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Chelyshkov matrix-collocation method for solving nonlinear quadratic integral equations

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Abstract. The main purpose of this paper is to use the Chelyshkov-collocation spectral method for solving nonlinear Quadratic integral equations of Volterra type. The method is based on the approximate solutions in terms of Chelyshkov polynomials with unknown coefficients. The Chelyshkov polynomials and their properties are employed to derive the operational matrices of integral and product. The application of these operational matrices for solving the mentioned problem is explained. The error analysis of the proposed method is investigated. Finally, some numerical examples are provided to demonstrate the efficiency of the method.

§1 Introduction

The theory of integral equations has many useful applications in describing numerous events and problems of the real world. For example, these equations are often applicable in engineering, mathematical physics, economics and biology. A special type of the integral equations, namely quadratic integral equations, provide an important tool for modeling processes engineering and consequently this type of integral equations have been used increasingly in different areas of applied science, such as the theory of radiative transfer, kinetic theory of gases, the traffic theory and the theory of neutron transport [5]. Moreover, the quadratic integral equations have been used in solving most of the boundary value problems of both ordinary and partial differential equations [1]. Thus, due to the above-mentioned applications, the development of numerical approaches for solving quadratic integral equations is very essential.

Recently, many analytical and numerical methods have been applied to solve quadratic integral equations. In 2001, Banas et al. [6] studied the solvability of the quadratic integral equation of Urysohn-Stieltjes type. In 2004, Banas and Maritinon [5] investigated the solvability of a nonlinear quadratic integral equation of Volterra type. In 2005, Darvish [11] considered the nonlinear quadratic integral equations of fractional orders and presented an existence theorem for them. In 2006, Banas et al. [4] investigated the existence and asymptotic behaviour of solutions of the nonlinear quadratic integral equations of Hammerstein type. In [1, 2, 12–14, 21, 33, 34], various numerical methods such as Adomian decomposition method,

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repeated trapezoidal method, Picard method, homotopy perturbation method and variational iteration method are used to solve the nonlinear quadratic integral equations. In 2009, Yousefi et al. [31] applied the variational iteration method to present the approximate solution of nonlinear mixed Volterra-Fredholm integral equations. In 2016, Mirzaee and Hadadiyan [24] presented a numerical method to solve nonlinear quadratic integral equations based on modification of hat functions and their operational matrices. In 2017, Al-Badrani [3] used the variational homotopy perturbation method to solve nonlinear quadratic integral equations. In 2018, Mirzaee and Alipour [23] described an approximate scheme based on piecewise hat function to solve nonlinear quadratic integral equations of fractional order. In 2019, Rabbani et al. [26] investigated the existence of a solution for nonlinear quadratic integral equations using the generalized form of Darbo fixed point theorem. In 2020, Zeghdane [32] solved the Volterra stochastic integral equations by the Chebyshev cardinal functions. Recently, Basseem and Alalyani [7] solved a mixed nonlinear quadratic integral equation with a singular kernel by a quadrature method.

In this paper, we introduce a matrix-based method for numerical solution of quadratic integral equations of Volterra type in the form:

$$y(x) = a(x) + f(x, y(x)) \int_0^x g(t, y(t))dt. \tag{1.1}$$

The proposed method is based on the spectral-collocation method [9]. The spectral methods play a significant role in various fields of applied science, especially in fluid dynamics where a large spectral hydrodynamics codes are now regularly used to study turbulence, transition, numerical weather prediction and ocean dynamics [9]. Recently, researchers have investigated the solutions of partial differential and integral equations that model many systems and processes in chemistry, physics, biology, engineering, astrophysics and space science. For example, in 2019, Hamid et al. [15] presented an efficient and precise computational algorithm based on a new kind of polynomials together with the collocation technique to solve time-space fractional partial differential equations with the Riesz derivative. Also, in 2021, Hamid et al. [16] solved a family of nonlinear evolution differential equations by an operational matrix-based spectral computational method coupled with the Picard technique. Spectral methods are built on approximating the series solutions for differential equations in terms of orthogonal polynomials, say $\sum a_k\phi_k$. There are three well-known versions that are used as popular techniques to determine the expansion coefficients, namely collocation, tau and Galerkin methods [9].

The major objective of this paper is to provide a new numerical method for solving Eq. (1.1) by the orthogonal Chelyshkov polynomials introduced in [10]. These polynomials have been used in the solution of weakly singular integral equations in [27] based on the product integration method and for solving the Sine-Gordon equation [29]. Sezer et al. derived the matrix method based on Chelyshkov polynomials for solving a class of mixed functional integro-differential equations [25]. Bazm et al. in [8] developed the matrix formulation for nonlinear Vlottera-Hammersian integral. In 2018, Talaei et al. [28, 30] proposed the Chelyskov matrix formulation based on collocation method for numerical solution of the multi-order fractional differential equations and two-dimensional Fredholm-Volterra integral equations. In 2020, Hamid et al. [17] analyzed the transport dynamics and anomalous diffusion in fractional model by a Chelyshkov polynomial-based algorithm. Also, in the same year, they [18, 19] used

Chelyshkov polynomials to analyze some other problems. Recently, Hamid et al. [20] used a Picard Chelyshkov polynomial method to attain nonlinear oscillatory problems of arbitrary orders.

The outline of this paper is as follows : in Section 2, we review existence theorem of Eq. (1.1) and some properties of Chelyshkov polynomials. The operational matrices of integration and product are derived and apply together with collocation method to reduce the problem to a system of algebraic linear equations in Section 3. The error analysis and algorithm of the method is presented in Section 4. The numerical results are given in Section 5. Finally, the conclusion of the paper is given in Section 6.

§2 Preliminaries

At first, we present the existence theorem of a unique solution for the nonolinear quadratic integral equation (1.1).

Theorem 2.1. *[12] Suppose that for Eq. (1.1), the following conditions hold:*
(i) $a : [0, 1] \to [0, \infty)$ *is a continuous function.*
(ii) The functions f *and* g *are continuous and also there exist positive constants* M_1 *and* M_2 *such that* $|f| \le M_1$ *and* $|g| \le M_2$.
(iii) The functions f *and* g *satisfy Lipschitz condition with Lipschitz constants* L_1 *and* L_2 *with respect to their second variables, i.e.,*

$$\begin{aligned} |f(t, x) - f(t, y)| &\le L_1|x - y|, \\ |g(t, x) - g(t, y)| &\le L_2|x - y|. \end{aligned}$$

If

$$L_1M_2 + M_1L_2 < 1, \tag{2.2}$$

then the nonlinear quadratic integral equation (1.1) has a unique solution $y \in C[0, 1]$.

In continuation, we recall the basic definition of Chelyshkov polynomials as well as some important properties. The orthogonal Chelyshkov polynomials [10] are sequences of polynomials which are orthogonal in the interval $[0, 1]$ with the weight function 1. These polynomials are explicitly defined by

$$C_{N,n}(x) = \sum_{j=0}^{N-n} (-1)^j \binom{N-n}{j} \binom{N+n+j+1}{N-n} x^{n+j}, \quad n = 0, 1, ..., N. \tag{2.3}$$

For example, if $N = 3$ we obtain

$$\begin{aligned} C_{3,0}(x) &= 4 - 30x + 60x^2 - 35x^3, \\ C_{3,1}(x) &= 10x - 30x^2 + 21x^3, \\ C_{3,2}(x) &= 6x^2 - 7x^3, \\ C_{3,3}(x) &= x^3. \end{aligned}$$

Also, Fig. 1 shows the behavior of Chelyshkov polynomials in the interval $[0, 1]$ for $N = 3$. Now, let us approximate the solution of Eq. (1.1) as

$$y(x) \simeq y_N(x) = \sum_{n=0}^{N} a_n C_{N,n}(x) = \mathbf{A\Phi}(x) = \mathbf{A}\ W\mathbf{X}(x), \tag{2.4}$$

where

$$\mathbf{A} = (a_0, a_1, ..., a_N), \qquad \mathbf{X}(x) = (1, x, ..., x^N)^T,$$

and

$$\mathbf{\Phi}(x) = (C_{N,0}(x), C_{N,1}(x), ..., C_{N,N}(x))^T. \tag{2.5}$$

Also W is a $(N+1)\times(N+1)$ upper triangular matrix as follows:
If N is odd,

$$W = \begin{pmatrix} \binom{N}{0}\binom{N+1}{N} & -\binom{N}{1}\binom{N+2}{N} & \dots & \binom{N}{N-1}\binom{2N}{N} & -\binom{N}{N}\binom{2N+1}{N} \\ 0 & \binom{N-1}{0}\binom{N+2}{N-1} & \dots & -\binom{N-1}{N-2}\binom{2N}{N-1} & \binom{N-1}{N-1}\binom{2N+1}{N-1} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & \binom{1}{0}\binom{2N}{1} & -\binom{1}{1}\binom{2N+1}{1} \\ 0 & 0 & \dots & 0 & 1 \end{pmatrix},$$

and if N is even,

$$W = \begin{pmatrix} \binom{N}{0}\binom{N+1}{N} & -\binom{N}{1}\binom{N+2}{N} & \dots & -\binom{N}{N-1}\binom{2N}{N} & \binom{N}{N}\binom{2N+1}{N} \\ 0 & \binom{N-1}{0}\binom{N+2}{N-1} & \dots & \binom{N-1}{N-2}\binom{2N}{N-1} & -\binom{N-1}{N-1}\binom{2N+1}{N-1} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & \binom{1}{0}\binom{2N}{1} & -\binom{1}{1}\binom{2N+1}{1} \\ 0 & 0 & \dots & 0 & 1 \end{pmatrix}.$$

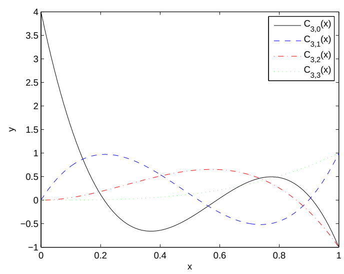

Figure 1. Behavior of Chelyshkov polynomials for $N = 3$.

These polynomials satisfy the orthogonality condition

$$\int_0^1 C_{N,p}(x)C_{N,q}(x)dx = \begin{cases} 0, & p \neq q, \\ \frac{1}{p+q+1}, & p = q, \end{cases} \tag{2.6}$$

where $p, q = 0, 1, ..., N$. Also, they can be connected to hypergeometric functions, orthogonal exponential polynomials and Jacobi polynomials $P_k^{\iota,\gamma}(x)$ by the following relation

$$C_{N,n}(x) = (-1)^{N-n} x^n P_{N-n}^{0,2n+1}(2x-1), \quad n = 0, 1, ..., N.$$

Hence, they keep distinctively attributes of the classical orthogonal polynomials and may be facilitated to different problems on approximation. Indeed, the family of orthogonal polynomials $\{C_{N,n}(x)\}_{n=0}^{N}$ have n multiple zeros $x = 0$ and $N - n$ distinct real zeros in the interval $[0, 1]$. Hence, for every N if the roots of the polynomial $C_{N,n}(x)$ are chosen as node points, then an accurate numerical quadrature can be derived.

§3 Matrix representation and Main results

In this section, we first introduce the operational matrix of integration.

Theorem 3.1. *The operational matrix of integration is defined as follows*

$$\int_0^x \mathbf{\Phi}(s)ds \simeq P\mathbf{\Phi}(x),$$

where P is an $(N+1) \times (N+1)$ operational matrix as follows

$$P = \begin{pmatrix} \Theta(0,0) & \Theta(0,1) & \dots & \Theta(0,N) \\ \Theta(1,0) & \Theta(1,1) & \cdots & \Theta(1,N) \\ \vdots & \vdots & \ddots & \vdots \\ \Theta(N,0) & \Theta(N,1) & \dots & \Theta(N,N) \end{pmatrix},$$

where

$$\Theta(n,k) = \sum_{j=n}^{N} (-1)^{j-n} \binom{N-n}{j-n} \binom{N+j+1}{N-n} \beta(1, j+1) \xi_{k,j}, \tag{3.7}$$

and

$$\xi_{k,j} = (2k+1) \sum_{l=k}^{N} \frac{(-1)^{l-k}}{j+l+2} \binom{N-k}{l-k} \binom{N+l+1}{N-k}.$$

Proof. Integrating of $C_{N,n}(x)$ from 0 to x yields

$$\int_0^x C_{N,n}(s)ds = \sum_{j=n}^{N} (-1)^{j-n} \binom{N-n}{j-n} \binom{N+j+1}{N-n} \beta(1, j+1) x^{j+1}. \tag{3.8}$$

Now, approximating $x^{(j+1)}$ by $N+1$ terms of Chelyshkov polynomials, we obtain

$$x^{(j+1)} \simeq \sum_{k=0}^{N} \xi_{k,j} C_{N,k}(x), \tag{3.9}$$

where $\xi_{k,j}$ is given as follows

$$\begin{aligned} \xi_{k,j} &= (2k+1) \int_0^1 x^{(j+1)} C_{N,k}(x) dx, \\ &= (2k+1) \sum_{l=k}^{N} (-1)^{l-k} \binom{N-k}{l-k} \binom{N+l+1}{N-k} \int_0^1 x^{j+l+1} dx, \\ &= (2k+1) \sum_{l=k}^{N} \frac{(-1)^{l-k}}{j+l+2} \binom{N-k}{l-k} \binom{N+l+1}{N-k}. \end{aligned} \tag{3.10}$$

By Eqs. (3.8) and (3.9) we obtain

$$\int_0^x C_{N,n}(s)ds = \sum_{k=0}^{N}\left(\sum_{j=n}^{N}(-1)^{j-n}\binom{N-n}{j-n}\binom{N+j+1}{N-n}B(1,j+1)\xi_{k,j}\right)C_{N,k}(x)$$
$$= \sum_{k=0}^{N}\Theta(n,k)C_{N,k}(x), \tag{3.11}$$

where $\Theta(n,k)$ are given in Eq. (3.7). Accordingly, Eq. (2.4) can be rewritten as the following vector form:

$$\int_0^x C_{N,n}(s)ds = (\Theta(n,0),\Theta(n,1),...,\Theta(n,N))\,\mathbf{\Phi}(x).$$

This leads to the desired result for $n=0,...,N$. □

Now, we introduce the operational matrix of product that great main role in the proposed method, since reduce the solution of Eq. (1.1) to the solution of an algebraic equation.

Theorem 3.2. *If $V=[v_0,v_1,...,v_N]^T$, then*

$$\mathbf{\Phi}(x)\mathbf{\Phi}^T(x)V \simeq \widehat{V}\mathbf{\Phi}(x), \tag{3.12}$$

with

$$\widehat{V} = [\widehat{v}_{i,j}]_{i,j=0}^{N}, \quad \widehat{v}_{i,j} = \sum_{l=0}^{N} v_l\mu_{i,l,j}. \tag{3.13}$$

Proof. Form left side of Eq. (3.12), we have

$$\mathbf{\Phi}(x)\mathbf{\Phi}^T(x)V = \begin{pmatrix} \sum_{j=0}^{N} v_jC_{N,0}(x)C_{N,j}(x) \\ \sum_{j=0}^{N} v_jC_{N,1}(x)C_{N,j}(x) \\ \vdots \\ \sum_{j=0}^{N} v_jC_{N,N}(x)C_{N,j}(x) \end{pmatrix}. \tag{3.14}$$

Using relation Eq. (2.4), one can approximate $C_{N,i}(x)C_{N,j}(x)$ for $i,j=0,...,N$ in the form

$$C_{N,i}(x)C_{N,j}(x) \simeq \sum_{k=0}^{N}\mu_{i,j,k}C_{N,k}(x),$$

Using the orthogonality of Chelyshkov polynomials, we have

$$\begin{aligned}\mu_{i,j,k} &= (2k+1)\int_0^1 C_{N,i}(x)C_{N,j}(x)C_{N,k}(x)dx \\ &= (2k+1)\int_0^1\left(\sum_{l=0}^{2N}\rho_{i,j,l}x^l\right)C_{N,k}(x)dx \\ &= (2k+1)\sum_{l=0}^{2N}\rho_{i,j,l}\left(\int_0^1 x^lC_{N,k}(x)dx\right) \\ &= (2k+1)\sum_{l=0}^{2N}\sum_{r=k}^{N}\frac{\rho_{i,j,l}(-1)^{r-k}}{(r+l+1)}\binom{N-k}{r-k}\binom{N+r+1}{N-k}.\end{aligned}$$

Therefore, for $i = 0, 1, ..., N$, we have

$$\begin{aligned}\sum_{j=0}^{N} v_j C_{N,i}(x) C_{N,j}(x) &\simeq \sum_{j=0}^{N} v_j \left(\sum_{k=0}^{N} \mu_{i,j,k} C_{N,k}(x) \right) \\ &= \sum_{k=0}^{N} \left(\sum_{j=0}^{N} v_j \mu_{i,j,k} \right) C_{N,k}(x) \\ &= \sum_{k=0}^{N} \widehat{v}_{i,k} C_{N,k}(x). \qquad (3.15)\end{aligned}$$

Substituting Eq. (3.15) into Eq. (3.14) leads to the desired result. □

For implementing the operational matrices method on Eq. (1.1), we find the collocation approximation in the form

$$\begin{cases} w_1(x) = f(x, y(x)) \simeq w_{N,1}(x) = \sum_{i=0}^{N} w_{i,1} C_{N,i}(x) = \mathbf{\Phi}^T(x)\mathbf{W}_1, \\ \\ w_2(x) = g(x, y(x)) \simeq w_{N,2}(x) = \sum_{i=0}^{N} w_{i,2} C_{N,i}(x) = \mathbf{\Phi}^T(x)\mathbf{W}_2, \end{cases} \qquad (3.16)$$

where

$$\mathbf{W}_i = (w_{i,0}, w_{i,1}, ..., w_{i,N})^T, \quad \mathbf{\Phi}(x) = (C_{N,0}(x), C_{N,1}(x), ..., C_{N,N}(x))^T.$$

Based Sloan's new collocation method [22], it is observed that $w_i(x)$ for $i = 1, 2$ satisfy in following integral equations

$$\begin{cases} w_1(x) = f\left(x, a(x) + w_1(x)\int_0^x w_2(t)dt\right), \\ \\ w_2(x) = g\left(x, a(x) + w_1(x)\int_0^x w_2(t)dt\right). \end{cases} \qquad (3.17)$$

Applying an approximation scheme to Eq. (3.17) will lead to an approximate solution of Eq. (1.1). By substituting the approximated Eq. (3.16) in Volterra integral part of Eq. (1.1) and using Theorems 3.1 and 3.2, we get

$$\begin{aligned} w_1(x)\int_0^x w_2(t)dt &\simeq \mathbf{W}_1^T \mathbf{\Phi}(x) \int_0^x \mathbf{\Phi}^T(x)\mathbf{W}_2 dt \\ &= \mathbf{W}_1^T \mathbf{\Phi}(x) \int_0^x \mathbf{\Phi}^T(x) dt \mathbf{W}_2 \\ &= \mathbf{W}_1^T \mathbf{\Phi}(x) \mathbf{\Phi}^T(x) P^T \mathbf{W}_2 \\ &\simeq \mathbf{\Phi}^T(x) \widehat{\mathbf{W}}_1^T P^T \mathbf{W}_2. \qquad (3.18)\end{aligned}$$

By substituting (3.18) and (3.16) in (3.17) we obtain

$$\begin{cases} \mathbf{\Phi}^T(x)\mathbf{W}_1 = f\left(x, a(x) + \mathbf{\Phi}^T(x)\widehat{\mathbf{W}}_1^T P^T \mathbf{W}_2\right), \\ \\ \mathbf{\Phi}^T(x)\mathbf{W}_2 = g\left(x, a(x) + \mathbf{\Phi}^T(x)\widehat{\mathbf{W}}_1^T P^T \mathbf{W}_2\right). \end{cases} \qquad (3.19)$$

Now, by collocation points $x_i,\ i = 0, 1, ..., N$, the zeros of the $C_{N+1,0}(x)$, we have

$$\begin{cases} \mathbf{\Phi}^T(x_i)\mathbf{W}_1 = f\left(x_i, a(x_i) + \mathbf{\Phi}^T(x_i)\widehat{\mathbf{W}}_1^T P^T \mathbf{W}_2\right), \\ \\ \mathbf{\Phi}^T(x_i)\mathbf{W}_2 = g\left(x_i, a(x_i) + \mathbf{\Phi}^T(x_i)\widehat{\mathbf{W}}_1^T P^T \mathbf{W}_2\right), \end{cases} \qquad (3.20)$$

then Eq. (3.20) is a set of $2(N+1)$ nonlinear algebraic equations with $2(N+1)$ unknowns which can be solved using any standard iteration technique, after solving this nonlinear system, we get $\mathbf{W}_i,\ i=1,2$. Hence, the solution can be written in the form

$$y_N(x) = a(x) + w_{N,1}(x)\int_0^x w_{N,2}(t)dt. \tag{3.21}$$

§4 Error analysis

In this section, an error bound is computed for the unknown function $y(x)$ via its expansion by means of Chelyshkov orthogonal polynomials.

Definition 4.1. *(Taylor's formula). Suppose that $D^k f(x) \in C[0,1]$ for $k = 0,1,...,N+1$. Then, we have*

$$f(x) = \sum_{i=0}^{N} \frac{x^i}{\Gamma(i+1)} D^i f(0) + \frac{x^{(N+1)}}{\Gamma(N+2)} D^{(N+1)} f(\xi), \tag{4.22}$$

with $0 < \xi \le x$, $\forall x \in (0,1]$. Also,

$$|f(x) - \sum_{i=0}^{N} \frac{x^i}{\Gamma(i+1)} D^i f(0)| \le M \frac{x^{(N+1)}}{\Gamma(N+2)}, \tag{4.23}$$

where $|D^{(N+1)} f(\xi)| \le M$.

Let $M_N = \{C_{N,0}(x), C_{N,1}(x), ..., C_{N,N}(x)\}$ be the set of polynomials of complete $L^2[0,1]$, since M_N finite dimensional then for every f be an arbitrary element in $L^2[0,1]$ be unique best approximation from $\widehat{f} \in M_N$ that is

$$\|f - \widehat{f}\| \le \|f - g\|, \quad \forall g \in M_N.$$

Moreover, there exist unique coefficients $a_0, a_1, ..., a_N$ such that

$$\widehat{f}(x) = f_N(x) = \sum_{n=0}^{N} a_n C_{N,n}(x) = \mathbf{A}\mathbf{\Phi}(x), \tag{4.24}$$

where

$$\mathbf{A} = (a_0, a_1, ..., a_N), \qquad \mathbf{\Phi}(x) = (C_{N,0}(x), C_{N,1}(x), ..., C_{N,N}(x))^T, \tag{4.25}$$

and

$$a_n = (2n+1)\langle C_{N,n}(x), \widehat{f}\rangle_w.$$

Let us define the error function as $e_N(x) = f(x) - f_N(x)$ where $f(x)$ and $f_N(x)$ are the exact and approximate solutions of Eq. (1.1), respectively. In the following lemma, we present an upper bound for estimating the error.

Lemma 4.2. *Suppose that $D^k f(x) \in C[0,1]$ for $k = 0,1,...,N$. if $f_N(x) := \mathbf{A}^T\mathbf{\Phi}(x)$ is the best approximation to $f(x)$, then the error bound is presented as follows*

$$\|f - \mathbf{A}^T\mathbf{\Phi}(x)\| \le \frac{M}{\Gamma(N+2)\sqrt{(2N+3)}}.$$

Proof. Considering the Taylor's formula

$$f(x) = \sum_{i=0}^{N} \frac{x^i}{\Gamma(i+1)} D^i f(0) + \frac{x^{(N+1)}}{\Gamma(N+2)} D^{(n+1)} f(\xi), \tag{4.26}$$

then we have

$$|f(x) - \sum_{i=0}^{N} \frac{x^i}{\Gamma(i+1)} D^i f(0)| \leq M \frac{x^{(N+1)}}{\Gamma(N+2)}. \tag{4.27}$$

Since $\mathbf{A}^T \mathbf{\Phi}(x)$ is the best approximation to $f(x)$ from M_N and

$$\sum_{i=0}^{N} \frac{x^i}{\Gamma(i+1)} D^i f(0) \in M_N,$$

hence

$$\|f - \mathbf{A}^T \mathbf{\Phi}(x)\|^2 \leq \|f - \sum_{i=0}^{N} \frac{x^i}{\Gamma(i+1)} D^i f(0)\|^2 \leq \frac{M^2}{\Gamma(N+2)^2} \int_0^1 x^{2(N+1)} dx$$

then

$$\|f - \mathbf{A}^T \mathbf{\Phi}(x)\|^2 \leq \frac{M^2}{\Gamma(N+2)^2 (2N+3)}.$$

□

Hence, an upper bound in L^2 is obtained for the approximating solution. The convergence of the proposed method depends on the above error bound.

In the following, we summarize the steps of the proposed method as an implementation algorithm as follows:

Step 1. Choose N as the degree of approximate solution.

Step 2. Compute the vector basis $\mathbf{\Phi}$ from (4.25).

Step 3. Compute the matrices P and $\widehat{V}$ from Theorems 2 and 3 and the matrices (3.16) and (3.18).

Step 4. Compute the collocation points x_i (the roots of $C_{N+1,0}(x)$).

Step 5. Solve the nonlinear system (3.20).

Step 6. Construct the approximate solution y_N from Eq. (3.21).

§5 Numerical Examples

In this section, we illustrate the presented method by giving some examples. The results are compared with the exact solutions by calculating the following maximum absolute error and

$$e_N(x) = max|y(x) - y_N(x)|, \quad x \in [0, 1],$$

where $y(x)$ denote the exact solution of the given examples and $y_N(x)$ be the approximate solution obtained by the presented method. All calculations are supported by Maple 12.

Example 5.1. *Consider the equation*

$$y(x) = x^3 - \frac{x^{10}}{35} + \frac{1}{5} y(x) \int_0^x y^2(t) dt, \tag{5.28}$$

where the exact solution is $y(x) = x^3$. We approximate the solution y(x) by $N = 5$ and obtain:

$$\begin{aligned} y_5(x) &= -\frac{492033989}{19503769752} x^{10} - \frac{313506513}{21743720771} x^9 + \frac{778572632}{4104895303} x^8 - \frac{2342373949}{8465223580} x^7 \\ &+ \frac{903144767}{5339862835} x^6 - \frac{470847353}{9672047107} x^5 + \frac{139556639}{21287342581} x^4 + \frac{3964583787}{3966484492} x^3 \\ &+ \frac{19325089}{462305901501} x^2 - \frac{3487712}{1412591261827} x + \frac{480189}{10290205568813}. \end{aligned}$$

Table 1. Absolute errors of Example 5.1.

x	$e_5(x)$	$e_{10}(x)$
0	4.67×10^{-8}	1.70×10^{-21}
0.1	5.07×10^{-8}	1.70×10^{-15}
0.2	6.10×10^{-8}	7.67×10^{-14}
0.3	2.69×10^{-7}	1.97×10^{-13}
0.4	1.70×10^{-6}	3.55×10^{-13}
0.5	8.65×10^{-6}	1.78×10^{-12}
0.6	7.23×10^{-6}	1.17×10^{-11}
0.7	3.36×10^{-6}	3.26×10^{-11}
0.8	2.13×10^{-6}	7.15×10^{-11}
0.9	3.86×10^{-5}	5.89×10^{-11}
1	1.15×10^{-4}	5.74×10^{-10}

The values of the absolute errors for $N = 5, 10$ *at specified points are reported in Table 1. Also, Figs. 2 and 3 show the absolute errors on a logarithmic scale for* $N = 5$ *and* $N = 10$*, respectively.*

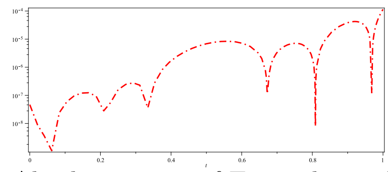

Figure 2. Absolute errors of Example 5.1 for $N = 5$.

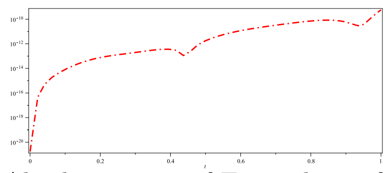

Figure 3. Absolute errors of Example 5.1 for $N = 10$.

Example 5.2. *Consider the following equation*

$$y(x) = x^2 + \frac{1}{80}x^{17} - \frac{1}{70}x^{16} + \frac{x^5}{10}y^2(x)\int_0^x (1-t)y^3(t)dt, \tag{5.29}$$

with the exact solution is $y(x) = x^2$. *For* $N = 5$, *we have:*

$$\begin{aligned} y_5(x) \quad = \quad & -\frac{1}{80}x^{17} - \frac{1}{70}x^{16} + \frac{3814567227}{1821007492}x^{10} - \frac{6933631193}{969169535}x^9 + \frac{11332960939}{1060775641}x^8 \\ - \quad & \frac{3636277708}{400432233}x^7 + \frac{6060872787}{1264362361}x^6 - \frac{48736302585}{30291481081}x^5 + \frac{2730752507}{8097081674}x^4 \\ - \quad & \frac{228911458}{5536285859}x^3 + \frac{1382050063}{1378563708}x^2 - \frac{61077871}{1274039906927}x. \end{aligned}$$

Table 2. Absolute errors of Example 5.2.

x	$e_5(x)$	$e_{10}(x)$
0	0	0
0.1	7.70×10^{-7}	7.81×10^{-17}
0.2	1.30×10^{-11}	4.09×10^{-14}
0.3	5.60×10^{-9}	1.36×10^{-12}
0.4	1.21×10^{-8}	2.11×10^{-11}
0.5	8.37×10^{-7}	1.56×10^{-10}
0.6	6.22×10^{-7}	5.81×10^{-10}
0.7	8.35×10^{-6}	4.25×10^{-9}
0.8	9.19×10^{-7}	5.48×10^{-9}
0.9	2.01×10^{-4}	5.52×10^{-8}
1	3.60×10^{-4}	1.41×10^{-6}

The values of the absolute errors for $N = 5, 10$ at specified points are reported in Table 2. This table shows that our method has an appropriate convergence rate for this problem. Also, Figs. 4 and 5 show the absolute errors on a logarithmic scale for $N = 5$ and $N = 10$, respectively.

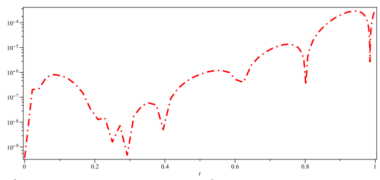

Figure 4. Absolute errors of Example 5.2 for $N = 5$.

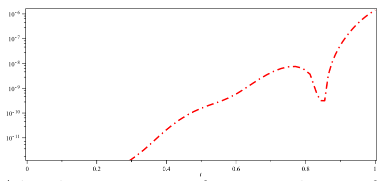

Figure 5. Absolute errors of Example 5.2 for $N = 10$.

§6 Conclusions

In this paper, we presented the spectral collocation method for solving nonlinear quadratic integral equations of Volterra type. According to the proposed method, the operational matrices of integral and product for Chelyshkov polynomials have been derived. These matrices are used to obtain approximation solutions for the nonlinear quadratic integral equations of Volterra type. The presented matrix method provides the following advantages: it is very simple to construct the main matrix and to do computer programming, simplicity of implementation besides good approximation results in low terms of basis by only a small number of Chelyshkov polynomials. The proposed method can be extended to related problems, such as system of nonlinear quadratic integral equations. This is possible for future works.

Declarations

Conflict of interest The authors declare no conflict of interest.

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