,t)/M(\theta,t)\\
=\;&e^{-(r-q)t}S(0)\exp\left(t\delta\left(\sqrt{\alpha^2-(\beta+\theta)^2}-\sqrt{\alpha^2-(\beta+\theta+1)^2}\right)+\mu t\right). \qquad (7)\end{aligned}$$

From (6) and (7), θ is the solution to the following equation:

$$r-q-\mu=\delta\left(\sqrt{\alpha^2-(\beta+\theta)^2}-\sqrt{\alpha^2-(\beta+\theta+1)^2}\right). \tag{8}$$

In what follows, we will show that Equation (8) has a unique solution under some mild conditions. ***Lemma 2.3*** *When $\alpha > \frac{1}{2}$ and $-\sqrt{2\alpha-1} < (r-q-\mu)/\delta < \sqrt{2\alpha-1}$, Equation (8) has a unique solution on interval $(-\alpha-\beta, \alpha-\beta-1)$.*

Proof. Let

$$f(x) = \sqrt{\alpha^2 - x^2} - \sqrt{\alpha^2 - (x+1)^2} - (r-q-\mu)/\delta.$$

It is equivalent to proving that $f(x)$ has a unique solution on the interval $(-\alpha, \alpha-1)$.

Obviously $f(x)$ is continuous on the interval $[-\alpha, \alpha-1]$ and

$$f(-\alpha) = -\sqrt{2\alpha-1} - (r-q-\mu)/\delta < 0,$$
$$f(\alpha-1) = \sqrt{2\alpha-1} - (r-q-\mu)/\delta > 0.$$

Hence, $f(x)$ has a zero point on $(-\alpha, \alpha-1)$. Moreover,

$$\left(\frac{x}{\sqrt{\alpha^2 - x^2}}\right)' = \frac{\alpha^2}{(\alpha^2 - x^2)^{\frac{3}{2}}} > 0,$$

so $\frac{x}{\sqrt{\alpha^2 - x^2}}$ is an increasing function. Thus

$$f'(x) = \frac{x+1}{\sqrt{\alpha^2 - (x+1)^2}} - \frac{x}{\sqrt{\alpha^2 - x^2}} > 0,$$

and hence $f(x)$ is a strictly increasing function on $(-\alpha, \alpha-1)$. This implies the zero point of $f(x)$ is unique. Now we have proven Lemma 2.3. $\square$

In the following, for convenience, we denote the risk neutral equivalent martingale measure P_θ by Q.

Proposition 2.4 *Under the risk neutral equivalent martingale measure Q, $\{L(t)\}_{0\le t\le T}$ is the NIG process with parameters α, $\beta+\theta$, δ, μ. Moreover, the characteristic function $\varphi^{(\theta)}(z,t)$ of $L(t)$, under Q, is given by*

$$\varphi^{(\theta)}(z,t) = \exp\left(t\delta\left(\sqrt{\alpha^2 - (\beta+\theta)^2} - \sqrt{\alpha^2 - (\beta+\theta+\mathrm{i}z)^2}\right) + \mathrm{i}\mu z t\right), \quad z \in \mathbf{R}. \tag{9}$$

Proof. Firstly, we will show $\{L(t)\}_{0\le t\le T}$ has independent and stationary increments under Q. Let $\mathbf{1}_{\{\cdot\}}$ be the indicator function. Noticing that $\{L(t)\}_{0\le t\le T}$ has independent increments under P, then by Lemma 2.1 and Lemma 2.2, for any $0 \le s < t \le T$, $F_s \in \mathcal{F}_s$ and every Borel subset B of $\mathbf{R}$, we have

$$\begin{aligned}
& Q\left(\{L(t)-L(s)\in B\} \cap F_s\right) \\
= \ & \mathbb{E}_Q\left[\mathbf{1}_{\{L(t)-L(s)\in B\}}\mathbf{1}_{\{F_s\}}\right] \\
= \ & \mathbb{E}_P\left[\Lambda(\theta,t)\mathbf{1}_{\{L(t)-L(s)\in B\}}\mathbf{1}_{\{F_s\}}\right] \\
= \ & \mathbb{E}_P\left[\frac{\Lambda(\theta,t)}{\Lambda(\theta,s)}\mathbf{1}_{\{L(t)-L(s)\in B\}}\Lambda(\theta,s)\mathbf{1}_{\{F_s\}}\right] \\
= \ & \mathbb{E}_P\left[\frac{\Lambda(\theta,t)}{\Lambda(\theta,s)}\mathbf{1}_{\{L(t)-L(s)\in B\}}\right]\mathbb{E}_P\left[\Lambda(\theta,s)\mathbf{1}_{\{F_s\}}\right] \\
= \ & \mathbb{E}_P\left[\Lambda(\theta,s)\right]\mathbb{E}_P\left[\frac{\Lambda(\theta,t)}{\Lambda(\theta,s)}\mathbf{1}_{\{L(t)-L(s)\in B\}}\right]\mathbb{E}_P\left[\Lambda(\theta,s)\mathbf{1}_{\{F_s\}}\right] \\
= \ & \mathbb{E}_P\left[\Lambda(\theta,t)\mathbf{1}_{\{L(t)-L(s)\in B\}}\right]\mathbb{E}_P\left[\Lambda(\theta,s)\mathbf{1}_{\{F_s\}}\right] \\
= \ & \mathbb{E}_Q\left[\mathbf{1}_{\{L(t)-L(s)\in B\}}\right]\mathbb{E}_Q\left[\mathbf{1}_{\{F_s\}}\right] \\
= \ & Q\left(\{L(t)-L(s)\in B\}\right)Q\left(F_s\right),
\end{aligned}$$

which yields the independence of the increments of $\{L(t)\}_{0\le t\le T}$ under Q. Similarly, using that $\{L(t)\}_{0\le t\le T}$ has independent and stationary increments under P, we have

$$\begin{aligned}
& Q\left(\{L(t)-L(s)\in B\}\right)\\
= \;& \mathbb{E}_P\left[\Lambda(\theta,t)\mathbf{1}_{\{L(t)-L(s)\in B\}}\right]\\
= \;& \mathbb{E}_P\left[\frac{\Lambda(\theta,t)}{\Lambda(\theta,s)}\mathbf{1}_{\{L(t)-L(s)\in B\}}\Lambda(\theta,s)\right]\\
= \;& \mathbb{E}_P\left[\frac{\Lambda(\theta,t)}{\Lambda(\theta,s)}\mathbf{1}_{\{L(t)-L(s)\in B\}}\right]\\
= \;& \mathbb{E}_P\left[\frac{e^{\theta(L(t)-L(s))}}{M(\theta,t-s)}\mathbf{1}_{\{L(t)-L(s)\in B\}}\right]\\
= \;& \mathbb{E}_P\left[\frac{e^{\theta L(t-s)}}{M(\theta,t-s)}\mathbf{1}_{\{L(t-s)\in B\}}\right]\\
= \;& \mathbb{E}_P\left[\Lambda(\theta,t-s)\mathbf{1}_{\{L(t-s)\in B\}}\right]\\
= \;& Q\left(\{L(t-s)\in B\}\right),
\end{aligned}$$

which yields the stationarity of the increments of $\{L(t)\}_{0\le t\le T}$ under Q.

Secondly, to prove $L(t)$ has $NIG(\alpha,\beta+\theta,t\delta,t\mu)$ distribution under Q, we only need to determine the characteristic function of $L(t)$ under Q. By Lemma 2.1, and Equations (2), (4) and (5), we have

$$\begin{aligned}
\varphi^{(\theta)}(z,t) &= \mathbb{E}_Q[e^{\mathrm{i}zL(t)}]=\mathbb{E}_P[e^{\mathrm{i}zL(t)}\Lambda(\theta,t)]\\
&= \mathbb{E}_P[e^{\mathrm{i}(z-\mathrm{i}\theta)L(t)}]/M(\theta,t)\\
&= \exp\left(t\delta\left(\sqrt{\alpha^2-(\beta+\theta)^2}-\sqrt{\alpha^2-(\beta+\theta+\mathrm{i}z)^2}\right)+\mathrm{i}\mu zt\right).
\end{aligned}$$

This also explains that $L(t)$ has $NIG(\alpha,\beta+\theta,t\delta,t\mu)$ distribution under Q.

Thus Proposition 2.4 has been established. □

§3 Pricing Power Option by Fourier Transform

In this section, we first derive the analytical pricing formulas of power options by Fourier Transform and its inverse. These analytical pricing formulas are expressed in the form of Fourier integrals. The application of the fast Fourier transform (FFT) to calculate these formulas is also discussed.

According to Zhang [26], power options are classified into asymmetric and symmetric power options. In the asymmetric case, the power is given only to the underlying asset price at maturity, while in the symmetric case, the power is given to the difference between the underlying asset price at maturity and the strike price. Specifically, the payoffs of asymmetric power call and put options at maturity T are

$$((S(T))^p-K)^+\quad\text{and}\quad(K-(S(T))^p)^+ \tag{10}$$

respectively, while the payoffs of symmetric power call and put options at maturity T are

$$((S(T)-K)^p)^+\quad\text{and}\quad((K-S(T))^p)^+ \tag{11}$$

respectively, where K is the strike price, $x^+=\max\{x,0\}$ and the power parameter $p\in\mathbf{R}$

and $p > 0$. In particular, if $p = 1$ in either asymmetric or symmetric case, the power option degenerates to the standard European option.

The following theorems present the pricing formulas of power options.

Theorem 3.1 *Let $C_a^p(t;K)$ and $P_a^p(t;K)$ be the prices of asymmetric power call and put option with strike price K at time t $(0 \leq t \leq T)$ respectively. Assume that the parameters satisfy $\alpha > 1/2$ and $-\sqrt{2\alpha-1} < (r-q-\mu)/\delta < \sqrt{2\alpha-1}$. Set $\tau = T-t$, $k = \ln[K/(S(t))^p]$.*

i) For any $0 < p < \alpha-\beta-\theta$, we have

$$C_a^p(t;K) = \frac{e^{-(r\tau+\lambda_1 k)}(S(t))^p}{2\pi}\int_{-\infty}^{+\infty} e^{-\mathrm{i}vk}F_1(v,\tau)\mathrm{d}v \tag{12}$$

with modified parameter $\lambda_1 \in (0, (\alpha-\beta-\theta)/p-1]$;

ii) for any $p > 0$, we have

$$P_a^p(t;K) = \frac{e^{-(r\tau+\lambda_2 k)}(S(t))^p}{2\pi}\int_{-\infty}^{+\infty} e^{-\mathrm{i}vk}F_2(v,\tau)\mathrm{d}v \tag{13}$$

with modified parameter $\lambda_2 \in [(-\alpha-\beta-\theta)/p-1, -1)$, where

$$F_m(v,\tau) = \frac{\varphi^{(\theta)}(p(v-\mathrm{i}(\lambda_m+1)),\tau)}{(\mathrm{i}v+\lambda_m)(\mathrm{i}v+\lambda_m+1)}, \qquad m = 1,2.$$

Proof. By the risk neutral pricing principle and (10), (3), we have

$$\begin{aligned} C_a^p(t;K) &= e^{-r(T-t)}\mathbb{E}_Q[C_a^p(T;K)|\mathcal{F}_t] \\ &= e^{-r(T-t)}\mathbb{E}_Q[((S(T))^p-K)^+|\mathcal{F}_t] \\ &= e^{-r(T-t)}(S(t))^p\mathbb{E}_Q\left[\left(e^{p(L(T)-L(t))}-e^k\right)^+\middle|\mathcal{F}_t\right]. \end{aligned}$$

Since $L(T)-L(t)$ is independent of $\mathcal{F}_t$ and has the same distribution as $L(T-t) = L(\tau)$, $k = \ln\frac{K}{(S(t))^p}$ is $\mathcal{F}_t-$ measurable, we get

$$C_a^p(t;K) = e^{-r\tau}(S(t))^p\int_{\frac{k}{p}}^{+\infty}(e^{px}-e^k)f^{(\theta)}(x,\tau)\mathrm{d}x, \tag{14}$$

where $f^{(\theta)}(x,t)$ denotes the density function of $L(t)$ under Q. Let

$$g_1^p(k) = \int_{\frac{k}{p}}^{+\infty}(e^{px}-e^k)f^{(\theta)}(x,\tau)\mathrm{d}x, \tag{15}$$

where $g_1^p(k)$ is not integrable with respect to k over the negative axis because

$$\lim_{k\to-\infty} g_1^p(k) = \mathbb{E}_Q[(e^{pL(\tau)})] = \varphi^{(\theta)}(-\mathrm{i}p,\tau) \neq 0.$$

The Fourier transform of $g_1^p(k)$ does not exist. In order to apply the Fourier transform we multiply $g_1^p(k)$ by exponentially decaying term $e^{\lambda_1 k}$ with $\lambda_1 > 0$ as follows:

$$G_1^p(k) = e^{\lambda_1 k}g_1^p(k),$$

where $G_1^p(k)$ may not be integrable with respect to k over the positive axis if λ_1 is too large.

Therefore, we need to find the upper bound on λ_1. Note that when $\lambda_1 > 0$

$$\begin{aligned}\int_{-\infty}^{+\infty}|G_1^p(k)|\mathrm{d}k &= \int_{-\infty}^{+\infty}e^{\lambda_1 k}\int_{\frac{k}{p}}^{+\infty}(e^{px}-e^k)f^{(\theta)}(x,\tau)\mathrm{d}x\mathrm{d}k\\ &= \int_{-\infty}^{+\infty}f^{(\theta)}(x,\tau)\int_{-\infty}^{px}(e^{px+\lambda_1 k}-e^{(\lambda_1+1)k})\mathrm{d}k\mathrm{d}x\\ &= \frac{1}{\lambda_1(\lambda_1+1)}\int_{-\infty}^{+\infty}e^{p(\lambda_1+1)x}f^{(\theta)}(x,\tau)\mathrm{d}x.\end{aligned}$$

From (9) we know that $G_1^p(k)$ is absolutely integrable if $p(\lambda_1+1)\le\alpha-\beta-\theta$, that is, $\lambda_1\le(\alpha-\beta-\theta)/p-1$. Hence when $0<\lambda_1\le(\alpha-\beta-\theta)/p-1$, the Fourier transform $F_1(v,\tau)$ of $G_1^p(k)$ exists and

$$\begin{aligned}F_1(v,\tau) &= \int_{-\infty}^{+\infty}e^{ivk}G_1^p(k)\mathrm{d}k\\ &= \int_{-\infty}^{+\infty}e^{(iv+\lambda_1)k}\int_{\frac{k}{p}}^{+\infty}(e^{px}-e^k)f^{(\theta)}(x,\tau)\mathrm{d}x\mathrm{d}k\\ &= \int_{-\infty}^{+\infty}f^{(\theta)}(x,\tau)\int_{-\infty}^{px}(e^{px+(\mathrm{i}v+\lambda_1)k}-e^{(\mathrm{i}v+\lambda_1+1)k})\mathrm{d}k\mathrm{d}x\\ &= \frac{1}{(\mathrm{i}v+\lambda_1)(\mathrm{i}v+\lambda_1+1)}\int_{-\infty}^{+\infty}e^{p(\mathrm{i}v+\lambda_1+1)x}f^{(\theta)}(x,\tau)\mathrm{d}x\\ &= \frac{\varphi^{(\theta)}(p(v-\mathrm{i}(\lambda_1+1)),\tau)}{(\mathrm{i}v+\lambda_1)(\mathrm{i}v+\lambda_1+1)}.\end{aligned}$$

Applying the inverse Fourier transform we can obtain

$$g_1^p(k)=\frac{e^{-\lambda_1 k}}{2\pi}\int_{-\infty}^{+\infty}e^{-\mathrm{i}vk}F_1(v,\tau)dv. \tag{16}$$

Equation (12) follows from Equations (14), (15) and (16).

Similarly, we can get

$$P_a^p(t;K)=e^{-r\tau}(S(t))^p\int_{-\infty}^{\frac{k}{p}}(e^k-e^{px})f^{(\theta)}(x,\tau)\mathrm{d}x\equiv e^{-r\tau}(S(t))^p g_2^p(k).$$

In order to apply Fourier transform we multiply $g_2^p(k)$ by exponential term $e^{\lambda_2 k}$. When $(-\alpha-\beta-\theta)/p-1\le\lambda_2<-1$, the Fourier transform $F_2(v,\tau)$ of $G_2^p(k)=e^{\lambda_2 k}g_2^p(k)$ exists and

$$F_2(v,\tau)=\int_{-\infty}^{+\infty}e^{\mathrm{i}vk}G_2^p(k)\mathrm{d}k=\frac{\varphi^{(\theta)}(p(v-\mathrm{i}(\lambda_2+1)),\tau)}{(\mathrm{i}v+\lambda_2)(\mathrm{i}v+\lambda_2+1)}.$$

By the inverse Fourier transform, we obtain Equation (13).

Theorem 3.2 *Let $C_s^p(t;K)$ and $P_s^p(t;K)$ be the prices of symmetric power call and put option with strike price K at time t $(0\le t\le T)$ respectively. Assume that the parameters satisfy $\alpha>1/2$ and $-\sqrt{2\alpha-1}<(r-q-\mu)/\delta<\sqrt{2\alpha-1}$. Set $\tau=T-t$, $k=\ln[K/S(t)]$.*

i) If p is an integer with $0<p<\alpha-\beta-\theta$, then

$$C_s^p(t;K) = \frac{e^{-(r\tau+\lambda_3 k)}(S(t))^p}{2\pi}\int_{-\infty}^{+\infty}e^{-\mathrm{i}vk}F_3(v,\tau)\mathrm{d}v \tag{17}$$

with modified parameter $\lambda_3 \in (0, \alpha - \beta - \theta - p]$ *and*

$$F_3(v,\tau) = \frac{p!}{\prod_{m=0}^{p}(\lambda_3 + m + \mathrm{i}v)}\varphi^{(\theta)}(v - \mathrm{i}(p+\lambda_3), \tau);$$

ii) if p *is an integer with* $p > 0$, *then*

$$P_s^p(t;K) = \frac{e^{-(r\tau+\lambda_4 k)}(S(t))^p}{2\pi}\int_{-\infty}^{+\infty} e^{-\mathrm{i}vk}F_4(v,\tau)\mathrm{d}v \tag{18}$$

with modified parameter $\lambda_4 \in [-\alpha - \beta - \theta - p, -p)$ *and*

$$F_4(v,\tau) = (-1)^{p+1}\frac{p!}{\prod_{m=0}^{p}(\lambda_4 + m + \mathrm{i}v)}\varphi^{(\theta)}(v - \mathrm{i}(p+\lambda_4), \tau).$$

Proof. By (3) and (11), similar to the proof of Theorem 3.1, we get

$$C_s^p(t;K) = e^{-r\tau}(S(t))^p\int_k^{+\infty}(e^x - e^k)^p f^{(\theta)}(x,\tau)\mathrm{d}x \equiv e^{-r\tau}(S(t))^p g_3^p(k).$$

In order to apply the Fourier transform, we multiply $g_3^p(k)$ by exponential term $e^{\lambda_3 k}$. Let $G_3^p(k) = e^{\lambda_3 k}g_3^p(k)$, we have

$$\begin{aligned}\int_{-\infty}^{+\infty}|G_3^p(k)|\mathrm{d}k &= \int_{-\infty}^{+\infty}e^{\lambda_3 k}\int_k^{+\infty}(e^x - e^k)^p f^{(\theta)}(x,\tau)\mathrm{d}x\mathrm{d}k \\ &= \int_{-\infty}^{+\infty}f^{(\theta)}(x,\tau)\int_{-\infty}^{x}e^{\lambda_3 k}e^{px}(1-e^{k-x})^p\mathrm{d}k\mathrm{d}x.\end{aligned}$$

Taking $y = e^{k-x}$, then $e^k = ye^x$, $\mathrm{d}k = \frac{1}{y}\mathrm{d}y$, and making the various substitutions into the integral, we get

$$\int_{-\infty}^{+\infty}|G_3^p(k)|\mathrm{d}k = \int_{-\infty}^{+\infty}e^{(p+\lambda_3)x}f^{(\theta)}(x,\tau)\mathrm{d}x\int_0^1 y^{\lambda_3 - 1}(1-y)^p\,\mathrm{d}y.$$

Hence, when $0 < \lambda_3 \le \alpha - \beta - \theta - p$, $G_3^p(k)$ is absolutely integrable and

$$\int_{-\infty}^{+\infty}|G_3^p(k)|\mathrm{d}k = B(\lambda_3, p+1)\varphi^{(\theta)}(-\mathrm{i}(p+\lambda_3), \tau),$$

where $B(\cdot,\cdot)$ is the Euler Beta function. Then the Fourier transform $F_3(v,\tau)$ of $G_3^p(k)$ exists and if p is a positive integer

$$\begin{aligned}F_3(v,\tau) &= \int_{-\infty}^{+\infty}e^{\mathrm{i}vk}G_3^p(k)\mathrm{d}k \\ &= \int_{-\infty}^{+\infty}e^{(\lambda_3+\mathrm{i}v)k}\int_k^{+\infty}(e^x - e^k)^p f^{(\theta)}(x,\tau)\mathrm{d}x\mathrm{d}k \\ &= \int_{-\infty}^{+\infty}f^{(\theta)}(x,\tau)\int_{-\infty}^{x}e^{(\lambda_3+\mathrm{i}v)k}\sum_{m=0}^{p}\begin{pmatrix}p\\ m\end{pmatrix}(-1)^m e^{mk}e^{(p-m)x}\mathrm{d}k\mathrm{d}x \\ &= \sum_{m=0}^{p}\begin{pmatrix}p\\ m\end{pmatrix}(-1)^m\int_{-\infty}^{+\infty}e^{(p-m)x}f^{(\theta)}(x,\tau)\int_{-\infty}^{x}e^{(m+\lambda_3+\mathrm{i}v)k}\mathrm{d}k\mathrm{d}x \\ &= \left(\sum_{m=0}^{p}\begin{pmatrix}p\\ m\end{pmatrix}(-1)^m\frac{1}{m+\lambda_3+\mathrm{i}v}\right)\int_{-\infty}^{+\infty}e^{(p+\lambda_3+\mathrm{i}v)x}f^{(\theta)}(x,\tau)\mathrm{d}x \\ &= \frac{p!}{\prod_{m=0}^{p}(\lambda_3 + m + \mathrm{i}v)}\varphi^{(\theta)}(v - \mathrm{i}(p+\lambda_3), \tau),\end{aligned}$$

where the last equality can be deduced by induction. Then by the inverse Fourier transform we can obtain Equation (17).

Similarly, we can get

$$P_s^p(t;K) = e^{-r\tau}(S(t))^p \int_{-\infty}^{k} (e^k - e^x)^p f^{(\theta)}(x,\tau)\mathrm{d}x \equiv e^{-r\tau}(S(t))^p g_4^p(k).$$

In order to apply the Fourier transform, we multiply $g_4^p(k)$ by exponential term $e^{\lambda_4 k}$. Let $G_4^p(k) = e^{\lambda_4 k} g_4^p(k)$. When $-\alpha - \beta - \theta \leq p + \lambda_4 < 0$, we have

$$\int_{-\infty}^{+\infty} |G_4^p(k)|\mathrm{d}k = B(-(p+\lambda_4), p+1)\varphi^{(\theta)}(-\mathrm{i}(p+\lambda_4),\tau) < +\infty.$$

Then the Fourier transform $F_4(v,\tau)$ of $G_4^p(k)$ exists and if p is a positive integer

$$\begin{aligned} F_4(v,\tau) &= \int_{-\infty}^{+\infty} e^{\mathrm{i}vk} G_4^p(k)\mathrm{d}k \\ &= \sum_{m=0}^{p} \begin{pmatrix} p \\ m \end{pmatrix} (-1)^{p-m} \int_{-\infty}^{+\infty} e^{(p-m)x} f^{(\theta)}(x,\tau) \int_{x}^{+\infty} e^{(m+\lambda_4+\mathrm{i}v)k}\mathrm{d}k\mathrm{d}x \\ &= (-1)^{p+1} \frac{p!}{\prod_{m=0}^{p}(\lambda_4 + m + \mathrm{i}v)} \varphi^{(\theta)}(v - \mathrm{i}(p+\lambda_4),\tau). \end{aligned}$$

By the inverse of the Fourier transform, we obtain Equation (17).

Remark 3.3 Equations (12), (13), (17) and (18) present the pricing formulas for asymmetric power call and put, symmetric power call and put options, respectively. They look exactly the same, but the ranges of modified parameters λ_m $(m = 1,2,3,4)$ in four equations are different. These parameters play very important role in deriving pricing formulas since they constitute exponentially decaying terms to make sure the existence of Fourier transform.

Remark 3.4 In the proofs of Equations (12) and (17), we impose the upper bound condition $p < \alpha - \beta - \theta$ to guarantee that $\mathbb{E}_Q[(S(T))^p] = (S(0))^p \mathbb{E}_Q[e^{pL(T)}] < +\infty$ holds for call options. However, in the proofs of (13) and (18), the upper bound condition is removed since power put option always has finite payoffs for any $p > 0$.

Following that, we use the FFT algorithm to approximate the Fourier integrals in Theorems 3.1-3.2. Let

$$\rho_\xi^p(k) = \int_{-\infty}^{+\infty} e^{-\mathrm{i}vk} F_\xi(v,\tau)dv, \quad \xi = 1,2,3,4.$$

Truncating $\rho_\xi^p(k)$ at points $(-\frac{N}{2} - \frac{1}{2})\eta$ and $(\frac{N}{2} - \frac{1}{2})\eta$ and applying the midpoint rule, we get

$$\begin{aligned} \rho_\xi^p(k) &\approx \int_{(-\frac{N}{2}-\frac{1}{2})\eta}^{(\frac{N}{2}-\frac{1}{2})\eta} e^{-\mathrm{i}vk} F_\xi(v,\tau)dv \\ &= \sum_{n=-\frac{N}{2}}^{\frac{N}{2}-1} \int_{(n-\frac{1}{2})\eta}^{(n+\frac{1}{2})\eta} e^{-\mathrm{i}vk} F_\xi(v,\tau)dv \\ &\approx \sum_{n=-\frac{N}{2}}^{\frac{N}{2}-1} e^{-\mathrm{i}n\eta k} F_\xi(n\eta,\tau)\eta \end{aligned}$$

$$= \sum_{m=1}^{N} e^{-\mathrm{i}(m-\frac{N}{2}-1)\eta k} F_{\xi}\left(\left(m-\frac{N}{2}-1\right)\eta, \tau\right)\eta.$$

Recall that FFT is an efficient algorithm to calculate

$$\omega(l) = \sum_{j=1}^{N} e^{-\mathrm{i}\frac{2\pi}{N}(j-1)(l-1)} x(j), \quad l = 1, 2, \cdots, N,$$

where N is a power of 2. FFT algorithm takes N complex numbers as input and gives back N complex numbers as output [20]. To apply FFT, we take $\eta = \frac{2\pi}{Nh}$ and

$$k_u = \left(u - \frac{N}{2} - 1\right)h, \quad u = 1, 2, \cdots, N,$$

where h is the step size about k. Then we have

$$\begin{aligned}
\rho_{\xi}^{p}(k_u) &\approx \sum_{m=1}^{N} e^{-\mathrm{i}h\eta(m-1)(u-1)} e^{-\mathrm{i}\pi(m+u-2+\frac{N\pi}{2})} F_{\xi}\left(\left(m-\frac{N}{2}-1\right)\eta, \tau\right)\eta \\
&= \sum_{m=1}^{N} e^{-\mathrm{i}\frac{2\pi}{N}(m-1)(u-1)} (-1)^{m+u} F_{\xi}\left(\left(m-\frac{N}{2}-1\right)\eta, \tau\right)\eta \\
&= \eta(-1)^u \sum_{m=1}^{N} e^{-\mathrm{i}\frac{2\pi}{N}(m-1)(u-1)} X(m),
\end{aligned}$$

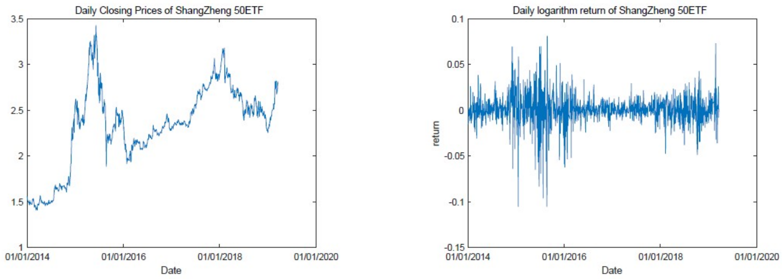
where $X(m) = (-1)^m F_{\xi}((m - \frac{N}{2} - 1)\eta, \tau)$, $u = 1, 2, \cdots, N$. Then we can compute $\rho_{\xi}^{p}(k_u)$ by FFT. This completes the model.

§4 Numerical Examples

In this section we test the NIG process using the daily logarithm return of Shangzheng 50ETF and use the proposed method in Section 3 to calculate the prices of power options on Shangzheng 50ETF. All the calculations are carried out with MATLAB R2017a.

Figure 1(a) shows the closing prices of Shangzheng 50ETF from 2 Jan. 2014 to 20 Mar. 2019 downloaded from the DZH365 website (`http://www.gw.com.cn`). Figure 1(b) illustrates the continuously compounded daily log returns associated with the price series. Table 1 lists the statistical characteristics of the daily log return for Shangzheng 50ETF. These values in Table 1 can be obtained by the built-in function of MATLAB R2017a. The empirical skewness is negative, which shows that the empirical distribution has a longer tail to the left than to the right, and the empirical kurtosis is greater than 3, which implies that the empirical distribution is more peaked and the tail approaches to zero more slowly than normal distribution whose kurtosis is 3.

Using the $NIG(\alpha, \beta, \delta, \mu)$ distribution to model daily log return for Shangzheng 50ETF, the estimated values and standard deviations for parameters $\alpha, \beta, \delta, \mu$ are obtained via maximum likelihood estimation, as documented in Table 2. Figure 2 shows the empirical probability density function (blue line), the normal probability density function (black line) and the NIG probability density function (red line). It can be seen that the NIG distribution is more suitable than normal distribution to model daily log returns of Shangzheng 50ETF because the red line is more consistent with the blue line.



(a) (b)

Figure 1. Daily closing prices and daily log return of Shangzheng 50ETF.

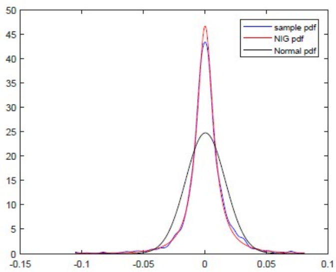


Figure 2. Sample pdf, normal pdf and NIG pdf of daily return of Shangzheng 50ETF.

Table 1. Statistical characteristics of daily log return for Shangzheng 50ETF.

Minimum	Maximum	Mean	Std. Dev.	Skewness	Kurtosis
-0.1052	0.0809	0.0005	0.0161	-0.6295	10.6484

Table 2. Parameter estimator in the NIG distribution of daily log return.

Parameter	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\delta}$	$\hat{\mu}$
Estimated Value	30.5780	1.0011	0.0082	0.0002
Standard-Deviation	4.2336	2.2671	0.0005	0.0004

In order to compute the prices of power options we need to find the solution θ to equation (8), which is related to the risk neutral equivalent martingale measure Q. Choose the dividend yield in years $q = 0.0201$ of Shangzheng 50ETF quoted on 06 Dec. 2018 and the risk free interest rates $r = 0.0224$ corresponding to maturities $\tau = 0.5139$ years. The data is obtained from the website (`http://www.gw.com.cn`). Then we get the solution $\theta = -2.2228$. Assume $t = 0$, $S(0) = 2.794$, which is the price of Shangzheng 50ETF quoted on 06 Dec. 2018. Set $\lambda_1 = \lambda_3 = 5$, $\lambda_2 = \lambda_4 = -5$ with the power parameter $p = 1, 2, 3$ respectively. We obtain the prices of power options for different strike prices by Equations (12), (13), (17) and (18). Figure 3 depicts the prices of power options against the strike prices K, where Figures 3(a) and 3(c) show that the prices of the power call options are decreasing as K is increasing, while Figures 3(b) and 3(d) show that the prices of the power put options are increasing as K is increasing.

For strike prices between 2 and 3, the asymmetric power call option prices are between 0.8036 and 0.1657 when $p = 1$, between 6.1523 and 5.1728 when $p = 2$, between 23.1360 and 22.1497 when $p = 3$, respectively. In order to put the three cases into one graph, the value of the vertical axis unit is set to be relatively large, which makes Figure 3(a) look like a linear payoff.

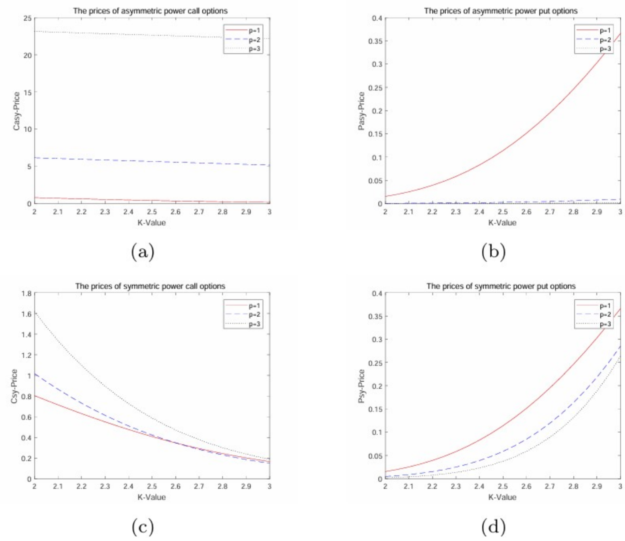


Figure 3. The prices of power options.

It takes about 15 seconds to compute 21 asymmetric power call option prices directly by Equation (12). However, the FFT algorithm dramatically improves the speed of calculation. Taking $N = 4096$, $h = \pi/2000$, we can obtain the prices of 4096 power options in about 0.78 seconds, in one go.

In Table 3, we list the prices of power call option with $p = 1$ (standard European call option) for different strike prices. The first column gives strike prices, the second column lists market prices (the actual prices for power call options traded) downloaded from the DZH365 website (`http://www.gw.com.cn`), the third column lists NIG prices directly calculated by Equation (12), the fourth column lists NIGFFT prices which are calculated by linear interpolations using the data obtained via FFT under NIG model, and the fifth column lists BSM prices which are calculated by Formula (18.4) on page 373 of [10]. It takes about 7.07 and 1.11 seconds to compute the third and fourth columns in Table 3, respectively. This shows that FFT algorithm is more efficient.

To compare accuracy, we compute the average absolute error (AAE), the average relative percentage error (ARPE) and the root-mean-square error (RMSE) which are defined by Schoutens in [22].

$$\begin{aligned} \text{AAE} &= \sum_{\text{options}} \frac{|\text{market price} - \text{model price}|}{\text{number of options}} \\ \text{ARPE} &= \frac{1}{\text{number of options}} \sum_{\text{options}} \frac{|\text{market price} - \text{model price}|}{\text{market price}} \\ \text{RMSE} &= \sqrt{\sum_{\text{options}} \frac{(\text{market price} - \text{model price})^2}{\text{number of options}}} \end{aligned}$$

Table 4 gives the relevant measures. From Table 4 we can see that the errors between the NIG price and the market price are lowest and consequently the prices of options under NIG model are much closer to the market prices than that under BSM model.

Table 3. The Prices of the Power Call Options for different exercise price with $p = 1$.

	Option Price			
Strike Price	Market Price	NIG Price	NIGFFT Price	BSM Price
2.50	0.4174	0.4078	0.4078	0.4045
2.55	0.3857	0.3762	0.3763	0.3729
2.60	0.3535	0.3464	0.3463	0.3429
2.65	0.3224	0.3182	0.3182	0.3147
2.70	0.2921	0.2916	0.2916	0.2881
2.75	0.2660	0.2667	0.2668	0.2632
2.80	0.2420	0.2435	0.2435	0.2399
2.85	0.2185	0.2218	0.2218	0.2182
2.90	0.1985	0.2016	0.2017	0.1981
2.95	0.1810	0.1829	0.1930	0.1795
3.00	0.1645	0.1657	0.1657	0.1623

Table 4. AAE, ARPE and RMSE of NIG price, NIGFFT price and BSM price.

	NIG	NIGFFT	BSM
AAE	0.003873	0.0039	0.005209
ARPE	1.2709%	1.2841%	1.5992%
RMSE	0.0050362	0.0050426	0.00699

§5 Conclusion

In this paper we price power option with underlying price process that follows an exponential NIG process. We first find the equivalent martingale measure by the Esscher transform. Then the pricing formulas of power options are derived by the Fourier transform and its inverse transform. In order to reduce computing time, the FFT algorithm is utilized to approximate the Fourier integral. Finally, we choose Shangzheng 50ETF options for empirical study. It is shown that the NIG distribution is more suitable than normal distribution to model daily log returns of Shangzheng 50ETF since the options prices under NIG model are closer to the market prices than that under BSM model. The results show that the NIG model is more accurate and the FFT algorithm is more effective.

Acknowledgement

We would like to thank two anonymous referees for valuable comments and suggestions which lead to the improvement of this article.

Declarations

Conflict of interest The authors declare no conflict of interest.

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