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A neural network based on novel equivalent model for linear complementarity problems

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Abstract. A family of neural networks is proposed to solve linear complementarity problems (LCP). The neural networks are constructed from the novel equivalent model of LCP, which is reformulated by utilizing the modulus and smoothing technologies. Some important properties of the proposed novel equivalent model are summarized. In addition, the stability properties of the proposed steepest descent-based neural networks for LCP are analyzed. In order to illustrate the theoretical results, we provide some numerical simulations and compare the proposed neural networks with existing neural networks based on the NCP-functions. Numerical results indicate that the performance of the proposed neural networks is effective and robust.

§1 Introduction

Many problems in scientific computing and engineering applications need to find the solution of complementary problems, such as the convex quadratic programming, the bimatrix game, the contact problems, the network equilibrium problems, the free boundary problems of fluid dynamics, and other (see [8,10] and the references therein).

The linear complementarity problem, abbreviated as LCP, is to find a pair of real vectors $z, w \in \mathbb{R}^n$ such that

$$z \geq 0, \quad w := Az + q \geq 0 \quad \text{and} \quad z^T w = 0, \tag{1}$$

where $A \in \mathbb{R}^{n\times n}$ and $q \in \mathbb{R}^n$ are the given matrix and vector, respectively. Here, z^T denotes the transpose of the vector z. We denote the LCP (1) by the LCP(A, q).

The nonlinear complementarity problem, abbreviated as NCP, is to find a real vector $z \in \mathbb{R}^n$ such that

$$z \geq 0, \quad F(z) \geq 0, \quad z^T F(z) = 0, \tag{2}$$

where $F : \mathbb{R}^n \to \mathbb{R}^n$ is a nonlinear mapping.

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When $F(z) = Az + q$, the NCP (2) reduces to the LCP (1). Recall that a function $\phi : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is called an NCP-function if it is satisfies

$$\phi(a, b) = 0 \Leftrightarrow a \geq 0, \quad b \geq 0, \quad ab = 0. \tag{3}$$

Due to the wide range of applications in many areas, a lot of research efforts have been devoted to the study of complementarity problems, including the solvability conditions and numerical algorithms. There are many numerical algorithms for solving the complementarity problems. These algorithms may be categorized into two classes based on the so-called NCP-function or not. For example, the merit function approach [11, 14, 22], the nonsmooth Newton method [9, 30], the smoothing method [5, 25] and the regularization approach [13, 27] exploit NCP-functions. However, the well-known interior-point method [23, 24] and proximal point algorithm [26] do not utilize NCP-functions.

Recently, utilizing the modulus technology, Bai in [2] established a class of modulus-based matrix splitting for solving the linear complementarity problem (1). This class of iteration methods is essentially based on an equivalent transformation of the LCP into a system of fixed-point equations involving only the absolute value of certain vector. Since the modulus-based iteration methods take advantage of the linear part of LCP(A, q), they solve the LCP fast and economically. Therefore, the modulus-based iteration methods have attracted a lot of attention; see [3, 16, 17, 20, 31, 32].

The numerical algorithms mentioned above can efficiently solve the LCP, but it is often desirable to obtain a real-time solution in scientific and engineering applications. The neural network method is an ideal method for solving real-time optimization problems, which were first introduced in optimization by Hopfield and Tank in the 1980s [12, 28]. Neural networks based on the well-known Fischer-Burmeister (FB) function [19] and the generalized Fischer-Burmeister function [4] have already been studied for solving the nonlinear complementarity problems.

The neural network methods based on NCP-functions for solving NCP (2) are as follows. Let ϕ be an NCP-function. Definite $\Phi : \mathbb{R}^n \to \mathbb{R}^n$ by

$$\Phi(z) := \begin{pmatrix} \phi(z_1, F_1(z)) \\ \vdots \\ \phi(z_n, F_n(z)) \end{pmatrix}$$

and definite $f : \mathbb{R}^n \to \mathbb{R}_+$ by

$$f(z) := \frac{1}{2}\|\Phi(z)\|^2 = \frac{1}{2}\sum_{i=1}^{n} \phi(z_i, F_i(z))^2.$$

Then, solving the NCP is translated to solving the minimization problem.

$$\min_{z \in \mathbb{R}^n} f(z).$$

Moreover, consider the steepest descent-based neural network

$$\frac{\mathrm{d}z(t)}{\mathrm{d}t} = -\rho \nabla f(z), \quad z(t_0) \in \mathbb{R}^n \tag{4}$$

to solve the above minimization problem.

Motivated by the preceding discussion, we construct a new family of neural networks based on the modulus and smoothing technologies for solving LCP (1), which reformulates the LCP(A, q) as a novel equivalent model. We show that the proposed neural networks offer stability and effective simulation results.

The outline of the paper is as follows. In Section 2, we revisit equivalent reformulations of the LCP(A, q) based on the modulus and smoothing technologies. We also review some mathematical preliminaries related to the stability analysis of first-order differential equations. In Section 3, we summarize some important properties of our proposed novel equivalent model that we will use in constructing the neural networks. In Section 4, we look at the stability and convergence analysis of the novel neural networks, which include the characterization of stationary points of the induced merit functions. In Section 5, we present the results of our numerical simulations. Finally, in Section 6, we end this paper with some concluding remarks.

Notions. $\mathbb{R}^n$ denotes the space of n-dimensional real column vectors, $\mathbb{R}^{m\times n}$ denotes the space of $m \times n$ real matrices, and A^T denotes the transpose of the matrix $A \in \mathbb{R}^{m\times n}$. For any differentiable function $f : \mathbb{R}^n \to \mathbb{R}$, $\nabla f(x)$ means the gradient of f at x. We use $|x| = (|x_1|, |x_2|, \ldots, |x_n|)^T$ to denote the absolute value of the vector $x \in \mathbb{R}^n$. Here, we use (x, y) to replace the column vector $(x^T, y^T)^T$.

§2 Preliminaries

2.1 NCP-function

In this subsection, we present some well-known NCP-functions, which will be used in the following numerical simulation.

- The well-known Fischer-Burmeister (FB) function (see [19]):
$$\phi_{\mathrm{FB}}(a, b) = \sqrt{a^2 + b^2} - (a + b). \tag{5}$$

- The generalized Fischer-Burmeister (GFB) function (see [4]):
$$\phi^p_{\mathrm{GFB}}(a, b) = \|(a, b)\|_p - (a + b), \quad \text{where } p \in (1, +\infty). \tag{6}$$

- The generalized natural-residual function (see [1]):
$$\phi^p_{\mathrm{GNR}}(a, b) = a^p - (a - b)^p_+, \quad \text{where } p > 1 \text{ is an odd integer}. \tag{7}$$

- The first natural symmetrization of ϕ^p_{GNR} (see [1]):
$$\phi^p_{\text{S1-GNR}}(a, b) = \begin{cases} a^p - (a - b)^p, & \text{if} \quad a > b, \\ a^p, & \text{if} \quad a = b, \\ b^p - (b - a)^p, & \text{if} \quad a < b. \end{cases} \tag{8}$$

- The second natural symmetrization of ϕ^p_{GNR} (see [1]):
$$\phi^p_{\text{S2-GNR}}(a, b) = \begin{cases} a^p b^p - (a - b)^p b^p, & \text{if} \quad a > b, \\ a^p b^p, & \text{if} \quad a = b, \\ a^p b^p - (b - a)^p a^p, & \text{if} \quad a < b. \end{cases} \tag{9}$$

2.2 Reformulations for LCP

In [2], Bai provided the following equivalent expression of the LCP(A, q), which is essential and useful for establishing the neural network for the LCP(A, q).

Theorem 2.1. ([2]) Let $A = M - N$ be a splitting of the matrix $A \in \mathbb{R}^{n\times n}$ with M being nonsingular, Σ_1 and Σ_2 be $n \times n$ nonnegative diagonal matrices, and Σ and Γ be $n \times n$ positive diagonal matrices such that $\Sigma = \Sigma_1 + \Sigma_2$. For the LCP$(A, q)$, the following statements hold true:
(1) if (w, z) is a solution of the LCP(A, q), then $x = \frac{1}{2}(\Gamma^{-1}z - \Sigma^{-1}w)$ satisfies the implicit fixed-point equation

$$(M\Gamma + \Sigma_1)x = (N\Gamma - \Sigma_2)x + (\Sigma - A\Gamma)|x| - q. \tag{10}$$

(2) if x satisfies the implicit fixed-point equation (10), then

$$z = \Gamma(|x| + x) \quad \text{and} \quad w = \Sigma(|x| - x) \tag{11}$$

is a solution of the LCP(A, q).
From (10), we can obtain that the LCP(A, q) is equivalent to solving the following nonlinear equation

$$(A\Gamma + \Sigma)x - (\Sigma - A\Gamma)|x| + q = 0. \tag{12}$$

Let $y = |x|$, then the equation (12) is equivalent to

$$\begin{cases} (A\Gamma + \Sigma)x - (\Sigma - A\Gamma)y + q = 0, \\ |x| = y. \end{cases} \tag{13}$$

We investigate the following class of functions:

$$\theta_\mu(a, b) = \sqrt{(1-\mu)a^2 + \mu b^2} - b, \tag{14}$$

where μ is a fixed parameter such that $\mu \in (0, 1)$. It is obvious that the expression inside the square root in (14) is always nonnegative, i.e.,

$$(1-\mu)a^2 + \mu b^2 \geq 0, \quad \forall a, b \in \mathbb{R}.$$

Hence, θ_μ is at least well-defined. Moreover, it is easy to get that

$$\theta_\mu(a, b) = 0 \Leftrightarrow b = |a|. \tag{15}$$

Now, we define the operator $\Phi_\mu : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ by

$$\Phi_\mu(x, y) := \begin{pmatrix} (A\Gamma + \Sigma)x - (\Sigma - A\Gamma)y + q \\ \theta_\mu(x_1, y_1) \\ \vdots \\ \theta_\mu(x_n, y_n) \end{pmatrix}, \tag{16}$$

then it follows immediately from Theorem 2.1 and the relational expression (15) that x^* solves the equation (12) if and only if (x^*, y^*) solves the equation $\Phi_\mu(x, y) = 0$.

For the given parameter μ with $\mu \in (0, 1)$ and the given positive diagonal matrices Σ and Γ, define the function $f_\mu : \mathbb{R}^{2n} \to \mathbb{R}_+$ by

$$f_\mu(x, y) := \frac{1}{2}\|\Phi_\mu(x, y)\|^2$$

$$= \frac{1}{2}\|(A\Gamma + \Sigma)x - (\Sigma - A\Gamma)y + q\|^2 + \frac{1}{2}\sum_{i=1}^{n}(\theta_\mu(x_i, y_i))^2. \tag{17}$$

Then, the LCP(A, q) can be reformulated as a minimization problem

$$\min_{(x,y)\in\mathbb{R}^{2n}} \ f_\mu(x, y).$$

Hence, f_μ is a merit function for the LCP(A, q) and its global minimizer coincides with the solution of the LCP(A, q).

2.3 The first-order differential equations

In this subsection, we recall some materials about the first-order differential equations (ODE):

$$\dot{v}(t) = H(v(t)), \quad v(t_0) = v_0 \in \mathbb{R}^n, \tag{18}$$

where $H : \mathbb{R}^n \to \mathbb{R}^n$ is a mapping. The following materials can be found in ODE textbooks; see [21, 29].

Definition 2.1. A point $v^* = v(t^*)$ is called an equilibrium point or a steady state of the dynamic system (18) if $H(v^*) = 0$. If there is a neighborhood $\Delta^* \subseteq \mathbb{R}^n$ of v^* such that $H(v^*) = 0$ and $H(v) \neq 0$ for any $v \in \Delta^*\backslash\{v^*\}$, then v^* is called an isolated equilibrium point.

Lemma 2.1. Assume that H is a continuous mapping from $\mathbb{R}^n$ to $\mathbb{R}^n$. Then for arbitrary $t_0 \geq 0$ and $v_0 \in \mathbb{R}^n$, there exists a local solution $v(t)$ for (18) with $t \in [t_0, \tau)$ for some $\tau > t_0$. In addition, if H is locally Lipschitz continuous at v_0, then the solution is unique; if H is Lipschitz continuous in $\mathbb{R}^n$, then τ can be extended to ∞.

If a local solution defined on $[t_0, \tau)$ cannot be extended to a local solution on a larger interval $[t_0, \tau_1)$ with $\tau_1 > \tau$, then it is called a maximal solution, and the interval $[t_0, \tau)$ is the maximal interval of existence. Clearly, any local solution has an extension to a maximal one. We denote $[t_0, \tau(v_0))$ by the maximal interval of existence associated with v_0.

Lemma 2.2. Assume that $H : \mathbb{R}^n \to \mathbb{R}^n$ is continuous. If $v(t)$ with $t \in [t_0, \tau(v_0))$ is a maximal solution and $\tau(v_0) < \infty$, then $\lim\limits_{t\to\tau(v_0)} \|v(t)\| = \infty$.

Definition 2.2. (Stability in the sense of Lyapunov) Let $v(t)$ be a solution of (18). An isolated equilibrium point v^* is Lyapunov stable if for any $v_0 = v(t_0)$ and any scalar $\varepsilon > 0$, there exists a real number $\delta > 0$ such that $\|v(t) - v^*\| < \varepsilon$ for all $t \geq t_0$ and $\|v(t_0) - v^*\| < \delta$.

Definition 2.3. (Asymptotic stability) An isolated equilibrium point v^* is said to be asymptotically stable if in addition to being Lyapunov stable, it has the property that $v(t) \to v^*$ as $t \to +\infty$ for all $\|v(t_0) - v^*\| < \delta$.

Definition 2.4. (Lyapunov function) Let $\Delta \subseteq \mathbb{R}^n$ be an open neighborhood of $\overline{v}$. A continuously differentiable function $W : \mathbb{R}^n \to \mathbb{R}^n$ is said to be a Lyapunov function at the state $\overline{v}$ over the set Δ for (18) if

$$\begin{cases} W(\overline{v}) = 0, \quad W(v) > 0, \quad \forall v \in \Delta\backslash\{\overline{v}\}, \\ \dfrac{\mathrm{d}W(v(t))}{\mathrm{d}t} = \nabla W(v(t))^T H(v(t)) \leq 0, \quad \forall v \in \Delta. \end{cases}$$

Lemma 2.3. (1) An isolated equilibrium point v^* is Lyapunov stable if there exists a Lyapunov function over some neighborhood Δ^* of v^*.

(2) An isolated equilibrium point v^* is asymptotically stable if there exists a Lyapunov function over some neighborhood Δ^* of v^* such that $\dfrac{\mathrm{d}W(v(t))}{\mathrm{d}t} < 0$ for all $v \in \Delta^* \backslash \{v^*\}$.

§3 Properties of the merit function f_μ

In this section, we investigate some properties of the merit function f_μ defined in (17).

Lemma 3.1. Let $\vartheta_\mu : \mathbb{R}^2 \to \mathbb{R}$ be defined by

$$\vartheta_\mu(a,b) = \sqrt{(1-\mu)a^2 + \mu b^2}.$$

Then there is a constant $c_\mu := \mu^2 - \mu + 1 \in (\frac{3}{4}, 1)$ such that

$$\|\nabla \vartheta_\mu(a,b)\|^2 \le c_\mu$$

for all the vectors $(a,b) \neq (0,0)$.

Proof. Let $(a,b) \in \mathbb{R}^2$ be any nonzeros vector, then we can get that ϑ_μ is continuously differentiable at (a,b). By computation, we obtain

$$\begin{aligned}
\|\nabla \vartheta_\mu(a,b)\|^2 &= (\frac{\partial \vartheta_\mu(a,b)}{\partial a})^2 + (\frac{\partial \vartheta_\mu(a,b)}{\partial b})^2 \\
&= \Big(\frac{(1-\mu)a}{\sqrt{(1-\mu)a^2+\mu b^2}}\Big)^2 + \Big(\frac{\mu b}{\sqrt{(1-\mu)a^2+\mu b^2}}\Big)^2 \\
&= \frac{(1-\mu)^2 a^2 + \mu^2 b^2}{(1-\mu)a^2+\mu b^2} \\
&= 1 - \mu(1-\mu)\frac{a^2+b^2}{(1-\mu)a^2+\mu b^2}.
\end{aligned}$$

Note that $\mu \in (0,1)$, then $a^2 + b^2 \ge (1-\mu)a^2 + \mu b^2$, then it can get that

$$0 \le \|\nabla \vartheta_\mu(a,b)\|^2 \le 1 - \mu(1-\mu) := c_\mu.$$

This completes the proof. □

Based on Lemma 3.1, we present the Clarke's generalized Jacobian (see [6]) of Φ_μ (16) at any arbitrary point $(x,y) \in \mathbb{R}^{2n}$.

Lemma 3.2. For the given parameter μ with $\mu \in (0,1)$ and the given positive diagonal matrices Σ and Γ, we can get the generalized Jacobian of Φ_μ at any arbitrary point $(x,y) \in \mathbb{R}^{2n}$ is

$$J(x,y) = \begin{pmatrix} A\Gamma + \Sigma & A\Gamma - \Sigma \\ D_x & D_y - I_n \end{pmatrix}, \tag{19}$$

where

$$D_x = \mathrm{diag}(d_x^1, d_x^2, \ldots, d_x^n) \quad \text{and} \quad D_y = \mathrm{diag}(d_y^1, d_y^2, \ldots, d_y^n)$$

are diagonal matrices with the i-th diagonal element being given by

$$d_x^i = \begin{cases} \dfrac{(1-\mu)x_i}{\sqrt{(1-\mu)x_i^2 + \mu y_i^2}}, & \text{if } x_i^2 + y_i^2 > 0, \\ \xi, & \text{if } x_i^2 + y_i^2 = 0 \end{cases} \tag{20}$$

and

$$d_y^i = \begin{cases} \dfrac{\mu y_i}{\sqrt{(1-\mu)x_i^2 + \mu y_i^2}}, & \text{if } x_i^2 + y_i^2 > 0, \\ \eta, & \text{if } x_i^2 + y_i^2 = 0. \end{cases} \tag{21}$$

Here, the real numbers ξ and η satisfy that $\xi^2 + \eta^2 \leq c_\mu$ for any $\xi, \eta \in \mathbb{R}$ while $x_i^2 + y_i^2 = 0$, where $c_\mu = \mu^2 - \mu + 1 \in (\frac{3}{4}, 1)$.

We recall that an $n \times n$ matrix A is called a P-matrix if every principal minor of A is positive. Equivalently, for every nonzero vector $x \in \mathbb{R}^n$, there is an index i_0 with $x_{i_0} \neq 0$ such that $x_{i_0}(Ax)_{i_0} > 0$. For a number of equivalent formulations, we refer the interested reader to the excellent book [7] by Cottle, Pang and Stone.

A square matrix $A \in \mathbb{R}^{n\times n}$ is called a Z-matrix if its off-diagonal entries are nonpositive. A nonsingular matrix $A \in \mathbb{R}^{n\times n}$ is called an M-matrix if it is a Z-matrix and $A^{-1} \geqslant 0$; and an H-matrix if its comparison matrix $\langle A\rangle = (\langle a\rangle_{ij}) \in \mathbb{R}^{n\times n}$ is an M-matrix, where

$$\langle a\rangle_{ij} = \begin{cases} |a_{ij}| & \text{for } i = j, \\ -|a_{ij}| & \text{for } i \neq j, \end{cases} \quad i, j = 1, 2, \ldots, n.$$

In particular, an H-matrix having positive diagonal entries is called an H_+-matrix.

According to the definition of P-matrix, if a matrix A is a P-matrix, then A^T is also a P-matrix. And it follows that the matrix A is a P-matrix if and only if the LCP(A, q) has a unique solution for all $q \in \mathbb{R}^n$. A sufficient condition for the matrix A to be a P-matrix is that A is a positive-definite matrix or an H_+-matrix.

Lemma 3.3. ([15]) A matrix of the form

$$D_a + D_b M$$

is nonsingular for all positive (negative) semidefinite diagonal matrices $D_a, D_b \in \mathbb{R}^n$ such that $D_a + D_b$ is positive (negative) definite if and only if $M \in \mathbb{R}^{n\times n}$ is a P-matrix.

Lemma 3.4. For the given parameter μ with $\mu \in (0, 1)$ and the given positive diagonal matrices Σ and Γ, if the matrix A is a P-matrix, then the generalized Jacobian matrix $J(x, y)$ defined in (19) is nonsingular for any vector $(x, y) \in \mathbb{R}^{2n}$.

Proof. Let $r = (r_1, r_2) \in \mathbb{R}^{2n}$ such that

$$\begin{pmatrix} A\Gamma + \Sigma & A\Gamma - \Sigma \\ D_x & D_y - I_n \end{pmatrix} \begin{pmatrix} r_1 \\ r_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

According to (20) and (21), we can obtain that $-1 < d_x^i < 1$ and $-1 < d_y^i < 1$ for $i = 1, 2, \ldots, n$. Hence, the diagonal matrix $D_y - I_n$ is nonsingular. Denote

$$S = -(D_y - I_n)^{-1} D_x \quad \text{and} \quad G = (A\Gamma + \Sigma) + (A\Gamma - \Sigma)S.$$

Thus, $r_2 = Sr_1$ and r_1 satisfies that $Gr_1 = 0$.

Denote

$$\alpha(x, y) = \{i : x_i \neq 0, y_i = 0\},$$
$$\beta(x, y) = \{i : x_i = 0, y_i \neq 0\},$$
$$\gamma(x, y) = \{i : x_i \neq 0, y_i \neq 0\},$$
$$\delta(x, y) = \{i : x_i = 0, y_i = 0\}.$$

Let s_i be the i-th diagonal element of the diagonal matrix S. Now, we consider four cases.

Case 1: when $i \in \alpha(x, y)$, then $-1 < d_x^i < 1$ and $d_y^i = 0$. It follows that

$$s_i = -(d_y^i - 1)^{-1} d_x^i = d_x^i.$$

Then, $-1 < s_i < 1$.

Case 2: when $i \in \beta(x,y)$, then $d_x^i = 0$ and $-1 < d_y^i < 1$. It follows that

$$s_i = -(d_y^i - 1)^{-1} d_x^i = 0.$$

Case 3: when $i \in \gamma(x,y)$, it follows that

$$\begin{aligned} s_i &= -(d_y^i - 1)^{-1} d_x^i \\ &= -\Big(\frac{\mu y_i}{\sqrt{(1-\mu)x_i^2 + \mu y_i^2}} - 1\Big)^{-1} \frac{(1-\mu)x_i}{\sqrt{(1-\mu)x_i^2 + \mu y_i^2}} \\ &= \frac{(1-\mu)x_i}{\sqrt{(1-\mu)x_i^2 + \mu y_i^2} - \mu y_i}. \end{aligned}$$

Since

$$\begin{aligned} &\big[(1-\mu)x_i\big]^2 - \big[\sqrt{(1-\mu)x_i^2 + \mu y_i^2} - \mu y_i\big]^2 \\ &= (1-\mu)^2 x_i^2 - ((1-\mu)x_i^2 + \mu y_i^2)^2 - \mu^2 y_i^2 + 2\mu y_i \sqrt{(1-\mu)x_i^2 + \mu y_i^2} \\ &= -\mu\Big[(1-\mu)x_i^2 + (1+\mu)y_i^2 - 2y_i\sqrt{(1-\mu)x_i^2 + \mu y_i^2}\Big] \\ &= -\mu\Big[(1-\mu)x_i^2 + \mu y_i^2 - 2y_i\sqrt{(1-\mu)x_i^2 + \mu y_i^2} + y_i^2\Big] \\ &= -\mu\Big[\sqrt{(1-\mu)x_i^2 + \mu y_i^2} - y_i\Big]^2 \le 0, \end{aligned}$$

then $-1 \le s_i \le 1$.

Case 4: when $i \in \delta(x,y)$, it follows that

$$s_i = -(d_y^i - 1)^{-1} d_x^i = -(\eta_i - 1)^{-1}\xi_i = \frac{\xi_i}{1-\eta_i},$$

where ξ_i and η_i satisfy $\xi_i^2 + \eta_i^2 \le c_\mu$ with $c_\mu \in (\frac{3}{4}, 1)$. Since $\xi_i^2 + \eta_i^2 \le c_\mu$, we can get that

$$(\sqrt{c_\mu} - \frac{1}{2})^2 \le (\eta_i - \frac{1}{2})^2 \le (\sqrt{c_\mu} + \frac{1}{2})^2$$

and

$$\begin{aligned} &\xi_i^2 - (1-\eta_i)^2 = \xi_i^2 - \eta_i^2 + 2\eta_i - 1 \le c_\mu - 2\eta_i^2 + 2\eta_i - 1 \\ &= c_\mu - 2(\eta_i - \frac{1}{2})^2 - \frac{1}{2} \le c_\mu - 2(\sqrt{c_\mu} - \frac{1}{2})^2 - \frac{1}{2} \\ &= -(\sqrt{c_\mu} - 1)^2 \le 0. \end{aligned}$$

Thus, $-1 \le s_i \le 1$. Therefore, we can obtain that $-I \le S \le I$.

According to the above analysis, the matrices $(I-S)\Sigma$ and $(I+S)\Gamma$ are positive semidefinite diagonal matrices. Note that

$$\Big[(I-S)\Sigma + (I+S)\Gamma\Big]_{ii} = (1-s_i)\Sigma_{ii} + (1+s_i)\Gamma_{ii},$$

(1) when $-1 < s_i < 1$, it has

$$(1-s_i)\Sigma_{ii} + (1+s_i)\Gamma_{ii} > 0;$$

(2) when $s_i = -1$, it has

$$(1-s_i)\Sigma_{ii} + (1+s_i)\Gamma_{ii} = 2\Sigma_{ii} > 0;$$

(3) when $s_i = 1$, it has

$$\big(1 - s_i\big)\Sigma_{ii} + \big(1 + s_i\big)\Gamma_{ii} = 2\Gamma_{ii} > 0.$$

Then, $\big(I - S\big)\Sigma + \big(I + S\big)\Gamma$ is a positive definite diagonal matrix.

Moreover, A is a P-matrix, then A^T is also a P-matrix. According to Lemma 3.3, the matrix

$$G^T = (I - S)\Sigma + (I + S)\Gamma A^T$$

is nonsingular. Therefore, $r_1 = 0$ and $r_2 = 0$. This completes the proof. $\square$

Define $\widetilde{\theta}_\mu : \mathbb{R}^2 \to \mathbb{R}_+$ by

$$\widetilde{\theta}_\mu(a, b) := \frac{1}{2}(\theta_\mu(a, b))^2 = \frac{1}{2}(\sqrt{(1-\mu)a^2 + \mu b^2} - b)^2.$$

In order to establish some important results for the merit function f_μ, we will take a closer look at the properties of $\widetilde{\theta}_\mu$ and give the following result.

Lemma 3.5. The function $\widetilde{\theta}_\mu$ is continuously differentiable with $\nabla\widetilde{\theta}_\mu(0, 0) = (0, 0)^T$.

Proof. The continuous differentiability of $\widetilde{\theta}_\mu$ can be verified by direct calculation. Since the origin is a global minimizer of the function $\widetilde{\theta}_\mu$, from the first-order optimality conditions in unconstrained minimization, we can get that $\nabla\widetilde{\theta}_\mu(0, 0) = (0, 0)^T$. $\square$

According to Lemmas 3.2 and 3.5, we obtain the continuously differentiable for the function f_μ.

Lemma 3.6. The function f_μ is continuously differentiable with the gradient

$$\nabla f_\mu(x, y) = J(x, y)^T\Phi_\mu(x, y),$$

for any $J(x, y) \in \partial\Phi_\mu(x, y)$.

§4 Stability and convergence analysis

In this section, we consider the steepest descent-based neural network for solving the LCP(A, q) with its dynamical equation being given by

$$\frac{\mathrm{d}v(t)}{\mathrm{d}t} = -\rho\nabla f_\mu(v(t)), \tag{22}$$

where $v(t) = (x(t), y(t))$ and $\nabla f_\mu(x, y) = J(x, y)^T\Phi_\mu(x, y)$ for any $J(x, y) \in \partial\Phi_\mu(x, y)$.

Now, we look at the stability and convergence analysis of the neural network (22).

It is easy to obtain the following results, which can be seen in [19].

Proposition 4.1. For the given $n \times n$ positive diagonal matrices Σ and Γ, let $f_\mu : \mathbb{R}^{2n} \to \mathbb{R}_+$ be defined as in (17). Then the following results hold.

(1) For any $v = (x, y) \in \mathbb{R}^{2n}$, it holds $f_\mu(v) \geq 0$.

(2) If the solution set of LCP(A, q) is nonempty, the point $v = (x, y)$ is a global minimizer of $f_\mu(v)$ if and only if $z = \Gamma(y + x)$ and $w = \Sigma(y - x)$ solves of the LCP(A, q).

(3) $f_\mu(v(t))$ is nonincreasing function of t, where $v(t)$ is a solution of the neural network (22).

Proposition 4.2. (1) If the solution set of the LCP(A, q) is nonempty, then every solution of the LCP(A, q) corresponds to an equilibrium point of the neural network (22).

(2) If A is a P-matrix, then the unique equilibrium point of (22) corresponds to the unique solution of the LCP(A, q).

Proof. (1) If (z^*, w^*) is a solution of the LCP(A, q), we have $\Phi_\mu(v^*) = 0$ at the point $v^* = (x^*, |x^*|)$ with $x^* = \dfrac{1}{2}(\Gamma^{-1}z^* - \Sigma^{-1}w^*)$. From Lemma 3.6, we can obtain that $\nabla f_\mu(v^*) = 0$.

Hence, $\frac{\mathrm{d}v(t)}{\mathrm{d}t} = 0$.
(2) Let $v^* = (x^*, y^*)$ is an equilibrium point of (22), then $\nabla f_\mu(v^*) = 0$. Since A is a P-matrix, by Lemma 3.4, the generalized Jacobian matrix $J(x^*, y^*)$ defined in (19) is nonsingular. With $\nabla f_\mu(v^*) = J(x^*, y^*)^T \Phi_\mu(x^*, y^*) = 0$, it yields $\Phi_\mu(x^*, y^*) = 0$. From Proposition 4.1 (2), we can get that $z^* = \Gamma(y^* + x^*)$ and $w^* = \Sigma(y^* - x^*)$ solve of the LCP(A, q). This completes the proof. □

Now, we establish the following proposition about the existence of the solution trajectory of the neural network (22), which is similar to [19].

Proposition 4.3. (1) For any initial state $v_0 = v(t_0)$, there exists exactly one maximal solution $v(t)$ with $t \in [t_0, \tau(v_0))$ for the neural network (22).
(2) If the level sets $\mathcal{L}(v_0) = \{v = (x, y) \in \mathbb{R}^{2n} : f_\mu(v) \le f_\mu(v_0)\}$ are bounded, then $\tau(v_0) = \infty$.

Theorem 4.1. If the level set $\mathcal{L}(v_0) = \{v = (x, y) \in \mathbb{R}^{2n} : f_\mu(v) \le f_\mu(v_0)\}$ is bounded, then the trajectory $v(t)$ of (22) for any $v_0 \in \mathbb{R}^{2n}$ converges to an equilibrium point.

With Proposition 4.2 and Theorem 4.1, we can obtain the following result.

Theorem 4.2. If A is a P-matrix and the level sets $\mathcal{L}(v_0) = \{v = (x, y) \in \mathbb{R}^{2n} : f_\mu(v) \le f_\mu(v_0)\}$ are bounded, then the unique accumulation point of the trajectory $v(t)$ solves the LCP(A, q).

From Proposition 4.2 (1), every solution v^* to the LCP is an equilibrium point of the neural network (22). If, in addition, v^* is an isolated equilibrium point of (22), then we can show that v^* is not only Lyapunov stable but also asymptotically stable.

Theorem 4.3. If v^* is an isolated equilibrium point of the neural network (22), then v^* is Lyapunov stable for (22), and furthermore, it is asymptotically stable.

Proof. Since $v^* = (x^*, y^*)$ is a solution to the LCP, $f_\mu(x^*, y^*) = 0$ and $\Phi_\mu(x^*, y^*) = 0$. In addition, since v^* is an isolated equilibrium point of (22), there exists a neighborhood $\Delta^* \subseteq \mathbb{R}^{2n}$ of v^* such that

$$\nabla f_\mu(x^*, y^*) = 0 \quad \text{and} \quad \nabla f_\mu(x, y) \neq 0, \quad \forall (x, y) \in \Delta^* \backslash \{v^*\}.$$

Next, we argue that $f_\mu(x, y)$ is indeed a Lyapunov function at (x, y) over the set Δ^* for (22) by showing that the conditions in Definition 2.4 are satisfied. Firstly, notice that $f_\mu(x, y) \ge 0$. Suppose that there is a point $\overline{v} = (\overline{x}, \overline{y}) \in \Delta^* \backslash \{v^*\}$ such that $f_\mu(\overline{x}, \overline{y}) = 0$. Thus, $\Phi_\mu(\overline{x}, \overline{y}) = 0$. By Lemma 3.6, we have $\nabla f_\mu(\overline{x}, \overline{y}) = 0$, i.e., $\overline{v} = (\overline{x}, \overline{y})$ is also an equilibrium point of (22), which clearly contradicts the assumption that v^* is an isolated equilibrium point in Δ^*. Therefore, we have that $f_\mu(x, y) > 0$ for any $(x, y) \in \Delta^* \backslash \{v^*\}$. Again that

$$\frac{\mathrm{d}f_\mu(v(t))}{\mathrm{d}t} = \nabla f_\mu(v(t))^T \frac{\mathrm{d}v(t)}{\mathrm{d}t} = -\rho \nabla f_\mu(v(t))^T \nabla f_\mu(v(t)) = -\rho \|\nabla f_\mu(v(t))\|^2 \le 0. \tag{23}$$

Hence, the conditions of Definition 2.4 are satisfied. It follows that f_μ is a Lyapunov function at the point v^* over the set Δ^*. From Lemma 2.3 (1), f_μ is Lyapunov stable.

Now, we show that v^* is asymptotically stable. Since v^* is isolated, from (23) we have

$$\frac{\mathrm{d}f_\mu(v(t))}{\mathrm{d}t} = -\rho \|\nabla f_\mu(v(t))\|^2 < 0, \quad \forall v(t) \in \Delta^* \backslash \{v^*\}.$$

This, by Lemma 2.3 (2), implies that v^* is asymptotically stable. □

§5 Simulation results

We consider the neural network based on different NCP-functions for solving LCP(A, q); see Table 1 for abbreviations of the corresponding methods.

Table 1. Abbreviations of testing methods.

Method	Description
FB	the neural network (4) based on ϕ_{FB} (5)
GFB	the neural network (4) based on ϕ^p_{GFB} (6)
GNR	the neural network (4) based on ϕ^p_{GNR} (7)
S1-GNR	the neural network (4) based on $\phi^p_{\text{S1-GNR}}$ (8)
S2-GNR	the neural network (4) based on $\phi^p_{\text{S2-GNR}}$ (9)
MNN	the neural network (22) based on Φ_μ (16)

In our simulations, we use MATLAB ordinary differential equation solvers *ode15s*, *ode23t*, *ode23t* and *ode23tb*. The stopping criterion in simulating is default, i.e., the scalar relative error tolerance less 1e-3. For MNN, we choose $\Gamma = \Sigma = I_n$.

Example 1. Consider the LCP(A, q) with

$$A = \begin{pmatrix} 4 & -1 & 1 & 0 & 0 \\ -1 & 4 & -1 & 1 & 0 \\ 1 & -1 & 4 & -1 & 1 \\ 0 & 1 & -1 & 4 & -1 \\ 0 & 0 & 1 & -1 & 4 \end{pmatrix}, \quad q = \begin{pmatrix} -7 \\ 5 \\ -18 \\ 9 \\ -23 \end{pmatrix}.$$

For Example 1, since the system matrix A is symmetric positive definite, we know that the LCP(A, q) has a unique solution that

$$z^* = (1, 0, 3, 0, 5)^T \quad \text{and} \quad w^* = (0, 1, 0, 1, 0)^T.$$

We simulate the neural network (22) with different values of μ, ρ and different initial points $v_0 = (x_0, y_0)$ to see the influence on the convergence of trajectories to the LCP solution. Here, we use the solver *ode23t*.

Figure 1 shows the trajectories of the neural network (22) for different values of ρ to solve Example 1 when the initial point $x_0 = (1, 0.5, -1.5, -0.5, 1)^T$, $y_0 = (-1, 2, -1, 2, -1)^T$ and $\mu = 0.1$. Figure 2 depicts the comparison of the different values of μ when the initial point $x_0 = (1, 0.5, -1.5, -0.5)^T$, $y_0 = (1, -1, 2, -1, 2, -1)^T$ and $\rho = 100$. Among these five values $\mu = 0.1, 0.3, 0.5, 0.7, 0.9$, the neural network (22) with $\mu = 0.1$ has the best numerical performance. In other words, a small value of μ yields a faster convergence.

In Figure 3, we simulate the neural network (22) with $\mu = 0.1$ and $\rho = 100$ using the following four initial points:

$$\begin{aligned}
v_0^1 &= (0, 0, 0, 0, 0, 0, 0, 0, 0, 0)^T, \\
v_0^2 &= (0, 0, 0, 0, 0, 1000, 1000, 1000, 1000, 1000)^T, \\
v_0^3 &= (1000, 1000, 1000, 1000, 1000, 1000, 1000, 1000, 1000, 1000)^T, \\
v_0^4 &= (-1000, -1000, -1000, -1000, -1000, 1000, 1000, 1000, 1000, 1000)^T.
\end{aligned}$$

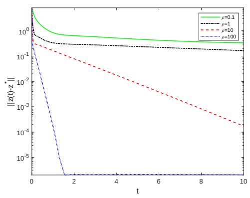


Figure 1. Convergence behavior of the error $\|z(t) - z^*\|$ in Example 1 using the neural network (22) for different values of ρ.

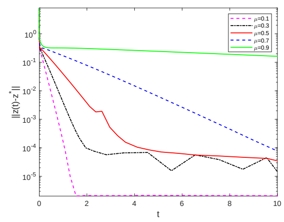


Figure 2. Convergence behavior of the error $\|z(t) - z^*\|$ in Example 1 using the neural network (22) for different values of μ.

Note that these initial points show the similar performances and all converge to the unique solution very fast. Transient behavior of $z(t)$ in Example 1 of the neural network (22) are presented in Figure 4. It shows that the neural network (22) is efficient in solving the LCP.

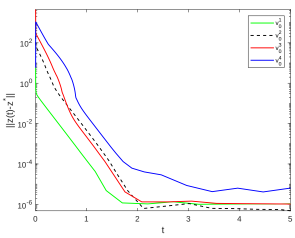


Figure 3. Convergence behavior of the error $\|z(t) - z^*\|$ in Example 1 using the neural network (22) with $\mu = 0.1$ and $\rho = 100$ for different initial points.

Example 2. Let m be a prescribed positive integer and $n = m^2$. Consider the LCP (1), where $A = \widehat{A} + \nu I \in \mathbb{R}^{n\times n}$ with

$$\widehat{A} = \mathrm{Tridiag}(-1.5I_m, S, -0.5I_m) \in \mathbb{R}^{n\times n}$$

being a block-tridiagonal matrix,

$$S = \mathrm{tridiag}(-1.5, 4, -0.5) \in \mathbb{R}^{m\times m}$$

being a tridiagonal matrix and

$$z^* = (1, 0, 1, 0, \ldots, 1, 0)^T \in \mathbb{R}^n$$

being the unique solution of the LCP(A, q).

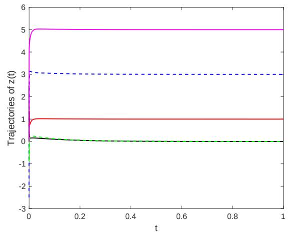
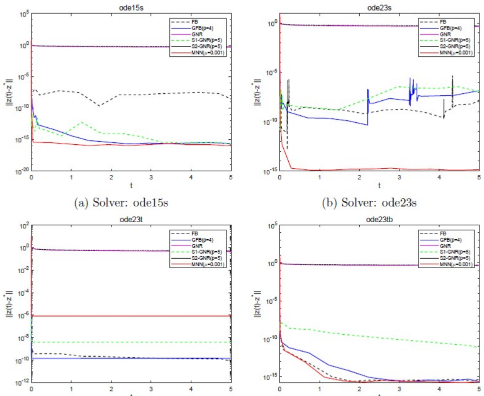


Figure 4. Transient behavior of $z(t)$ in Example 1 of the neural network (22) with $x_0 = (1, 0.5, -1.5, -0.5, 1)^T$, $y_0 = (-1, 2, -1, 2, -1)^T$, $\mu = 0.1$ and $\rho = 100$.



(a) Solver: ode15s (b) Solver: ode23s

(c) Solver: ode23t (d) Solver: ode23tb

Figure 5. Convergence behavior of the error $\|z(t) - z^*\|$ in Example 2 with $m = 6$, $\nu = 1$ and $\rho = 10^6$ for different solvers.

For Example 2, let $z_0 = ones(n, 1)$ be the initial point for FB, GFB, GNR, S1-GNR and S2-GNR and $v_0 = ones(2n, 1)$ be the initial point for MNN. Figure 5 depicts the comparison of different solvers and test methods with $\rho = 10^6$, $m = 6$, $\nu = 1$. It can be seen that GNR and S2-GNR don't converge to the solution of Example 2. When the solver 'ode23t' is used to solve the ordinary differential equation, the GFB with $p = 4$ method is the best. However, it can be observed in Figure 5, in which the numerical stability of MNN is the best for different solvers used to solve the ordinary differential equation.

§6 Concluding remarks

We have presented a new equivalent reformulation of the LCP(A, q) and constructed a family of neural networks. The characterization of stationary points of the corresponding merit functions has been established. Moreover, we proved the stability properties of the proposed steepest descent-based neural networks. Finally, numerical simulations indicate that the proposed neural networks have significantly faster convergence rates and better stability than those neural networks based on NCP-functions as their merit functions. Further discussions on the matrix A or the smoothing technology such that the level sets being bounded will be interesting topics in our future study.

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Declarations

Conflict of interest The authors declare no conflict of interest.

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