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Dynamical study of discrete prey-predator system incorporating proportional prey refuge with interval parameters

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Abstract. This paper presents the dynamical properties of a discrete-time prey-predator model with refuge in prey under imprecise biological parameters. We consider the refuge concept of prey, which is proportional to the density of prey species with interval parameters. The model develops with natural interval parameters since the uncertainties of parameters of any ecological system are a widespread phenomenon in nature. The equilibria of the model are obtained, and the dynamic behaviours of the proposed system are examined. Simulations of the model are performed for different parameters of the model. Numerical simulations show that the proposed discrete model exhibits rich dynamics of a chaotic and complex nature. Our study, through analytical derivation and numerical example, presents the effect of refuge on population dynamics under imprecise biological parameters.

§1 Introduction

In the classical Lotka-Volterra model ([1],[2]), multiple species ecological problem was presented first time with general interaction. One of the significant interactivities among species in bio-mathematics is the prey-predator correlation, which has been studied on a large scale ([3]-[14]) for its universal existence. Many factors are affecting the predator-prey dynamics. The theory of population dynamics is divided into two kinds of mathematical models: continuous and discrete-time models. The discrete-time models described by difference equations ([15]-[25]), and the discrete-time models are more appropriate when populations have a small number of population, or non-overlapping generations. In addition, exact numerical simulation results can be presented for discrete-time models. Furthermore, the numerical simulations of continuous-time models are derived by discretizing the models. Again, the discrete-time models ([32],[41],[48]) have richer dynamical attributes than the continuous-time models.

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The refuge is an exciting concept in the prey-predator model. It can be defined as the change in the density of prey attached per unit time per predator as the prey density changes. Many papers ([4],[12]) in continuous models with refuge exist in the literature, which presented the effect in the dynamics of prey-predator interaction. Kar [3] presented the stability analysis of a prey-predator system by incorporating a prey refuge. This paper proposes to study the discrete-time prey-predator model by considering the fact that prey refuge is proportional to the density of prey species. The different types of refuge concepts demonstrate the various prey dynamics for studying the prey-predator system. The effect of different prey refuge has more importance on the dynamics of discrete-time models than the continuous prey-predator systems. Some examples of predator and prey are rat and cat, lion and zebra, bear and fish, and fox and rabbit. Rats have refuges such as tall grass, allowing them to hide from predators such as owls and cats.

Hu and Cao [47] presented a discrete-time predator-prey system of Holling and Leslie type functional responses for bifurcation and chaos analysis. Cheng and Cao [44] presented a discrete-time ratio dependent on the predator-prey model for bifurcation analysis incorporating the Allee effect. Cui et al. [46] studied the dynamics of a discrete-time prey-predator system with Holling I functional response. Gámez et al. [42] presented a discrete-time prey-predator model for ecological monitoring. Chen and Chen [43] presented a discrete predator-prey model with stage structure and harvesting to study complex dynamic behaviours.

The discrete-time model in general is developed by researchers ([33],[34],[35]) with the assumption that the biological parameters are fixed and precisely known, however, it is different in practical situations. All biological parameters may not be fixed in reality, and rather it may deviate due to various causes ([26],[28],[29],[30],[31]). The biological parameters are sensible and more appropriate to be treated as imprecise quantity ([36],[40]) instead of a definite real number. Peixoto et al. [30] studied the predator-prey fuzzy model. First time in the study of the prey-predator system, Pal et al. [26] proposed the interval biological parameters for optimal harvesting of prey-predator bio-economic model. Liang and Zhao [38] presented the optimal harvesting of a Gompertz population model with a marine protected area using interval-value biological parameters. Pal et al. [39] studied the bifurcation of prey-predator system with time delay and harvesting using fuzzy parameters.

This paper considers one prey and one predator for the proposed discrete model and then develops and discusses the dynamical behaviour of the system for equilibrium points, stability, bifurcation, and chaotic situations, where some of the biological parameters of the system are interval numbers. The interval parameters of the system are present in the parametric functional form to study the proposed prey-predator discrete-time model. A parametric prey-predator mathematical model is formulated to find different system behaviour for different values of the parameters. The proposed procedure is more effective and exciting since we get different model behaviour using the functional form of interval parameters. The advantage of the proposed approach is that this study can present different system characteristics for changes of the parameters at a time in a single framework. This paper presents a discrete-time model of two species with the concept of prey refuge for imprecise biological parameters for dynamical behaviour.

The rest of the paper is organized as follows: in the second section, we introduce the prerequisite mathematics which is used in this paper. Section 3 presents the formulation of a discrete-time prey-predator model under non-overlapping generation incorporating the refuge under imprecise biological parameters. Section 4 offers the proposed discrete-time prey-predator system's equilibrium points and deals with the stability analysis of the proposed model around the interior equilibrium. We present Neimark-Sacker bifurcation and Filp bifurcation of the proposed model in section 5. Section 6 gives the chaos control procedure of the proposed prey-predator system. Section 7 gives a numerical simulation to support the proposed model's theoretical and analytical outcomes. Finally, the conclusion is provided in Section 8.

§2 Prerequisite Mathematics

An interval number is denoted by closed interval $a = [a_l, a_r]$ and defined by $a = [a_l, a_r] = \{x : a_l \leq x \leq a_r,\ x \in R\}$, R is the set of real numbers.

Definition: Interval-valued function: Let the interval is of the form $[a_l, a_r]$, where $a_l, a_r > 0$, the interval-valued function can be defined as $h(p) = a_l^{1-p} a_r^p$ for $p \in [0,\ 1]$.

Here, the arithmetic operations for two interval number $a = [a_l, a_u]$ and $b = [b_l, b_u]$ where $a_l, b_l > 0$, using the concept of parametric interval valued functions [27] are as follows:

Addition: The sum of the intervals $a + b = [a_l + b_l, a_u + b_u]$ in parametric form of the interval $a+b$ as interval-valued function is $h(p) = a_L^{1-p} a_U^p$, where $a_L = a_l + b_l$ and $a_U = a_u + b_u$.

Subtraction: The substraction of the intervals $a-b = [a_l - b_u, a_u - b_l]$, given that $a_l - b_u > 0$, in the form of interval valued function is $h(p) = b_L^{1-p} b_U^p$ where $b_L = a_l - b_u$ and $b_L = a_u - b_l$.

Scalar Multiplication: The scalar multiplication with the interval a is given by $\alpha A = \alpha [a_l, a_u] = \begin{cases} [\alpha a_l, \alpha a_u] & \text{if } \alpha \geq 0 \\ [\alpha a_u, \alpha a_l] & \text{if } \alpha < 0 \end{cases}$ such that $a_l > 0$. The interval αA is given by $h(p) = \begin{cases} c_L^{1-p} c_U^p & \text{if } \alpha \geq 0 \\ -d_L^{1-p} d_U^p & \text{if } \alpha < 0 \end{cases}$ where $c_{\scriptscriptstyle L} = \alpha a_l$, $c_{\scriptscriptstyle U} = \alpha a_u$, $d_L = |\alpha| a_u$ and $d_U = |\alpha| a_l$.

§3 Modeling of Discrete Prey-Predator System with Interval Coefficient

The reasonable perception of two species is essential in ecology, where the ambience has diversity for species to hide. The Sundarban Tiger Reserve is a mangrove biological system with 31 mammalian species, 14 turtle and tortoises species, seven amphibian species, more than 200 species of fishes, birds, insect, crustaceans, more than 50 species of snake, annelids, protozoa, reptiles, and 143 molluscs species, 104 nematode species, 40 significant mangrove species, 32 minor mangrove species, 30 back mangroves and associates species, and three mangrove habitat ferns species in West Bengal, India. Here, the topmost predator is the Tiger on land and estuarine crocodile in the water. And chital, sambar, barasingha, gaur, water buffalo, nilgai, serow, wild boar, hog deer, monkeys, rabbits, peafowl, wild pig, flying creatures, household domesticated animals, people, bison, gaurs, and bats are prey. Mangroves provide a refuge region for prey, we proposed a mathematical model of discrete-time prey-predator system considers that the densities of prey (x) and predator (y) populations change with time and have no age structure for both species. We consider populations with the non-overlapping generation, where all adults die after birth. The general form of the prey-predator system in discrete-time incorporating logistic growth and refuge on the prey population is described by the following format:

$$\begin{aligned} x_{n+1} &= f(x_n, y_n) = ax_n(1-x_n) - b(x_n - x_R)y_n \\ y_{n+1} &= g(x_n, y_n) = c(x_n - x_R)y_n - dy_n \end{aligned} \tag{1}$$

Where $\frac{df}{dy_n} \leq 0$ and $\frac{dg}{dx_n} \geqslant 0$, we consider here x_R as a refuge quantity of the prey population. Here, a, b, c, d and m are the non-negative parameters of the proposed discrete prey-predator system.

Here, x_R is considered based on the point of view $x_R = mx_n$, i.e. the quantity of hiding prey is proportional to the density of the prey species. Based on the assumption that the prey population as refuge is proportional to the density of the prey, the model (1) should be changed with the relation $x_R = mx_n$. Most of the prey-predator models are studied in a precise environment, but the data cannot be recorded or collected precisely due to several reasons in reality. Since the biological background of populations is not entirely predictable, the biological parameters of modelling the prey-predator system should be considered imprecise [36]. Hence analysis of the system with an uncertain growth rate of prey populations, interspecific competition rates of prey species [34], predation coefficient, and reduced rates of predator species is usually considered an effect of environmental fluctuations. The reproduction of the species depends on various factors, such as temperature, humidity, parasites and pathogens, and environmental pollution. Let $\widehat{a}$, $\widehat{b}$, $\widehat{c}$ and $\widehat{d}$ be the interval counterparts of a, b, c and d, respectively. Then the modified prey-predator model of (1) with the situation that the prey population as refuge is proportional to the density as follows:

$$\begin{aligned} x_{n+1} &= \widehat{a}x_n(1-x_n) - \widehat{b}(1-m)x_n y_n \\ y_{n+1} &= \widehat{c}(1-m)x_n y_n - \widehat{d}y_n \end{aligned} \tag{2}$$

where $\widehat{a} \in [a_l, a_u]$, $\widehat{b} \in [b_l, b_u]$, $\widehat{c} \in [c_l, c_u]$ and $\widehat{d} \in [d_l, d_u]$. Also $a_l > 0$, $b_l > 0$, $c_l > 0$ and $d_l > 0$.

Here we present the parametric form of the proposed discrete prey-predator system, which will be considered for the dynamical study. Based on section 2, the interval parameters can be presented as interval-valued function, and the prey-predator equations (2) can be written in the parametric prey-predator model as follows:

$$\begin{aligned} x_{n+1} &= {a_l}^{1-p}{a_u}^p x_n(1-x_n) - {b_l}^{1-p}{b_u}^p(1-m)x_n y_n \\ y_{n+1} &= {c_l}^{1-p}{c_u}^p(1-m)x_n y_n - {d_l}^{1-p}{d_u}^p y_n \end{aligned} \tag{3}$$

for $p \in [0,1]$.

§4 Dynamics of Proposed Discrete Prey-Predator Model

This section presents the dynamical analysis of the proposed parametric prey-predator system.

4.1 Fixed points of discrete prey-predator model

In this section, we present all possible equilibrium points of the proposed discrete prey-predator system. To study the stability of the fixed points of the model, first we present the following lemma:

Lemma 1. *Let $F(\tau)=\tau^2-B\tau+C$, assume that $F(1)>0$, τ_1 and τ_2 are roots of $F(\tau)=0$. Then*

(i) $|\tau_1|>1$ and $|\tau_2|>1$ if and only if $F(-1)>0$ and $C>1$;
(ii) $|\tau_1|<1$ and $|\tau_2|>1$, or, $|\tau_1|>1$ and $|\tau_2|<1$, if and only if $F(-1)<0$;
(iii) $|\tau_1|<1$ and $|\tau_2|<1$ if and only if $F(-1)>0$ and $C<1$;
(iv) $\tau_1=-1$ and $|\tau_2|\neq 1$ if and only if $F(-1)=0$ and $B\neq 0,2$;
(v) τ_1 and τ_2 are complex and $|\tau_1|=|\tau_2|=1$ if and only if $B^2-4C<0$ and $C=1$.

Here $\tau_1$and τ_2 are the eigenvalues of the fixed point (x,y). We recall some definitions of topological types for a fixed point (x,y).

A fixed point (x,y) is called
(i) a sink if $|\tau_1|<1$ and $|\tau_2|<1$, hence the sink is locally asymptotically stable.
(ii) a source if $|\tau_1|>1$ and $|\tau_2|>1$, so the source is locally unstable.
(iii) a saddle if $|\tau_1|>1$ and $|\tau_2|<1$, or, $|\tau_1|<1$ and $|\tau_2|>1$.
(iv) non-hyperbolic if either $|\tau_1|=1$ or $|\tau_2|=1$.

Fixed points of the system are determined by solving the following non-linear system of equations:
$x=a_l^{1-p}a_u^p x(1-x)-b_l^{1-p}b_u^p(1-m)xy$
$y=c_l^{1-p}c_u^p(1-m)xy-d_l^{1-p}d_u^p y$

Simple calculation gives the following three non-negative fixed points:
(i) $P_{10}=(0,0)$, (ii) $P_{11}=\left(\frac{a_l^{1-p}a_u^p-1}{a_l^{1-p}a_u^p},0\right), a_l^{1-p}a_u^p>1$, (iii) $P_{12}=(x^*,y^*)$ where $x^*=\frac{d_l^{1-p}d_u^p+1}{c_l^{1-p}c_u^p(1-m)}$ and $y^*=\frac{c_l^{1-p}c_u^p(1-m)\left(a_l^{1-p}a_u^p-1\right)-a_l^{1-p}a_u^p\left(d_l^{1-p}d_u^p+1\right)}{b_l^{1-p}b_u^p c_l^{1-p}c_u^p(1-m)^2}$, where $m<1$ and $c_l^{1-p}c_u^p(1-m)\left(a_l^{1-p}a_u^p-1\right)>a_l^{1-p}a_u^p\left(d_l^{1-p}d_u^p+1\right)$.

Now $\frac{dx^*}{dm}=\frac{d_l^{1-p}d_u^p+1}{c_l^{1-p}c_u^p(1-m)^2}>0$, then x^* is strictly increasing function of parameter m. So increasing the amount of prey refuge leads to the increasing density of the prey species.

$\frac{dy^*}{dm}=\frac{c_l^{1-p}c_u^p(1-m)\left(a_l^{1-p}a_u^p-1\right)-2a_l^{1-p}a_u^p\left(d_l^{1-p}d_u^p+1\right)}{b_l^{1-p}b_u^p c_l^{1-p}c_u^p(1-m)^3}$

Here $\frac{dy^*}{dm}>0$ when $m<1-\frac{2a_l^{1-p}a_u^p\left(d_l^{1-p}d_u^p+1\right)}{c_l^{1-p}c_u^p\left(a_l^{1-p}a_u^p-1\right)}$ then y^* is a strictly increasing function of m.

And $\frac{dy^*}{dm}<0$ when $m>1-\frac{2a_l^{1-p}a_u^p\left(d_l^{1-p}d_u^p+1\right)}{c_l^{1-p}c_u^p\left(a_l^{1-p}a_u^p-1\right)}$ then y^* is a strictly decreasing function of m.

If y^* is a strictly increasing (decreasing) function of parameter m, then increasing the amount of prey refuge leads to the increasing (decreasing) density of the predator species.

The above fixed points of the prey-predator system will be considered for dynamical study in the following sections for each of the equilibrium points.

4.2 Dynamics of prey-predator model with proportion prey refuge

Now we study the local behavior of the discrete-time prey-predator system at each equilibrium point. The map given by equation (3) is a non-invertible map of the plane. The study of the dynamical properties of the above map provides information about the long-run behavior of prey-predator populations. Starting from the initial condition (x_0, y_0), the iteration of (3) uniquely determines a trajectory of the states of the population output. The stability of the imprecise system (3) is carried out by computing the Jacobian Matrix corresponding to each equilibrium point. The Jacobian matrix J for the system (3) is

$$J = \begin{bmatrix} a_l^{1-p}a_u^p(1-2x) - b_l^{1-p}b_u^p(1-m)y & -b_l^{1-p}b_u^p(1-m)x \\ c_l^{1-p}c_u^p(1-m)y & c_l^{1-p}c_u^p(1-m)x - d_l^{1-p}d_u^p \end{bmatrix}$$

The characteristic equation of matrix J is $\tau^2 - Tr\,(J)\,\tau + Det\,(J) = 0$ where

$Tr\,(J) =$ Trace of matrix $J = a_l^{1-p}a_u^p - d_l^{1-p}d_u^p + \left[c_l^{1-p}c_u^p(1-m) - 2a_l^{1-p}a_u^p\right]x - b_l^{1-p}b_u^p(1-m)y$.

$Det(J) =$ Determinant of matrix $J = a_l^{1-p}a_u^p\left[c_l^{1-p}c_u^p(1-m) + 2d_l^{1-p}d_u^p\right]x + b_l^{1-p}b_u^p d_l^{1-p}d_u^p(1-m)y - 2a_l^{1-p}a_u^p c_l^{1-p}c_u^p(1-m)x^2 - a_l^{1-p}a_u^p d_l^{1-p}d_u^p$.

Hence, the model (3) is a dissipative dynamical system if

$$\Big| a_l^{1-p}a_u^p\left(c_l^{1-p}c_u^p(1-m) + 2d_l^{1-p}d_u^p\right)x + b_l^{1-p}b_u^p d_l^{1-p}d_u^p(1-m)y - a_l^{1-p}a_u^p\left(2c_l^{1-p}c_u^p(1-m)x^2 + d_l^{1-p}d_u^p\right)\Big| < 1.$$

The model (3) is a conservative dynamical one, if and only if

$$\Big| a_l^{1-p}a_u^p\left[c_l^{1-p}c_u^p(1-m) + 2d_l^{1-p}d_u^p\right]x + b_l^{1-p}b_u^p d_l^{1-p}d_u^p(1-m)y - a_l^{1-p}a_u^p\left(2c_l^{1-p}c_u^p(1-m)x^2 + d_l^{1-p}d_u^p\right)\Big| = 1.$$

The model (3) is an un-dissipated dynamical system if not dissipative neither conservative.

4.2.1 Stability and dynamic behavior of P_{10}

The dynamical behavior is discussed here using the Jacobian matrix at fixed point $P_{10} = (0,0)$. The Jacobian matrix at $P_{10} = (0,0)$ is $J = \begin{bmatrix} a_l^{1-p}a_u^p & 0 \\ 0 & -d_l^{1-p}d_u^p \end{bmatrix}$. The trivial equilibrium point $P_{10} = (0,0)$ is

(i) sink if $a_l^{1-p}a_u^p < 1$ and $d_l^{1-p}d_u^p < 1$,

(ii) source if $a_l^{1-p}a_u^p > 1$ and $d_l^{1-p}d_u^p > 1$,

(iii) saddle if $a_l^{1-p}a_u^p > 1$ and $d_l^{1-p}d_u^p < 1$, or, $a_l^{1-p}a_u^p < 1$ and $d_l^{1-p}d_u^p > 1$,

(iv) non-hyperbolic if $a_l^{1-p}a_u^p = 1$ or $d_l^{1-p}d_u^p = 1$.

4.2.2 Stability and dynamic behavior of P_{11}

The Jacobian matrix at $P_{11}=\left(\frac{a_l^{1-p}a_u^p-1}{a_l^{1-p}a_u^p},0\right)$ to discuss the dynamical behavior is

$$J=\begin{bmatrix} 2-a_l^{1-p}a_u^p & \frac{b_l^{1-p}b_u^p(1-a_l^{1-p}a_u^p)(1-m)}{a_l^{1-p}a_u^p} \\ 0 & \frac{(1-m)c_l^{1-p}c_u^p(a_l^{1-p}a_u^p-1)}{a_l^{1-p}a_u^p}-d_l^{1-p}d_u^p \end{bmatrix}$$

The equilibrium point P_{11} is

sink if $a_l^{1-p}a_u^p\in(1,3)$ and $m\in\left(1-\frac{a_l^{1-p}a_u^p(d_l^{1-p}d_u^p+1)}{c_l^{1-p}c_u^p(a_l^{1-p}a_u^p-1)},1-\frac{a_l^{1-p}a_u^p(d_l^{1-p}d_u^p-1)}{c_l^{1-p}c_u^p(a_l^{1-p}a_u^p-1)}\right)$,

source if $a_l^{1-p}a_u^p\in(3,\infty)$ and $m\in\left(0,1-\frac{a_l^{1-p}a_u^p(d_l^{1-p}d_u^p+1)}{c_l^{1-p}c_u^p(a_l^{1-p}a_u^p-1)}\right)\cup\left(1-\frac{a_l^{1-p}a_u^p(d_l^{1-p}d_u^p-1)}{c_l^{1-p}c_u^p(a_l^{1-p}a_u^p-1)},1\right)$,

saddle if $a_l^{1-p}a_u^p\in(3,\infty)$ and $m\in\left(1-\frac{a_l^{1-p}a_u^p(d_l^{1-p}d_u^p+1)}{c_l^{1-p}c_u^p(a_l^{1-p}a_u^p-1)},1-\frac{a_l^{1-p}a_u^p(d_l^{1-p}d_u^p-1)}{c_l^{1-p}c_u^p(a_l^{1-p}a_u^p-1)}\right)$,

or $a_l^{1-p}a_u^p\in(1,3)$ and $m\in\left(0,1-\frac{a_l^{1-p}a_u^p(d_l^{1-p}d_u^p+1)}{c_l^{1-p}c_u^p(a_l^{1-p}a_u^p-1)}\right)\cup\left(1-\frac{a_l^{1-p}a_u^p(d_l^{1-p}d_u^p-1)}{c_l^{1-p}c_u^p(a_l^{1-p}a_u^p-1)},1\right)$,

non-hyperbolic if $a_l^{1-p}a_u^p=1$ or 3,

otherwise, $m=1-\frac{a_l^{1-p}a_u^p(d_l^{1-p}d_u^p+1)}{c_l^{1-p}c_u^p(a_l^{1-p}a_u^p-1)}$ or $1-\frac{a_l^{1-p}a_u^p(d_l^{1-p}d_u^p-1)}{c_l^{1-p}c_u^p(a_l^{1-p}a_u^p-1)}$

4.2.3 Stability and dynamic behavior at the interior fixed point P_{12}

The dynamical behavior of the system (3) at fixed point P_{12} is studied here. From the Jacobian matrix J for the interior equilibrium point $P_{12}\,(x^*,y^*)$, we have

$1-Tr\,(J)+Det\,(J)=$
$1-\left[\left(a_l^{1-p}a_u^p-d_l^{1-p}d_u^p\right)+\left(c_l^{1-p}c_u^p(1-m)-\ 2a_l^{1-p}a_u^p\right)x-b_l^{1-p}b_u^p(1-m)y\right]+$
$a_l^{1-p}a_u^p\left[c_l^{1-p}c_u^p(1-m)+2d_l^{1-p}d_u^p\right]x+b_l^{1-p}b_u^pd_l^{1-p}d_u^p(1-m)y-2a_l^{1-p}a_u^pc_l^{1-p}c_u^p(1-m)x^2-a_l^{1-p}a_u^pd_l^{1-p}d_u^p$,

$1+Tr\,(J)+Det\,(J)=$
$1+\left[\left(a_l^{1-p}a_u^p-d_l^{1-p}d_u^p\right)+\left(c_l^{1-p}c_u^p(1-m)-\ 2a_l^{1-p}a_u^p\right)x-b_l^{1-p}b_u^p(1-m)y\right]+$
$a_l^{1-p}a_u^p\left[c_l^{1-p}c_u^p(1-m)+2d_l^{1-p}d_u^p\right]x+b_l^{1-p}b_u^pd_l^{1-p}d_u^p(1-m)y-2a_l^{1-p}a_u^pc_l^{1-p}c_u^p(1-m)x^2-a_l^{1-p}a_u^pd_l^{1-p}d_u^p$,

$Det\,(J)=a_l^{1-p}a_u^p\left(c_l^{1-p}c_u^p(1-m)+2d_l^{1-p}d_u^p\right)x+b_l^{1-p}b_u^pd_l^{1-p}d_u^p(1-m)y-2a_l^{1-p}a_u^pc_l^{1-p}c_u^p(1-m)x^2-a_l^{1-p}a_u^pd_l^{1-p}d_u^p$

At $P_{12}=(x^*,y^*)$ if $1-Tr\,(J)+Det\,(J)>0$ then interior equilibrium point is

sink if $1+Tr\,(J)+Det\,(J)>0$ and $Det\,(J)<1$,

source if $1+Tr\,(J)+Det\,(J)>0$ and $Det\,(J)>1$,

saddle if $1+Tr\,(J)+Det\,(J)<0$,

non hyperbolic if $1+Tr\,(J)+Det\,(J)=0$ and $Tr\,(J)\neq 0,2$ or $[Tr\,(J)]^2-4Det\,(J)<0$ and $Det\,(J)=1$.

The proposed parametric discrete prey-predator model (3) experiences bifurcation at the positive interior equilibrium point (x^*,y^*). The dynamic matrix of the prey-predator system undergoes different bifurcations, which arises for the following conditions:

At $P_{12} = (x^*, y^*)$ if $1 - Tr\,(J) + Det\,(J) > 0$, $1 + Tr\,(J) + Det\,(J) = 0$, and $Tr\,(J) \neq 0, 2$ then (x^*, y^*) can undergo flip bifurcation.

At $P_{12} = (x^*, y^*)$ if $1 - Tr\,(J) + Det\,(J) > 0$, $(Tr\,(J))^2 - 4Det\,(J) < 0$ and $Det\,(J) = 1$ then (x^*, y^*) can undergo Neimark-Sacker bifurcation.

§5 Bifurcation Analysis

In this section, we discuss the Neimark-Sacker bifurcation and Filp bifurcation of the model.

5.1 Neimark-Sacker bifurcation of proposed model

In this section, we discuss the Neimark-Sacker bifurcation of the model (3) at $P\,(x^*, y^*)$ when parameters are located in the following set $A = \{(a_l, a_u, b_l, b_u, c_l, c_u, d_l, d_u, p, m) : 1 - Tr\,(J) + Det\,(J) > 0,\ (Tr\,(J))^2 - 4Det\,(J) < 0,\ Det\,(J) = 1 \text{ and } p \in [0, 1]\,.\}$

The Neimark-Sacker bifurcation is analyzed for p as the bifurcation parameter. Further p^* is the perturbation of p, we consider a perturbation of the model as follows:

$$\begin{aligned} x_{n+1} &= a_l \left(\frac{a_u}{a_l}\right)^{p+p^*} x_n(1 - x_n) - b_l \left(\frac{b_u}{b_l}\right)^{p+p^*} (1 - m)x_n y_n \equiv f(x_n, y_n, p^*) \\ y_{n+1} &= (1 - m)c_l \left(\frac{c_u}{c_l}\right)^{p+p^*} x_n y_n - d_l \left(\frac{d_u}{d_l}\right)^{p+p^*} y_n \equiv g(x_n, y_n, p^*) \end{aligned} \tag{4}$$

where $|p^*| \lll 1$

Let $u_n = x_n - x^*, v_n = y_n - y^*$, then the fixed point $P_{12}\,(x^*, y^*)$ is transformed into the origin, and further expanding f and g as a Taylor series about the point $(u_n, v_n) = (0, 0)$ to the third order, the model (4) becomes

$$\begin{aligned} u_{n+1} &= \alpha_1 u_n + \alpha_2 v_n + \alpha_{11} u_n^2 + \alpha_{12} u_n v_n + \alpha_{22} v_n^2 + \alpha_{111} u_n^3 + \alpha_{112} u_n^2 v_n \\ &+ \alpha_{122} u_n v_n^2 + \alpha_{222} v_n^3 + O((|u_n| + |v_n|)^4) \\ v_{n+1} &= \beta_1 u_n + \beta_2 v_n + \beta_{11} u_n^2 + \beta_{12} u_n v_n + \beta_{22} v_n^2 + \beta_{111} u_n^3 + \beta_{112} u_n^2 v_n \\ &+ \beta_{122} u_n v_n^2 + \beta_{222} v_n^3 + O((|u_n| + |v_n|)^4) \end{aligned} \tag{5}$$

Where $\alpha_1 = f_x(x^*, y^*, 0)$, $\alpha_2 = f_y(x^*, y^*, 0)$, $\alpha_{11} = f_{xx}(x^*, y^*, 0)$, $\alpha_{12} = f_{xy}(x^*, y^*, 0)$, $\alpha_{22} = f_{yy}(x^*, y^*, 0)$, $\alpha_{111} = f_{xxx}(x^*, y^*, 0)$, $\alpha_{112} = f_{xxy}(x^*, y^*, 0)$, $\alpha_{122} = f_{xyy}(x^*, y^*, 0)$, $\alpha_{222} = f_{yyy}(x^*, y^*, 0)$; $\beta_1 = g_x(x^*, y^*, 0)$, $\beta_2 = g_y(x^*, y^*, 0)$, $\beta_{11} = g_{xx}(x^*, y^*, 0)$, $\beta_{12} = g_{xy}(x^*, y^*, 0)$, $\beta_{22} = g_{yy}(x^*, y^*, 0)$, $\beta_{111} = g_{xxx}(x^*, y^*, 0)$, $\beta_{112} = g_{xxy}(x^*, y^*, 0)$, $\beta_{122} = g_{xyy}(x^*, y^*, 0)$, $\beta_{222} = g_{yyy}(x^*, y^*, 0)$.

Note that the characteristic equation associated with the linearization of the model (5) at $(u_n, v_n) = (0, 0)$ is given by $\tau^2 - Tr\,(J_1(p^*))\,\tau + Det\,(J_1(p^*)) = 0$. Therefore, the roots of the characteristic equation are

$\tau_{1,2}(p^*) = \frac{Tr(J_1(p^*)) \pm i\sqrt{4Det(J_1(p^*)) - (Tr(J_1(p^*)))^2}}{2}$

From $|\tau_{1,2}(p^*)| = 1$, when $p^* = 0$ we have $|\tau_{1,2}(p^*)| = [Det\,(J_1(p^*))]^{\frac{1}{2}}$

and $l = \left[\frac{d|\tau_{1,2}(p^*)|}{dp^*}\right]_{p^*=0} \neq 0$

It is required that when $p^*=0, \tau_{1,2}^i \neq 1, i=1,2,3,4$, which is equivalent to $Tr\,(J_1(0)) \neq -2,-1,1,2$.

To study the normal form, let $\gamma = Im(\tau_{1,2})$ and $\delta = Re(\tau_{1,2})$. We define $T = \begin{bmatrix} 0 & 1 \\ \gamma & \delta \end{bmatrix}$ and using the transformation $\begin{bmatrix} u_n \\ v_n \end{bmatrix} = T \begin{bmatrix} \overline{x}_n \\ \overline{y}_n \end{bmatrix}$, the model (5) becomes

$$\overline{x}_{n+1} = \delta\overline{x}_n - \gamma\overline{y}_n + f_1(\overline{x}_n, \overline{y}_n), \quad \text{and} \quad \overline{y}_{n+1} = \gamma\overline{x}_n + \delta\overline{y}_n + g_1(\overline{x}_n, \overline{y}_n) \tag{6}$$

where the functions f_1 and g_1 denote the terms in the model (6) in variables $(\overline{x}_n, \overline{y}_n)$ with the order at least two.

It is required that the following discriminatory quantity Ω be non zero in order to undergo Hopf Bifurcation:

$\Omega = -Re\left[\frac{(1-2\overline{\tau})\overline{\tau}^2}{1-\tau}\xi_{11}\xi_{20}\right] - \frac{1}{2}\left|\xi_{11}\right|^2 - \left|\xi_{02}\right|^2 + Re(\overline{\tau}\xi_{21})$

where $\overline{\tau}$ is complex conjugate of τ and

$\xi_{20} = \frac{1}{8}\delta\left(2\beta_{22} - \delta\alpha_{22} - \alpha_{12} + 4\gamma\alpha_{22}\right) + \frac{1}{4}\gamma\alpha_{12} + \frac{1}{8}\delta i\left(4\gamma\alpha_{22} - 2\alpha_{22} - 2\delta\alpha_{22}\right) + \frac{1}{8}i\left(4\gamma\beta_{22} + 2\gamma^2\alpha_{22} - 2\alpha_{11}\right) + \frac{1}{8}\beta_{12} + \frac{\delta\alpha_{11}-2\beta_{11}}{4\gamma} + \frac{\delta^3\alpha_{22}-\delta^2\beta_{22}}{4\gamma} - \frac{\delta^2\alpha_{12}-\delta\beta_{12}}{4\gamma}$,

$\xi_{11} = \frac{1}{2}\gamma(\beta_{22} - \delta\alpha_{22}) + \frac{1}{2}i(\gamma^2\alpha_{22} + \alpha_{11} + \delta\alpha_{12} + \delta^2\alpha_{22}) + \frac{\beta_{11}-\delta\alpha_{11}}{2\gamma} + \frac{\delta\beta_{12}-\delta^2\alpha_{12}}{2\gamma} - \frac{\delta^2\beta_{22}-\delta^3\alpha_{22}}{\gamma}$,

$\xi_{02} = \frac{1}{4}\gamma(2\delta\alpha_{22} + \alpha_{12} + \beta_{22}) + \frac{1}{4}i(\beta_{12} + 2\delta\beta_{22} - 2\delta\alpha_{12} - \alpha_{11}) - \frac{\beta_{11}-\delta\alpha_{11}}{4\gamma} - \frac{\delta\beta_{12}-\delta^2\alpha_{12}}{4\gamma} + \frac{1}{4}\alpha_{22}i(\gamma^2 - 3\delta^2) + \frac{\delta^2\beta_{22}-\delta^3\alpha_{22}}{4\gamma}$,

$\xi_{21} = \frac{3}{8}\beta_{222}(\gamma^2+\delta^2) + \frac{1}{8}\beta_{112} + \frac{1}{4}\delta\alpha_{112} + \frac{1}{4}\delta\beta_{122} + \alpha_{122}(\frac{1}{8}\gamma^2 + \frac{3}{8}\delta^2 - \frac{1}{4}\delta) + \frac{3}{8}\alpha_{111} + \frac{3}{8}\alpha_{222}i(\gamma^2 + 2\delta^2) + \frac{3}{8}\alpha_{122}\gamma\delta i - \frac{1}{8}\beta_{122}\gamma i - \frac{3}{8}\beta_{222}\gamma\delta i - \frac{3\beta_{111}-3\delta\alpha_{111}}{8\gamma}i - \frac{3\delta\beta_{112}-3\delta^2\alpha_{112}}{8\gamma}i - \frac{3\delta^2\beta_{122}-3\delta^3\alpha_{122}}{8\gamma}i - \frac{3\delta^3\beta_{222}-3\delta^4\alpha_{222}}{8\gamma}i$

Finally, from the above analysis , we have the following result:

Theorem 2. *If $\Omega \neq 0$, then the model (3) undergoes Neimark-Sacker bifurcation at $P_{12}\,(x^*, y^*)$ when the parameter p^* varies in a small neighborhood of the origin. Moreover, if $\Omega < 0$ (or, $\Omega > 0$), then an attracting (or, repelling) invariant closed curve bifurcates from $P_{12}\,(x^*, y^*)$ for $p^* > 0$ (or, $p^* < 0$).*

5.2 Flip bifurcation analysis of interior fixed point

This section investigates the possibility of filp bifurcation of interior fixed point $P_{12}\,(x^*, y^*)$ by taking the parametric parameter p as bifurcation parameter. One can observe that one of the eigenvalues of the positive fixed point $P_{12}\,(x^*, y^*)$ is $\lambda_1 = -1$ and the other (λ_2) is neither 1 nor -1, provided the parameters of the model are obtained within the following set $A = \{(a_l, a_u, b_l, b_u, c_l, c_u, d_l, d_u, p, m)\colon\ 1 - Tr\,(J) + Det\,(J) > 0,\ 1 + Tr\,(J) + Det\,(J) = 0,\ Tr\,(J) \neq 0, 2 \text{ and } p \in [0,1]\}$.

Here we discuss flip bifurcation of the model (3) at $P_{12}\,(x^*, y^*)$ when the parameters vary in a small neighborhood of A. In analyzing the flip bifurcation, p is used as the bifurcation parameter. Further $p^*(|p^*| \lll 1)$ is the perturbation of p, a perturbation of the model can be considered as follows:

$$\begin{aligned} x_{n+1} &= a_l\left(\frac{a_u}{a_l}\right)^{p+p^*} x_n(1-x_n) - b_l\left(\frac{b_u}{b_l}\right)^{p+p^*}(1-m)x_ny_n \equiv f(x_n,y_n,p^*) \\ y_{n+1} &= (1-m)c_l\left(\frac{c_u}{c_l}\right)^{p+p^*} x_ny_n - d_l\left(\frac{d_u}{d_l}\right)^{p+p^*} y_n \equiv g(x_n,y_n,p^*) \end{aligned} \tag{7}$$

Let $u_n = x_n - x^*, v_n = y_n - y^*$, then equilibrium point $P_{12}\,(x^*, y^*)$ is transformed into the origin, and further expanding f and g as a Taylor series at $(u_n, v_n, p^*) = (0, 0, 0)$ to the third order, the model (7) becomes

$$\begin{aligned} u_{n+1} &= \alpha_1u_n + \alpha_2v_n + \alpha_{11}u_n^2 + \alpha_{12}u_nv_n + \alpha_{13}u_np^* + \alpha_{23}v_np^* + \\ &\quad \alpha_{111}u_n^3 + \alpha_{112}u_n^2v_n + \alpha_{113}u_n^2p^* + \alpha_{123}u_nv_np^* + O((|u_n|+|v_n|+|p^*|)^4) \\ v_{n+1} &= \beta_1u_n + \beta_2v_n + \beta_{11}u_n^2 + \beta_{12}u_nv_n + \beta_{22}v_n^2 + \beta_{13}u_np^* + \beta_{23}v_np^* + \beta_{111}u_n^3 + \\ &\quad \beta_{112}u_n^2v_n + \beta_{113}u_n^2p^* + \beta_{123}u_nv_np^* + \beta_{223}v_n^2p^* + O((|u_n|+|v_n|+|p^*|)^4) \end{aligned} \tag{8}$$

Where $\alpha_1 = f_x(x^*, y^*, 0)$, $\alpha_2 = f_y(x^*, y^*, 0)$, $\alpha_{11} = f_{xx}(x^*, y^*, 0)$, $\alpha_{12} = f_{xy}(x^*, y^*, 0)$, $\alpha_{13} = f_{xp^*}(x^*, y^*, 0)$, $\alpha_{23} = f_{yp^*}(x^*, y^*, 0)$, $\alpha_{111} = f_{xxx}(x^*, y^*, 0)$, $\alpha_{112} = f_{xxy}(x^*, y^*, 0)$, $\alpha_{113} = f_{xxp^*}(x^*, y^*, 0)$, $\alpha_{123} = f_{xyp^*}(x^*, y^*, 0)$, $\beta_1 = g_x(x^*, y^*, 0)$, $\beta_2 = g_y(x^*, y^*, 0)$, $\beta_{11} = g_{xx}(x^*, y^*, 0)$, $\beta_{12} = g_{xy}(x^*, y^*, 0)$, $\beta_{22} = g_{yy}(x^*, y^*, 0)$, $\beta_{13} = g_{xp^*}(x^*, y^*, 0)$, $\beta_{23} = g_{yp^*}(x^*, y^*, 0)$, $\beta_{111} = g_{xxx}(x^*, y^*, 0)$, $\beta_{112} = g_{xxy}(x^*, y^*, 0)$, $\beta_{113} = g_{xxp^*}(x^*, y^*, 0)$, $\beta_{123} = g_{xyp^*}(x^*, y^*, 0)$, $\beta_{223} = g_{yyp^*}(x^*, y^*, 0)$.

We define $T = \begin{bmatrix} \alpha_2 & \alpha_2 \\ -1-\alpha_1 & \lambda_2-\alpha_1 \end{bmatrix}$, the invertible nature of T is evident. Using the transformation $\begin{bmatrix} u_n \\ v_n \end{bmatrix} = T \begin{bmatrix} \overline{x}_n \\ \overline{y}_n \end{bmatrix}$ the model (8) becomes

$$\begin{aligned} \overline{x}_{n+1} &= -\overline{x}_n + f_1(u_n, v_n, p^*) \\ \overline{y}_{n+1} &= \lambda_2\overline{y}_n + g_1(u_n, v_n, p^*) \end{aligned} \tag{9}$$

where the functions f_1 and g_1 denote the terms in the model (9) in variables (u_n, v_n, p^*) with order at least two.

From the center manifold theorem, we know that there exists a center manifold $W^c(0, 0, 0)$ of the model (9) at $(0, 0)$ in a small neighborhood of $p^* = 0$, which can be approximately described as follows:

$W^c(0,0,0) = \left\{(\overline{x}_n, \overline{y}_n, p^*)\,\epsilon R^3 : \overline{y}_{n+1} = \overline{\alpha}_1\overline{x}_n^2 + \overline{\alpha}_2\overline{x}_np^* + O((|\overline{x}_n| + |p^*|)^3)\right\}$

where $\overline{\alpha}_1 = \frac{\alpha_2[(1+\alpha_1)\alpha_{11}+\alpha_2\beta_{11}]}{1-\lambda_2^2} + \frac{\beta_{22}(1+\alpha_1)^2}{1-\lambda_2^2} - \frac{(1+\alpha_1)[\alpha_{12}(1+\alpha_1)+\alpha_2\beta_{12}]}{1-\lambda_2^2}$,

$\overline{\alpha}_2 = \frac{(1+\alpha_1)[\alpha_{23}(1+\alpha_1)+\alpha_2\beta_{23}]}{\alpha_2(1+\lambda_2)^2} - \frac{(1+\alpha_1)\alpha_{13}+\alpha_2\beta_{13}]}{(1+\lambda_2)^2}$

We obtain the model (9), which is restricted to the center manifold $W^c(0, 0, 0)$, has the following form

$\overline{x}_{n+1} = -\overline{x}_n + h_1\overline{x}_n^2 + h_2\overline{x}_np^* + h_3\overline{x}_n^2p^* + h_4\overline{x}_np^{*2} + h_5\overline{x}_n^3 + O((|\overline{x}_n| + |p^*|)^3) \equiv F(\overline{x}_n, p^*)$

$h_1 = \frac{\overline{\alpha}_2[(\lambda_2-\overline{\alpha}_1)\alpha_{11}-\overline{\alpha}_2\beta_{11}]}{1+\lambda_2} - \frac{\beta_{22}(1+\overline{\alpha}_1)^2}{1+\lambda_2} - \frac{(1+\overline{\alpha}_1)[(\lambda_2-\overline{\alpha}_1)\alpha_{12}-\overline{\alpha}_2\beta_{12}]}{1+\lambda_2}$,

$h_2 = \frac{(\lambda_2-\overline{\alpha}_1)\alpha_{13}-\overline{\alpha}_2\beta_{13}}{1+\lambda_2} - \frac{(1+\overline{\alpha}_1)[(\lambda_2-\overline{\alpha}_1)\alpha_{23}-\overline{\alpha}_2\beta_{23}]}{\overline{\alpha}_2(1+\lambda_2)}$,

$h_3 = \frac{(\lambda_2-\alpha_1)\overline{\alpha}_1\alpha_{13}-\alpha_2\beta_{13}}{1+\lambda_2} + \frac{[(\lambda_2-\alpha_1)\alpha_{23}-\alpha_2\beta_{23}](\lambda_2-\alpha_1)\overline{\alpha}_1}{\alpha_2(1+\lambda_2)}$

$$-\frac{(1+\alpha_1)[(\lambda_2-\alpha_1)\alpha_{123}-\alpha_2\beta_{123}]}{1+\lambda_2}+\frac{\alpha_2[(\lambda_2-\alpha_1)\alpha_{113}-\alpha_2\beta_{113}]}{1+\lambda_2}-\frac{\beta_{223}(1+\alpha_1)^2}{1+\lambda_2}$$
$$+\frac{2\alpha_2\overline{\alpha}_2[(\lambda_2-\alpha_1)\alpha_{11}-\alpha_2\beta_{11}]}{1+\lambda_2}-\frac{2\beta_{22}\overline{\alpha}_2(1+\alpha_1)(\lambda_2-\alpha_1)}{1+\lambda_2}+\frac{\overline{\alpha}_2[(\lambda_2-\alpha_1)\alpha_{12}-\alpha_2\beta_{12}](\lambda_2-1-2\alpha_1)}{1+\lambda_2},$$
$$h_4=\frac{\overline{\alpha}_2[(\lambda_2-\alpha_1)\alpha_{13}-\alpha_2\beta_{13}]}{1+\lambda_2}+\frac{[(\lambda_2-\alpha_1)\alpha_{23}-\alpha_2\beta_{23}](\lambda_2-\alpha_1)\overline{\alpha}_2}{\alpha_2(1+\lambda_2)}+\frac{2\alpha_2\overline{\alpha}_2[(\lambda_2-\alpha_1)\alpha_{11}-\alpha_2\beta_{11}]}{1+\lambda_2}$$
$$+\frac{2\beta_{22}\overline{\alpha}_2(1+\alpha_1)(\lambda_2-\alpha_1)}{1+\lambda_2}+\frac{\overline{\alpha}_2[(\lambda_2-\alpha_1)\alpha_{12}-\alpha_2\beta_{12}](\lambda_2-1-2\alpha_1)}{1+\lambda_2},$$
$$h_5=\frac{2\alpha_2\overline{\alpha}_1[(\lambda_2-\alpha_1)\alpha_{11}-\alpha_2\beta_{11}]}{1+\lambda_2}+\frac{2\beta_{22}\overline{\alpha}_1(\lambda_2-\alpha_1)(1+\alpha_1)}{1+\lambda_2}+\frac{[(\lambda_2-\alpha_1)\alpha_{11}-\alpha_2\beta_{11}](\lambda_2-1-2\alpha_1)\overline{\alpha}_1}{1+\lambda_2}$$
$$+\frac{\overline{\alpha}_2^2[(\lambda_2-\alpha_1)\alpha_{111}-\alpha_2\beta_{111}]}{1+\lambda_2}-\frac{\overline{\alpha}_2(1+\alpha_1)[(\lambda_2-\alpha_1)\alpha_{112}-\alpha_2\beta_{112}]}{1+\lambda_2}.$$

For flip bifurcation, we require the two discriminatory quantities ξ_1 and ξ_2 be non-zero,
$\xi_1=\left(\frac{\partial^2 F}{\partial\overline{x}\partial p^*}+\frac{1}{2}\frac{\partial F}{\partial p^*}\frac{\partial^2 F}{\partial\overline{x}^2}\right)|_{(0,0)}$ and $\xi_2=\left(\frac{1}{6}\frac{\partial^3 F}{\partial\overline{x}^3}+\left(\frac{1}{2}\frac{\partial^2 F}{\partial\overline{x}^2}\right)^2\right)|_{(0,0)}$

Finally, from the above analysis, we have the following result.

Theorem 3. *If $\xi_1\neq 0$ and $\xi_2\neq 0$ then the model (3) undergoes flip bifurcation at $P_{12}=(x^*,y^*)$ when the parameter p varies. If $\xi_2>0$ and $\xi_2<0$, then the period-2 points that bifurcation from $P_{12}=(x^*,y^*)$ are stable and unstable, respectively.*

§6 Chaos Control

The controlling chaos in discrete-time models is a topic of great interest for many researchers, however, it is intended that chaos be avoided. For chaos control, many practical methods can be used in many fields such as communications, ecological systems, physics laboratories, turbulence, and cardiology, etc. Chaos control of discrete-time models can be obtained using various methods. We present a hybrid control technique and feedback control method to stabilize chaotic orbits at an unstable fixed point of the proposed discrete prey-predator model (3).

6.1 Hybrid control technique

The corresponding controlled system of (3) can be written as:

$$\begin{aligned} x_{n+1} &= \tau x_n(a_l^{1-p}a_u^p(1-x_n)-b_l^{1-p}b_u^p(1-m)y_n)+(1-\tau)x_n \\ y_{n+1} &= \tau y_n(c_l^{1-p}c_u^p(1-m)x_n-d_l^{1-p}d_u^p)+(1-\tau)y_n \end{aligned} \tag{10}$$

where $0<\tau<1$ denotes the control parameter, and in (10), the controlled strategy is a combination of both parameter perturbation and feedback control. Chaos for the controlled system (10) can be delayed, advanced, or even completely eliminated by a suitable choice of controlled parameter τ.

6.2 Feedback control method

Consider the following controlled form of model (3):

$$\begin{aligned} x_{n+1} &= a_l^{1-p}a_u^p x_n(1-x_n)-b_l^{1-p}b_u^p(1-m)x_ny_n+S \\ y_{n+1} &= c_l^{1-p}c_u^p(1-m)x_ny_n-d_l^{1-p}d_u^p y_n \end{aligned} \tag{11}$$

with the following feedback control law as the control force:

$$S = -q_1\left(x_n - x^*\right) - q_2\left(y_n - y^*\right)$$

where q_1 and q_2 are the feedback gain and (x^*, y^*) is a positive fixed point of model.

The Jacobian Matrix J for the system (11) at (x^*, y^*) is

$J = \begin{bmatrix} a_{11} - q_1 & a_{12} - q_2 \\ a_{21} & a_{22} \end{bmatrix}$

where $a_{11} = a_l^{1-p} a_u^p (1-2x) - b_l^{1-p} b_u^p (1-m) y$, $a_{12} = -b_l^{1-p} b_u^p (1-m) x$, $a_{21} = c_l^{1-p} c_u^p (1-m) y$, $a_{22} = c_l^{1-p} c_u^p (1-m) x - d_l^{1-p} d_u^p$.

The corresponding characteristic equation of matrix J is

$\tau^2 - \left(a_{11} + a_{22} - q_1\right)\tau + a_{22}\left(a_{11} - q_1\right) - a_{21}\left(a_{12} - q_2\right) = 0$

Let τ_1 and τ_2 are the eigenvalues of the matrix J then we have

$$\tau_1 + \tau_2 = a_{11} + a_{22} - q_1 \text{ and } \tau_1\tau_2 = a_{22}\left(a_{11} - q_1\right) - a_{21}\left(a_{12} - q_2\right) \tag{12}$$

The lines of marginal stability are determined by solving the equation $\tau_1 = \pm 1$ and $\tau_1\tau_2 = 1$. These conditions guarantee that the eigenvalues τ_1 and τ_2 have modulus less than 1.

Suppose $\tau_1\tau_2 = 1$, then from (12) we have line l_1 as:

$a_{22}q_1 - a_{21}q_2 = a_{22}a_{11} - a_{21}a_{12} - 1$

Suppose $\tau_1 = \pm 1$ then from (12) we have line l_2 and l_3 as:

$\left(1 - a_{22}\right) q_1 + a_{21}q_2 = \left(a_{11} - 1\right)\left(1 - a_{22}\right) + a_{21}a_{12}$, and

$\left(1 + a_{22}\right) q_1 - a_{21}q_2 = \left(a_{11} + 1\right)\left(1 + a_{22}\right) - a_{21}a_{12}$.

The stable eigenvalues lie within a triangular region by the lines l_1, l_2 and l_3.

§7 Numerical Example and Simulations

This section presents numerical examples with different parameter values to investigate and illustrate the analytical result of the previous section for the proposed model. Since the values of the proposed model parameters are not taken from real world observations as no case study performs on the species. For the simulation experiments, we mainly use the software MATLAB R2018a. We present the time plots and phase portraits to illustrate the theoretical analysis and show the discrete prey-predator system's fascinating, complex dynamical behaviour.

We consider the parameters of the prey-predator model as $\widehat{a} \in [a_l, a_u] = [3.8, 4.2]$, $\widehat{b} \in [b_l, b_u] = [2.8, 3.2]$, $\widehat{c} \in [c_l, c_u] = [3.8, 4.2]$, $\widehat{d} \in [d_l, d_u] = [0.1, 0.2]$, $m = 0.3$ to calculate equilibrium points, eigenvalues, and stability of equilibrium points as shown in table 1.

We observe from table 1 that the equilibrium point $(0, 0)$ is a saddle point for all values of p. It is also noted that the predator free equilibrium points are unstable for all value of p. Interior equilibrium points are stable spiral for $p = 0.0, 0.2, 0.4, 0.6$, but it changes its stability when $p = 0.8$ and 1, which implies that for the upper limit of some interval parameters, the prey-predator system changes its stability nature.

Table 1. Equilibrium points, eigenvalues and stability of equilibrium points.

p	Equilibrium points	Eigenvalues	Stability
$p = 0.0$	$(0, 0)$	$-0.1,\ 3.8$	Saddle point
	$(0.74, 0)$	$-1.82,\ 1.87$	Unstable point
	$(0.41, 0.63)$	$0.22 + 0.87i,\ 0.22 - 0.87i$	Stable point
$p = 0.2$	$(0, 0)$	$-0.11,\ 3.88$	Saddle point
	$(0.74, 0)$	$-1.86, 1.89$	Unstable point
	$(0.41, 0.64)$	$0.20 + 0.90i,\ 0.20 - 0.90i$	Stable point
$p = 0.4$	$(0, 0)$	$-0.13,\ 3.96$	Saddle point
	$(0.75, 0)$	$-1.98, 1.94$	Unstable point
	$(0.41, 0.65)$	$0.18 + 0.92i,\ 0.18 - 0.92i$	Stable point
$p = 0.6$	$(0, 0)$	$-0.15,\ 4.04$	Saddle point
	$(0.75, 0)$	$-2.02,\ 1.97$	Unstable point
	$(0.41, 0.65)$	$0.18 + 0.95i,\ 0.18 - 0.95i$	Stable point
$p = 0.8$	$(0, 0)$	$-0.17,\ 4.12$	Saddle point
	$(0.76, 0)$	$-2.14,\ 2.02$	Unstable point
	$(0.41, 0.66)$	$0.15 + 0.99i,\ 0.15 - 0.99i$	Unstable point
$p = 1.0$	$(0, 0)$	$-0.20,\ 4.20$	Saddle point
	$(0.76, 0)$	$-2.18,\ 2.03$	Unstable point
	$(0.41, 0.66)$	$0.14 + 1.02i,\ 0.14 - 1.02i$	Unstable point

We present the time plots for the given parameters in figure 1 for initial value $(x_0, y_0) = (0.6, 0.5)$ for different values of p in sub-figure as (a) $p = 0.5$, (b) $p = 0.7$, (c) $p = 0.75$, (d) $p = 0.8$.

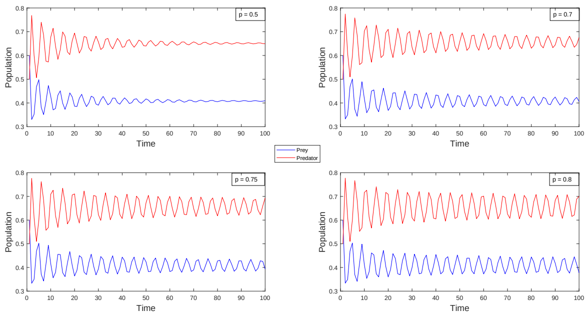

Figure 1. Time plot of the population with different values of p.

From the sub-plots of figure 1, we observe a damped oscillation, where the oscillations become smaller with time. We observe here interesting feature for different values of p, population take long time to stabilize the system after value of $p = 0.5$. The population does not stabilize and it oscillates in a systematic manner for the values of $p = 0.8$.

The phase portrait of the model for the given value of parameters and initial value $(x_0, y_0) = (0.6, 0.5)$ is shown in figure 2 for different values of p in sub-figure as (a) $p = 0.5$, (b) $p = 0.7$,

(c) $p = 0.75$, (d) $p = 0.8$. Figure 2 shows that the coexistence steady state is globally stable for lower values of p.

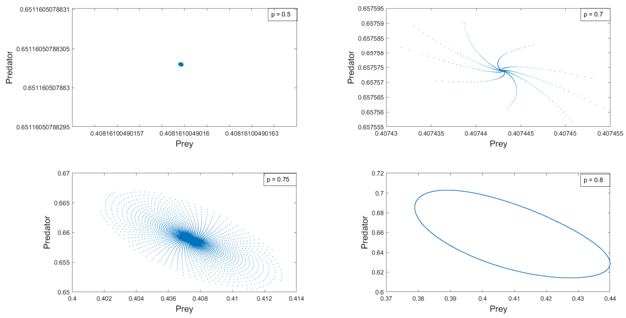

Figure 2. Phase portrait of the model with different values of p.

Figure 2 shows that all trajectories are spiral to the stable fixed point, and trajectories are attracted to a limit cycle about interior equilibrium points. Hence, there exists a bifurcation for p, and it is super critical - after the fixed point loses stability, it is surrounded by a stable limit cycle.

This paper projected the effect of prey refuge on the proposed prey- predator system, and the theoretical discussion of the prey refuge has been verified through the numerical demonstration. Figure 3 shows the effect of prey refuge in the population model for different ratios on the density of the prey species. Figure 3 is drawn with respect to the given interval valued parameters and $p = 0.4$ and for different prey ratios as (a) $m = 0.1$, (b) $m = 0.2$, (c) $m = 0.28$, (d) $m = 0.3$

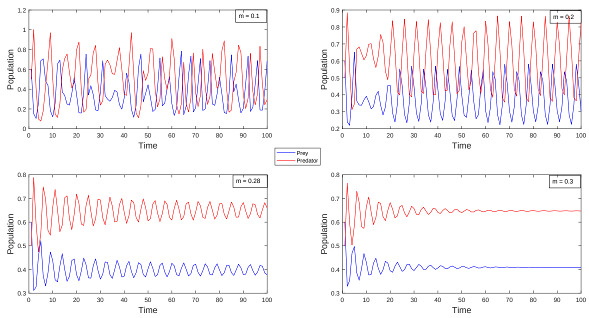

Figure 3. Effect of m in population system with time.

We observe from figure 3 that the effect on the system is more acceptable with increasing value of m. Refuge has a stabilizing effect in the proposed model, which is generally observed in the real world. We observe the chaotic behavior of the prey-predator system for $m = 0.1, 0.2$. We observe the damped oscillation of the species for $m = 0.28$ and $m = 0.3$.

Figure 4 is drawn for the different values of m in sub-plots on the basis of initial value

$(x_0, y_0) = (0.6, 0.5)$, $p = 0.4$, and the given interval parameter values of $\widehat{a}$, $\widehat{b}$, $\widehat{c}$, $\widehat{d}$. This is the phase portrait of the model for some specific value of parameters.

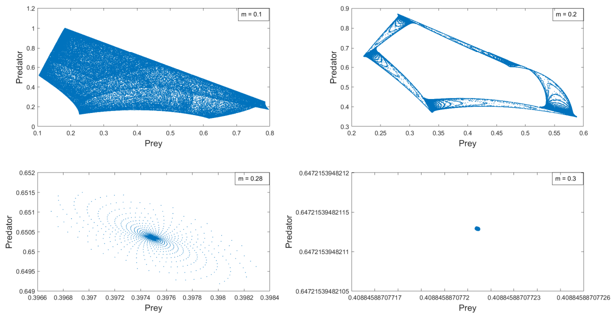

Figure 4. Phase portrait of the proposed system for different value of m.

Figure 5, 6 and 7 are drawn for the interval parameter values $\widehat{a} \in [a_l, a_u] = [3.8, 4.2]$, $\widehat{b} \in [b_l, b_u] = [2.8, 3.2]$, $\widehat{c} \in [c_l, c_u] = [3.8, 4.2]$, $\widehat{d} \in [d_l, d_u] = [0.1, 0.2]$, $m = 0.3$ and initial value $(0.6, 0.5)$.

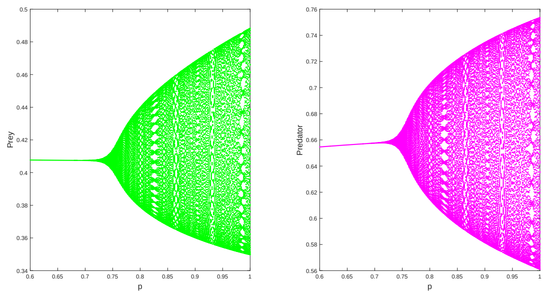

Figure 5. Bifurcation diagram with varing p.

Figure 5 depicts a smooth invariant circle bifurcates from a stable equilibrium. It is noticed that if p exceeds 0.75, there appears a circular curve enclosing equilibrium and its radius become larger with the increasing value of p.

Lyapunov exponents tell us the rate of divergence of nearby trajectories—a key component of chaotic dynamics. Negative Lyapunov exponents are characteristic of dissipative or non-conservative systems, and exhibit asymptotic stability; the more negative the exponent, the greater the stability. For positive Lyapunov, the orbit is unstable and chaotic, and will diverge at any arbitrary separation. In figure 6, The largest Lyapunov exponent L1 is greater than zero when $p > 0.95$, which implies that the system is chaotic. The phase portraits corresponding to figure 5 are presented in figure 7, which clearly depict the process of how a smooth invariant closed curve bifurcates from a stable fixed point.

The bifurcation analysis for the ratio of prey refuge, the Lyapunov exponents and phase

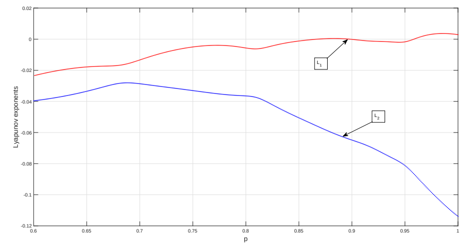

Figure 6. Lyapunov exponents of system with varing p.

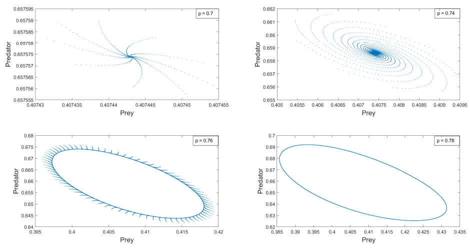

Figure 7. The phase portraits for various p corresponding to figure 5.

portraits with varying m for the proposed system are presented in figure 8, 9 and 10, which are drawn on the basis of parameter values $\widehat{a} \in [a_l, a_u] = [4.0, 4.2]$, $\widehat{b} \in [b_l, b_u] = [3.0, 3.2]$, $\widehat{c} \in [c_l, c_u] = [3.2, 3.4]$, $\widehat{d} \in [d_l, d_u] = [0.1, 0.2]$, and $p = 0.1$.

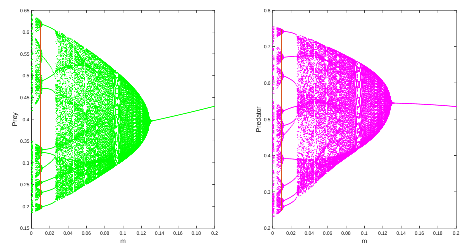

Figure 8. Bifurcation diagram of the system for m.

Figure 8 depicts the change of the system to a stable equilibrium from smooth invariant circle bifurcation. There appears a circular curve enclosing the equilibrium, and its radius becomes smaller with the increasing value of m and when the value of m exceeds 0.14, there appears an equilibrium point.

In figure 9, the largest Lyapunov exponent L1 is greater than zero when $m < 0.13$ except in periodic windows, which implies that the system is chaotic.

The phase portraits corresponding to figure 8 are presented in Figure 10, which clearly depict the process of how a stable fixed point bifurcates from chaos.

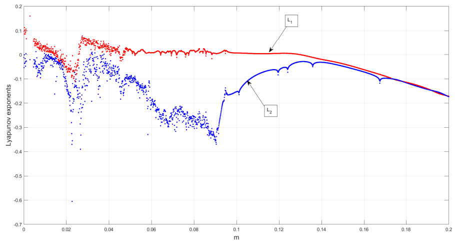

Figure 9. Lyapunov exponents of system for m.

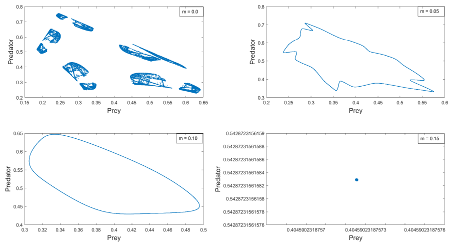

Figure 10. The phase portraits for various m corresponding to figure 8.

Figure 11 and 12, which are drawn on the basis of parameter values $\widehat{a} \in [a_l, a_u] = [3.8, 4.2]$, $\widehat{b} \in [b_l, b_u] = [2.8, 3.2]$, $\widehat{c} \in [c_l, c_u] = [3.8, 4.2]$, $\widehat{d} \in [d_l, d_u] = [0.1, 0.2]$, $p = 0.4, m = 0.1$ and initial value $(x_0, y_0) = (0.6, 0.5)$.

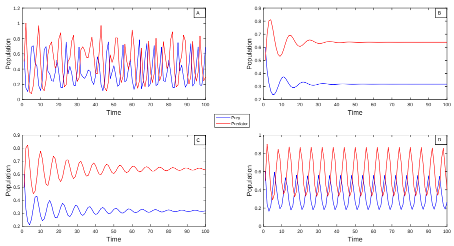

Figure 11. Hybrid control technique.

Hybrid control technique presented in figure 11, we observe the chaotic behaviour of the model in figure 11(A) and chaos control for different τ in (B) $\tau = 0.4$, (C) $\tau = 0.6$, (D) $\tau = 0.8$

We observe the chaotic behaviour of prey-predator shown in figure 12(A). In this case fixed point $(0.3007, 0.6371)$ is unstable. In the feedback control method for feedback gain $q_1 = -0.6$ and $q_2 = -0.6$, we observe the fixed point $(0.3007, 0.6371)$ that is stable as shown in figure 12(B).

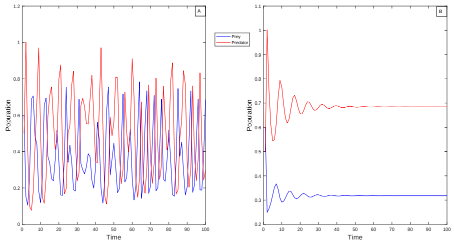

Figure 12. Feedback control method.

§8 Conclusion

We have studied the dynamical analysis of the proposed discrete prey-predator system with refuge proportion to prey density. In the Sundarban Tiger Reserve ecosystem, mangroves provide a refuge region for prey. This study has presented the qualitative behaviour of a discrete-time predator-prey system with imprecise biological parameters. We have found that the fixed points of the system incorporating the refuge concept on prey are proportional to the density of prey, and the stability has been discussed analytically. We have presented phase diagrams and time plots of the system for different values of the parameter, which exhibit the effects of prey refuge on the system. We have introduced a new concept in bifurcation analysis. The codimension of a bifurcation is the number of parameters that must be varied for the bifurcation to occur. When we have considered p as a bifurcation parameter, and ultimately we have presented here four bifurcation parameters in a particular range. Therefore, the interesting fact is that our technique in converting the four codimension bifurcations to one codimension. We have found different model reactions for different values of the parameters for prey refuge. This study will be beneficial for modelling and analysis of large-scale problems of predator-prey interactions. The mangrove is the controller in the Sundarban reserve ecosystem. If we destroy the mangrove, then we observe prey-predators imbalance that implies bifurcation and lastly chaos. A standard refuge is suitable for this ecosystem, more shelter is dangerous for predator species, and no sanctuary is difficult for prey. Our model is ideal for analysing this area's ecosystem. Also, the study of imprecise biological parameters for the predator-prey system will focus on the new dimensions of further research.

Declarations

Conflict of interest The authors declare no conflict of interest.

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