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The compound class of exponentiated power Lindley power series distribution: properties and applications

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Abstract. We introduce a new generalization of the exponentiated power Lindley distribution, called the exponentiated power Lindley power series (EPLPS) distribution. The new distribution arises on a latent complementary risks scenario, in which the lifetime associated with a particular risk is not observable; rather, we observe only the maximum lifetime value among all risks. The distribution exhibits decreasing, increasing, unimodal and bathtub shaped hazard rate functions, depending on its parameters. Several properties of the EPLPS distribution are investigated. Moreover, we discuss maximum likelihood estimation and provide formulas for the elements of the Fisher information matrix. Finally, applications to three real data sets show the flexibility and potentiality of the EPLPS distribution.

§1 Introduction

Modeling and analyzing lifetime data are important aspects of statistical research in many applied sciences such as engineering, medicine, economics and so on. Various recent probability distributions discussed modelling of such data by compounding well-known continuous distributions such as the exponential, Weibull, and exponentiated exponential distributions with the power series distribution that includes the Poisson, logarithmic, geometric and binomial distributions as particular cases. The compounding approach gives new distributions that extend well-known families of distributions. At the same time, they offer more flexibility for modelling lifetime data. The extensions, sometimes, provide reasonable parametric fits to practical applications as in lifetime and reliability studies. The flexibility of such compound distributions comes in terms of one or more hazard rate shapes that may be decreasing or increasing or bathtub shaped or unimodal shaped. Some prominent compound distributions introduced recently are the exponential logarithmic distribution due to Tahmasbi and Rezaei [1]; exponential power series distribution due to Chahkandi ad Ganjali [2]; Weibull geometric distribution due to

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Barreto-Souza et al. [3]; Weibull power series distribution due to Morais and Barreto-Souza [4]; exponentiated exponential Poisson distribution due to Barreto-Souza and Cribari-Neto [5]; complementary exponential geometric distribution due to Louzada-Neto et al. [6]; complementary Weibull geometric distribution due to Tojeiro et al. [7]; Burr XII negative binomial distribution due to Ramos et al. [8]; compound class of extended Weibull power series distributions due to Silva et al. [9]; compound class of linear failure rate power series distributions due to Mahmoudi and Jafari [12]; exponentiated Weibull logarithmic distribution due to Mahmoudi and Sepahdar [11]; generalized exponential power series distribution due to Mahmoudi and Jafari [10]; exponentiated Weibull power series distribution due to Mahmoudi and Shiran [13]; generalized modified Weibull power series distribution due to Bagheri et al. [14]; exponentiated power Lindley geometric distribution due to Alizadeh et al. [15]; Topp-Leone power series distribution due to Roozegar and Nadarajah [16]. See also Korkmaz et al. [17], Korkmaz and Erisoglu [18], Alizadeh et al. [19], Korkmaz et al. [20], Sen et al. [21] and Nasir et al. [22].

Lindley [23] introduced a one parameter distribution, known as the Lindley distribution, given by the probability density function (pdf)

$$f_L(x) = \frac{\lambda^2}{\lambda+1}(1+x)\, e^{-\lambda x}, \tag{1}$$

where $x > 0$ and $\lambda > 0$. Ghitany et al. [24] discussed various properties of this distribution and showed that in many ways the pdf given by (1) provides a better model for some applications than the exponential distribution. Bakouch et al. [25] proposed an extended Lindley distribution and discussed its various properties and applications. Ghitany et al. [26] developed a two parameter weighted Lindley distribution and discussed its applications to survival data. Nadarajah et al. [27] proposed a generalized Lindley distribution and discussed its various properties and applications. Merovci and Elbatal [28] used the quadratic rank transmutation map in order to generate a flexible family of probability distributions taking Lindley geometric distribution as the base value distribution. Asgharzadeh et al. [29] introduced a general family of continuous lifetime distributions by compounding any continuous distribution and the Poisson Lindley distribution. Oluyede and Yang [30] proposed a four parameter beta generalized Lindley distribution. This distribution contains the beta Lindley distribution and Lindley distribution as particular cases. A two parameter power Lindley distribution was proposed by Ghitany et al. [31]. A generalized power Lindley distribution with applications was proposed by Pararai et al. [32].

In this paper, we are motivated to introduce a new generalization of the Lindley distribution. It is shown to provide better fits than many of the known generalizations of the Lindley distribution, including those having more parameters. It exhibits a variety of bathtub shapes for its hazard rate function.

The aim of this paper is to propose a new class of lifetime distributions called exponentiated power Lindley power series distributions. The distributional properties are presented. The method of maximum likelihood is used to estimate the model parameters. Simulation study is presented to assess the performance and accuracy of maximum likelihood estimates (MLEs) of the EPLPS distribution. Some real data examples are discussed to illustrate the usefulness and applicability of the EPLPS distribution.

The contents of this paper are organized as follows. In Section 2, we define the new class of EPLPS distributions and derive the corresponding distributional properties. In Section 3, we provide detailed studies of the following particular cases of the EPLPS distribution: the exponentiated power Lindley Poisson and exponentiated power Lindley geometric distributions. We show that the hazard rate can be increasing, decreasing, bathtub shaped or unimodal. Section 4 derives various properties of the EPLPS distribution. We discuss MLE and Fisher information matrix in Section 5. Also given in Section 5 is a simulation study to assess the performance of MLEs. Section 6 gives real data applications to show the flexibility and potentiality of the EPLPS distribution. The paper is concluded in Section 7.

§2 The class of EPLPS distributions

A three parameter exponentiated power Lindley (EPL) distribution was introduced by Ashour and Eltehiwy [33]. Its cumulative distribution function (cdf) and pdf are

$$G(x) = \left[1 - \left(1 + \frac{\lambda x^\beta}{\lambda+1}\right) e^{-\lambda x^\beta}\right]^\alpha$$

and

$$g(x) = \frac{\alpha\beta\lambda^2 x^{\beta-1}}{\lambda+1}\left(1+x^\beta\right) e^{-\lambda x^\beta}\left[1 - \left(1 + \frac{\lambda x^\beta}{\lambda+1}\right) e^{-\lambda x^\beta}\right]^{\alpha-1},$$

respectively, for $\alpha, \beta, \lambda > 0$. This distribution can model monotone and non-monotone hazard rates, which are quite common in lifetime problems and reliability. Let N denote a power series random variable on positive integers with probability mass function given by

$$P(N=n) = \frac{a_n \theta^n}{C(\theta)},\ n = 1, 2, \ldots,\ \theta \in (0, S),$$

where $a_1, a_2, \ldots$ is a sequence of nonnegative real numbers, where at least one of them is strictly positive, S is a positive number not greater than the radius of convergence of the power series

$$\sum_{n=1}^{\infty} a_n \theta^n = C(\theta).$$

Suppose $Y_1, Y_2, \ldots, Y_N$ are independent and identical EPL random variables independent of N. Then $X = \max(Y_1, Y_2, \ldots, Y_N)$ is said to follow the EPLPS $(\alpha, \beta, \lambda, \theta)$ distribution. Its cdf, pdf, survival function and hazard rate function are

$$F(x) = \frac{C(\theta G(x))}{C(\theta)} = \frac{C\left(\theta\left[1 - \left(1 + \frac{\lambda x^\beta}{\lambda+1}\right) e^{-\lambda x^\beta}\right]^\alpha\right)}{C(\theta)}, \tag{2}$$

$$f(x) = \frac{\alpha\beta\theta\lambda^2 x^{\beta-1}}{\lambda+1}\left(1+x^\beta\right) e^{-\lambda x^\beta}\left[1 - \left(1 + \frac{\lambda x^\beta}{\lambda+1}\right) e^{-\lambda x^\beta}\right]^{\alpha-1} \frac{C'\left(\theta\left[1 - \left(1 + \frac{\lambda x^\beta}{\lambda+1}\right) e^{-\lambda x^\beta}\right]^\alpha\right)}{C(\theta)}, \tag{3}$$

$$S(x)=1-F(x)=\frac{C(\theta)-C\left(\theta\left[1-\left(1+\frac{\lambda x^{\beta}}{\lambda+1}\right)e^{-\lambda x^{\beta}}\right]^{\alpha}\right)}{C(\theta)},$$

and

$$h(x)=\frac{\frac{\alpha\beta\theta\lambda^{2}x^{\beta-1}}{\lambda+1}\left(1+x^{\beta}\right)e^{-\lambda x^{\beta}}\left[1-\left(1+\frac{\lambda x^{\beta}}{\lambda+1}\right)e^{-\lambda x^{\beta}}\right]^{\alpha-1}C'\left(\theta\left[1-\left(1+\frac{\lambda x^{\beta}}{\lambda+1}\right)e^{-\lambda x^{\beta}}\right]^{\alpha}\right)}{C(\theta)-C\left(\theta\left[1-\left(1+\frac{\lambda x^{\beta}}{\lambda+1}\right)e^{-\lambda x^{\beta}}\right]^{\alpha}\right)},\tag{4}$$

respectively.

Proposition 2.1. *The limit of the cdf of the EPLPS* $(\alpha,\beta,\lambda,\theta)$ *distribution as* $\theta\to 0^{+}$ *is*

$$\lim_{\theta\to 0^{+}}F(x)=\left[1-\left(1+\frac{\lambda x^{\beta}}{\lambda+1}\right)e^{-\lambda x^{\beta}}\right]^{\alpha c},$$

which is an EPL cdf with parameters αc, β *and* λ, *where* $c=\min\{n\in N; a_n>0\}$.

Proof. *The proof follows since*

$$\begin{aligned}\lim_{\theta\to 0^{+}}F(x) &= \lim_{\theta\to 0^{+}}\frac{C\left(\theta G(x)\right)}{C(\theta)}=\lim_{\theta\to 0^{+}}\frac{\sum_{n=1}^{\infty}a_n\theta^{n}[G(x)]^{n}}{\sum_{n=1}^{\infty}a_n\theta^{n}}\\ &= \lim_{\theta\to 0^{+}}\frac{a_c[G(x)]^{c}+\sum_{n=c+1}^{\infty}a_n\theta^{n-c}[G(x)]^{n}}{a_c+\sum_{n=c+1}^{\infty}a_n\theta^{n-c}}\\ &= [G(x)]^{c}=\left[1-\left(1+\frac{\lambda x^{\beta}}{\lambda+1}\right)e^{-\lambda x^{\beta}}\right]^{\alpha c}.\end{aligned}$$

The pdf of the EPLPS distribution can be expressed as a linear combination of pdfs:

$$f(x)=\theta g(x)\frac{C'\left(\theta G(x)\right)}{C(\theta)}=\sum_{n=1}^{\infty}P(N=n)g\left(x;\alpha n,\beta,\lambda\right),\tag{5}$$

where $g(x;\alpha n,\beta,\lambda)$ denotes the pdf of the EPL distribution with parameters $\alpha n,\beta,\lambda$. The equation (5) can be used to obtain some mathematical properties of the EPLPS distribution directly from those properties of the EPL distribution.

§3 Particular cases of the EPLPS distribution

In this section, we investigate the following particular cases of the EPLPS distribution: Exponentiated Power Lindley Poisson and Exponentiated Power Lindley logarithmic distributions. To illustrate the flexibility of these distributions, we plot the pdf, cdf and hazard rate function for different parameter values.

3.1 Exponentiated power Lindley Poisson distribution

The exponentiated power Lindley Poisson (EPLP) distribution is a particular case of the EPLPS distribution for $a_n = \frac{1}{n!}$ and $C(\theta) = e^{\theta} - 1$. Its cdf, pdf and hazard rate function are

$$F_{EPLP}(x) = \frac{e^{\theta\left[1-\left(1+\frac{\lambda x^{\beta}}{\lambda+1}\right)e^{-\lambda x^{\beta}}\right]^{\alpha}} - 1}{e^{\theta} - 1},$$

$$f_{EPLP}(x) = \frac{\alpha\beta\theta\lambda^2 x^{\beta-1}\left(1+x^{\beta}\right)e^{-\lambda x^{\beta}}}{\lambda+1}\left[1-\left(1+\frac{\lambda x^{\beta}}{\lambda+1}\right)e^{-\lambda x^{\beta}}\right]^{\alpha-1}\frac{e^{\theta\left[1-\left(1+\frac{\lambda x^{\beta}}{\lambda+1}\right)e^{-\lambda x^{\beta}}\right]^{\alpha}}}{e^{\theta}-1},$$

and

$$h_{EPLP}(x) = \frac{\frac{\alpha\beta\theta\lambda^2 x^{\beta-1}\left(1+x^{\beta}\right)e^{-\lambda x^{\beta}}}{\lambda+1}\left[1-\left(1+\frac{\lambda x^{\beta}}{\lambda+1}\right)e^{-\lambda x^{\beta}}\right]^{\alpha-1}\left(e^{\theta\left[1-\left(1+\frac{\lambda x^{\beta}}{\lambda+1}\right)e^{-\lambda x^{\beta}}\right]^{\alpha}} - 1\right)}{e^{\theta} - e^{\theta\left[1-\left(1+\frac{\lambda x^{\beta}}{\lambda+1}\right)e^{-\lambda x^{\beta}}\right]^{\alpha}}},$$

respectively, where $\alpha, \beta, \lambda, \theta > 0$.

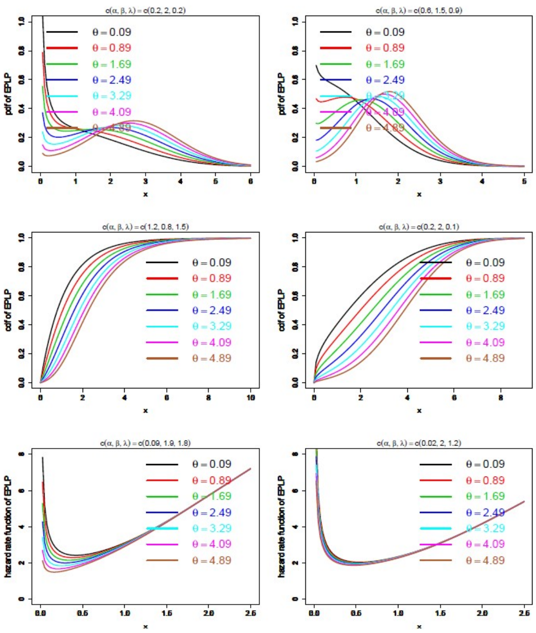

Figure 1. Pdf, cdf and hazard rate function of the EPLP distribution: pdf for $(\alpha, \beta, \lambda) = (0.2, 2, 0.2)$ (top left) and pdf for $(\alpha, \beta, \lambda) = (0.6, 1.5, 0.9)$ (top right); cdf for $(\alpha, \beta, \lambda) = (1.2, 0.8, 1.5)$ (middle left) and cdf for $(\alpha, \beta, \lambda) = (0.2, 2, 0.1)$ (middle right); hazard rate function for $(\alpha, \beta, \lambda) = (0.09, 1.9, 1.8)$ (bottom left) and hazard rate function for $(\alpha, \beta, \lambda) = (0.02, 2, 1.2)$ (bottom right).

Plots of pdf, cdf and hazard rate function of the EPLP distribution for different parameter values are given in Figure 1. The pdf becomes more unimodal with increasing values of θ. The bathtub shape of the hazard rate function becomes more concave with increasing values of θ.

3.2 Exponentiated power Lindley geometric distribution

The exponentiated power Lindley geometric (EPLG) distribution is a particular case of the EPLPS distribution for $a_n = 1$ and $C(\theta) = \theta(1-\theta)^{-1}$. Its cdf, pdf and hazard rate function are

$$F_{EPLG}(x) = \frac{(1-\theta)\left[1-\left(1+\frac{\lambda x^\beta}{\lambda+1}\right)e^{-\lambda x^\beta}\right]^\alpha}{1-\theta\left[1-\left(1+\frac{\lambda x^\beta}{\lambda+1}\right)e^{-\lambda x^\beta}\right]^\alpha},$$

$$f_{EPLG}(x) = \frac{\frac{\alpha\beta(1-\theta)\lambda^2 x^{\beta-1}}{\lambda+1}\left(1+x^\beta\right)e^{-\lambda x^\beta}\left[1-\left(1+\frac{\lambda x^\beta}{\lambda+1}\right)e^{-\lambda x^\beta}\right]^{\alpha-1}}{\left(1-\theta\left[1-\left(1+\frac{\lambda x^\beta}{\lambda+1}\right)e^{-\lambda x^\beta}\right]^\alpha\right)^2},$$

and

$$h_{EPLG}(x) = \frac{\frac{\alpha\beta(1-\theta)\lambda^2 x^{\beta-1}}{\lambda+1}\left(1+x^\beta\right)e^{-\lambda x^\beta}\left[1-\left(1+\frac{\lambda x^\beta}{\lambda+1}\right)e^{-\lambda x^\beta}\right]^{\alpha-1}}{\left(1-\left[1-\left(1+\frac{\lambda x^\beta}{\lambda+1}\right)e^{-\lambda x^\beta}\right]^\alpha\right)\left(1-\theta\left[1-\left(1+\frac{\lambda x^\beta}{\lambda+1}\right)e^{-\lambda x^\beta}\right]^\alpha\right)},$$

respectively, where $\alpha, \beta, \lambda > 0$ and $0 < \theta < 1$.

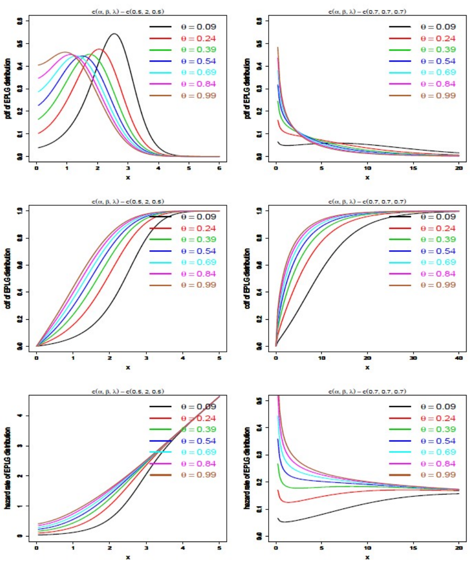

Figure 2. Pdf, cdf and hazard rate function of the EPLG distribution: pdf for $(\alpha,\beta,\lambda) = (0.5, 2, 0.5)$ (top left) and pdf for $(\alpha,\beta,\lambda) = (0.7, 0.7, 0.7)$ (top right); cdf for $(\alpha,\beta,\lambda) = (0.5, 2, 0.5)$ (middle left) and cdf for $(\alpha,\beta,\lambda) = (0.7, 0.7, 0.7)$ (middle right); hazard rate function for $(\alpha,\beta,\lambda) = (0.5, 2, 0.5)$ (bottom left) and hazard rate function for $(\alpha,\beta,\lambda) = (0.7, 0.7, 0.7)$ (bottom right).

Plots of pdf, cdf and hazard rate function of the EPLG distribution for some parameter values are given in Figure 2. The pdf becomes less unimodal with increasing values of θ. The hazard rate function either monotonically increases or monotonically decreases.

§4 Statistical distributional properties

In this section, we derive quantiles of the EPLPS distribution. Other properties including moments, moments of order statistics, Rényi and Shannon entropies and probability weighted moments can be obtained from the corresponding author.

Let X denote a random variable with the pdf (3). The quantile function, say $Q(p)$, defined by $F\left(Q(p)\right) = p$, for $0 < p < 1$, is the root of

$$\left(1+\frac{\lambda x^{\beta}}{\lambda+1}\right) e^{-\lambda x^{\beta}} = 1-\left(\underbrace{\frac{C^{-1}\left(p\; C(\theta)\right)}{\theta}}_{k_C}\right)^{\frac{1}{\alpha}} = 1-k_C^{\frac{1}{\alpha}}. \tag{6}$$

By substituting $Z(p) = -1-\lambda-\lambda\left[Q(p)\right]^{\beta}$, we may rewrite (6) as

$$Z(p)e^{Z(p)} = -(1+\lambda)\ \left(1-k_C^{\frac{1}{\alpha}}\right)\ e^{-1-\lambda}.$$

So, the solution for $Z(p)$ is

$$Z(p) = W\left(-(1+\lambda)\ \left(1-k_C^{\frac{1}{\alpha}}\right)\ e^{-1-\lambda}\right), \tag{7}$$

where $W(\cdot)$ denotes the Lambert W function, see Corless et al. [34] for detailed properties. Inverting (7), we can readily obtain

$$Q(p) = \left[-1-\frac{1}{\lambda}-\frac{1}{\lambda}W\left(-(1+\lambda)\ e^{-1-\lambda}\ \left(1-\left(\frac{C^{-1}\left(\theta C(\theta)\right)}{\theta}\right)^{\frac{1}{\alpha}}\right)\right)\right]^{\frac{1}{\beta}}.$$

Also the Galton' skewness defined by Galton [35] and the Moors' kurtosis defined by Moors [36] are given by

$$\text{skewness} = \frac{Q(6/8)-2Q(4/8)+Q(2/4)}{Q(6/8)-Q(2/8)}, \qquad \text{kurtosis} = \frac{Q(7/8)-Q(5/8)+Q(3/8)-Q(1/8)}{Q(6/8)-Q(2/8)}.$$

Figure 3 plots the behavior of Galton' skewness and Moors' kurtosis as functions of θ for representative values of α, β and λ.

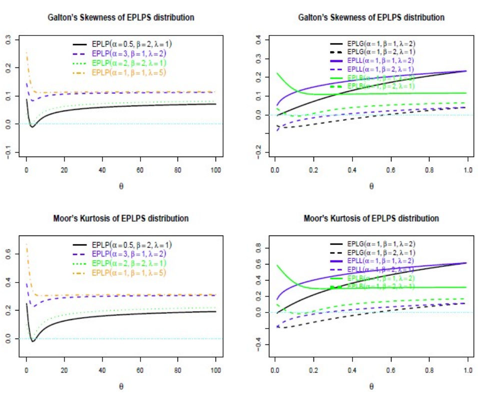

Figure 3. Galton's skewness and Moors' kurtosis for EPLP, EPLG, EPLL and EPLB (with $m = 20$) distributions for different values of α, β and λ.

The figure suggests that

- For the EPLP distribution, both Galton' skewness and Moors' kurtosis decrease and stabilize as θ increases and α, β, λ are not getting very large.

- For the EPLG and EPLL distributions, both Galton' skewness and Moors' kurtosis increase and stabilize as θ increases and α, β, λ are not getting very large.

- For the EPLB distribution, both Galton' skewness and Moors' kurtosis decrease and stabilize as θ increases and α, β, λ are not getting very large.

It can be concluded that these distributions behave differently as θ increases.

§5 Statistical inference and numerical results

5.1 Statistical inference

We estimate the parameters of the EPLPS distribution by the method of maximum likelihood. Suppose $x_1, x_2, \ldots, x_n$ is a random sample from the EPLPS distribution with all of its parameters unknown. The log-likelihood function is

$$\begin{aligned}\ln L &= n\ln(\alpha) + n\ln(\beta) + n\ln(\theta) \\ &\quad +2n\ln(\lambda) - n\ln(\lambda+1) - n\ln C(\theta) \\ &\quad +(\beta-1)\sum_{i=1}^{n}\ln(x_i) + \sum_{i=1}^{n}\ln\left(1+x_i^{\beta}\right) - \lambda\sum_{i=1}^{n}x_i^{\beta} \\ &\quad +(\alpha-1)\sum_{i=1}^{n}\ln\left[1-\left(1+\frac{\lambda x_i^{\beta}}{\lambda+1}\right)e^{-\lambda x_i^{\beta}}\right] \\ &\quad +\sum_{i=1}^{n}\ln C\left(\theta\left[1-\left(1+\frac{\lambda x_i^{\beta}}{\lambda+1}\right)e^{-\lambda x_i^{\beta}}\right]^{\alpha}\right).\end{aligned}$$

The first order partial derivatives of $\ln L$ with respect to α, β, λ and θ are

$$\begin{aligned}\frac{\partial \ln L}{\partial \alpha} &= \frac{n}{\alpha} + \sum_{i=1}^{n}\ln\left(1-\left(1+\frac{\lambda\, x_i{}^{\beta}}{\lambda+1}\right)e^{-\lambda\, x_i{}^{\beta}}\right) \\ &\quad +\theta\sum_{i=1}^{n}\frac{C'\left(\theta\left(1-\left(1+\frac{\lambda\, x_i{}^{\beta}}{\lambda+1}\right)e^{-\lambda\, x_i{}^{\beta}}\right)^{\alpha}\right)}{C\left(\theta\left(1-\left(1+\frac{\lambda\, x_i{}^{\beta}}{\lambda+1}\right)e^{-\lambda\, x_i{}^{\beta}}\right)^{\alpha}\right)} \\ &\quad \cdot\left(1-\left(1+\frac{\lambda\, x_i{}^{\beta}}{\lambda+1}\right)e^{-\lambda\, x_i{}^{\beta}}\right)^{\alpha} \\ &\quad \cdot\ln\left(1-\left(1+\frac{\lambda\, x_i{}^{\beta}}{\lambda+1}\right)e^{-\lambda\, x_i{}^{\beta}}\right),\end{aligned}$$

$$
\begin{aligned}
\frac{\partial \ln L}{\partial \beta} &= \frac{n}{\beta} + \sum_{i=1}^{n} \ln(x_i) + \sum_{i=1}^{n} \frac{x_i^{\beta} \ln(x_i)}{1 + x_i^{\beta}} - \lambda \left(\sum_{i=1}^{n} x_i^{\beta} \ln(x_i) \right) \\
&+ (\alpha - 1) \left\{ \sum_{i=1}^{n} \frac{-\frac{\lambda x_i^{\beta} \ln(x_i) e^{-\lambda x_i^{\beta}}}{\lambda+1} + \left(1 + \frac{\lambda x_i^{\beta}}{\lambda+1}\right) \lambda x_i^{\beta} \ln(x_i) e^{-\lambda x_i^{\beta}}}{1 - \left(1 + \frac{\lambda x_i^{\beta}}{\lambda+1}\right) e^{-\lambda x_i^{\beta}}} \right\} \\
&+ \alpha\theta \sum_{i=1}^{n} \left\{ \frac{C'\left(\theta \left[1 - \left(1 + \frac{\lambda x_i^{\beta}}{\lambda+1}\right) e^{-\lambda x_i^{\beta}}\right]^{\alpha}\right)}{C\left(\theta \left[1 - \left(1 + \frac{\lambda x_i^{\beta}}{\lambda+1}\right) e^{-\lambda x_i^{\beta}}\right]^{\alpha}\right)} \left[1 - \left(1 + \frac{\lambda x_i^{\beta}}{\lambda + 1}\right) e^{-\lambda x_i^{\beta}}\right]^{\alpha - 1} \right. \\
&\quad \cdot \left. \left[-\frac{\lambda x_i^{\beta} \ln(x_i) e^{-\lambda x_i^{\beta}}}{\lambda + 1} + \left(1 + \frac{\lambda x_i^{\beta}}{\lambda + 1}\right) \lambda x_i^{\beta} \ln(x_i) e^{-\lambda x_i^{\beta}} \right] \right\},
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial \ln L}{\partial \lambda} &= \frac{2n}{\lambda} - \frac{n}{\lambda + 1} - \sum_{i=1}^{n} x_i^{\beta} + (\alpha - 1) \\
&\quad \cdot \left\{ \sum_{i=1}^{n} \frac{-\left(\frac{x_i^{\beta}}{\lambda+1} - \frac{\lambda x_i^{\beta}}{(\lambda+1)^2}\right) e^{-\lambda x_i^{\beta}} + \left(1 + \frac{\lambda x_i^{\beta}}{\lambda+1}\right) e^{-\lambda x_i^{\beta}} x_i^{\beta}}{1 - \left(1 + \frac{\lambda x_i^{\beta}}{\lambda+1}\right) e^{-\lambda x_i^{\beta}}} \right\} \\
&\quad + \alpha\theta \sum_{i=1}^{n} \left\{ \frac{C'\left(\theta \left[1 - \left(1 + \frac{\lambda x_i^{\beta}}{\lambda+1}\right) e^{-\lambda x_i^{\beta}}\right]^{\alpha}\right)}{C\left(\theta \left[1 - \left(1 + \frac{\lambda x_i^{\beta}}{\lambda+1}\right) e^{-\lambda x_i^{\beta}}\right]^{\alpha}\right)} \right. \\
&\quad \cdot \left[1 - \left(1 + \frac{\lambda x_i^{\beta}}{\lambda + 1}\right) e^{-\lambda x_i^{\beta}}\right]^{\alpha - 1} \\
&\quad \cdot \left. \left(-\left(\frac{x_i^{\beta}}{\lambda + 1} - \frac{\lambda x_i^{\beta}}{(\lambda + 1)^2}\right) e^{-\lambda x_i^{\beta}} + \left(1 + \frac{\lambda x_i^{\beta}}{\lambda + 1}\right) e^{-\lambda x_i^{\beta}} x_i^{\beta} \right) \right\}
\end{aligned}
$$

and

$$
\frac{\partial \ln L}{\partial \theta} = \frac{n}{\theta} - \frac{nC'(\theta)}{C(\theta)} + \sum_{i=1}^{n} \left\{ \frac{C'\left(\theta \left[1 - \left(1 + \frac{\lambda x_i^{\beta}}{\lambda+1}\right) e^{-\lambda x_i^{\beta}}\right]^{\alpha}\right)}{C\left(\theta \left[1 - \left(1 + \frac{\lambda x_i^{\beta}}{\lambda+1}\right) e^{-\lambda x_i^{\beta}}\right]^{\alpha}\right)} \left[1 - \left(1 + \frac{\lambda x_i^{\beta}}{\lambda + 1}\right) e^{-\lambda x_i^{\beta}}\right]^{\alpha} \right\},
$$

respectively. To find out the MLEs of α, β, λ and θ, we have to solve $\frac{\partial \ln L}{\partial \alpha} = 0$, $\frac{\partial \ln L}{\partial \beta} = 0$, $\frac{\partial \ln L}{\partial \lambda} = 0$ and $\frac{\partial \ln L}{\partial \theta} = 0$ with respect to α, β, λ and θ. It is difficult to find closed form solutions for α, β, λ and θ. Therefore, a numerical technique such as the Newton-Raphson method can be adopted to obtain numerical estimates for these parameters. Let $\left(\widehat{\alpha}, \widehat{\beta}, \widehat{\lambda}, \widehat{\theta}\right)$ denote the MLEs of $(\alpha, \beta, \lambda, \theta)$. Approximate confidence intervals for the parameters can be based on asymptotic normality of the MLEs of α, β, λ and θ. Let $\mathbf{I}$ denote the observed information matrix of α, β, λ and θ. Explicit expressions for $\mathbf{I}$ can be obtained from the corresponding author. Let $\widehat{\mathbf{I}}$ denote $\mathbf{I}$ with $\left(\widehat{\alpha}, \widehat{\beta}, \widehat{\lambda}, \widehat{\theta}\right)$ replacing $(\alpha, \beta, \lambda, \theta)$. Furthermore, let $\mathbf{\Sigma} = \widehat{\mathbf{I}}^{-1}$. Then approximate

$100(1-\delta)\%$ confidence intervals for α, β, λ and θ are

$$\widehat{\alpha} \pm z_{\frac{\delta}{2}}\sqrt{\widehat{\Sigma}_{11}} \quad , \quad \widehat{\beta} \pm z_{\frac{\delta}{2}}\sqrt{\widehat{\Sigma}_{22}} \quad , \quad \widehat{\lambda} \pm z_{\frac{\delta}{2}}\sqrt{\widehat{\Sigma}_{33}} \quad , \quad \widehat{\theta} \pm z_{\frac{\delta}{2}}\sqrt{\widehat{\Sigma}_{44}},$$

where z_δ denotes the upper δth percentile of the standard normal distribution.

5.2 Numerical results

In this section, we perform a simulation study to investigate the performance of the MLEs of $(\alpha, \beta, \lambda, \theta)$. The following steps were performed for this purpose:

(1) Specify the sample size n and the values of the parameters α, β, λ and θ.

(2) Generate $u_i \sim Uniform(0,1)$, $i = 1, 2, \ldots, n$.

(3) Set

$$x_i = \left[-1 - \frac{1}{\lambda} - \frac{1}{\lambda} W\left(-(1+\lambda)\, e^{-1-\lambda}\, \left(1 - \left(\frac{C^{-1}\,(u_i C(\theta))}{\theta}\right)^{\frac{1}{\alpha}}\right)\right)\right]^{\frac{1}{\beta}}.$$

(4) Calculate the MLEs of all parameters.

(5) Repeat steps 2 and 3, N times.

(6) Calculate the bias and mean squared error (MSE) for each estimate.

The MSEs and biases of the estimates were computed by generating one thousand independent replications of samples of size $n = 20, 25, \ldots, 100$ from the EPLPS distribution with $(\alpha, \beta, \lambda, \theta) = (0.5, 1.2, 1.5, 0.8)$. Their plots are shown in Figure 4. Based on the simulation study, we observe that the MSE and bias for each parameter decrease when the sample size increases.

§6 Data analysis

To show the superiority of the EPLPS distribution, we use three real data sets. We compare the fit of the EPLPS distribution with some other known distributions, namely,

- The generalized Lindley (GL) distribution, introduced by Zakerzadeh and Dolati [37], with given pdf as

$$f_{GL}(x) = \frac{\lambda^2(\lambda x)^{\alpha-1}(\alpha+\gamma x)}{(\gamma+\lambda)\Gamma(\alpha+1)} e^{-\lambda x},\ x, \theta, \alpha, \gamma > 0.$$

- The new generalized Lindley (NGL) distribution, introduced by Nadarajah et al. [27], with given pdf as

$$f_{NGL}(x) = \frac{\alpha\lambda^2}{1+\lambda}(1+x)\, e^{-\lambda x}\left[1 - \left(1 + \frac{\lambda x}{\lambda+1}\right)\, e^{-\lambda x}\right]^{\alpha-1},\ x, \alpha, \lambda > 0.$$

- The two-parameter weighted Lindley (WL) distribution, introduced by Ghitany et al. [26], with given pdf as

$$f_{WL}(x) = \frac{\lambda^{c+1}}{(\lambda+c)\,\Gamma(c)} x^{c-1}(1+x)e^{-\lambda x},\ x, \alpha, \lambda > 0.$$

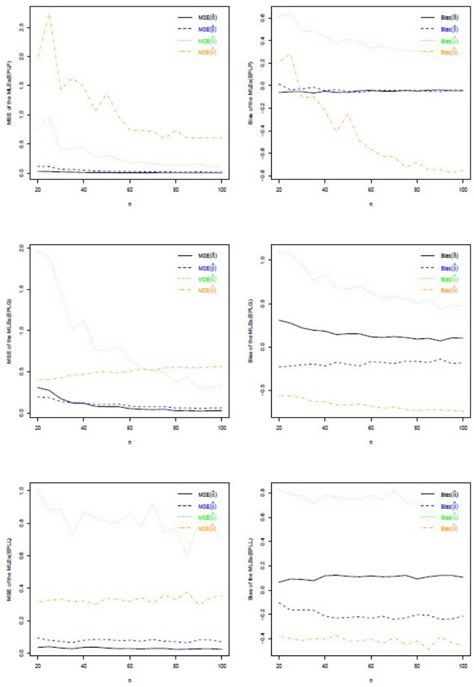

Figure 4. MSEs (left) and biases (right) of $\widehat{\alpha}$, $\widehat{\beta}$, $\widehat{\lambda}$ and $\widehat{\theta}$ for the EPLP (top), EPLG (middle) and EPLL (bottom) distributions with $(\alpha, \beta, \lambda, \theta) = (0.5, 1.2, 1.5, 0.8)$.

- The power Lindley (PL) distribution, introduced by Ghitany et al. [31], with given pdf as

$$f_{PL}(x) = \frac{\alpha\beta^2}{\beta+1}\left(1+x^{\alpha}\right)x^{\alpha-1}e^{-\beta x^{\alpha}},\ x, \alpha, \beta > 0.$$

- The exponentiated power Lindley (EPL) distribution, introduced by Ashour and Eltehiwy [33], with given pdf as

$$f_{EPL}(x) = \frac{\alpha\beta\lambda^2 x^{\beta-1}}{\lambda+1}\left(1+x^{\beta}\right)e^{-\lambda x^{\beta}}\left[1-\left(1+\frac{\lambda x^{\beta}}{\lambda+1}\right)\ e^{-\lambda x^{\beta}}\right]^{\alpha-1},\ x, \alpha, \beta, \lambda > 0.$$

- The Lindley-Poisson (LP) distribution, introduced by Gui et al. [38], with given pdf as

$$f_{LP}(x) = \frac{\lambda^2\theta(x+1)e^{\frac{\theta e^{-\lambda x}(\lambda+\lambda x+1)}{\lambda+1}-\lambda x}}{(\lambda+1)\left(e^{\theta}-1\right)},\ x, \theta, \lambda > 0.$$

- The Lindley-geometric (LG) distribution, introduced by Zakerzadeh and Mahmoudi [39], with given pdf as

$$f_{LG}(x) = \frac{\lambda^2}{\lambda+1}(1-\theta)(1+x)e^{-\lambda x}\left[1-\theta\left(1+\frac{\lambda x}{\lambda+1}\right)e^{-\lambda x}\right]^{-2},\ 0<\theta<1, x, \lambda > 0.$$

- The beta-generalized Lindley (BGL) distribution, introduced by Oluyede and Yang [30], with given pdf as

$$f_{BGL}(x) = \frac{\alpha\lambda^2}{B(a,b)\ (1+\lambda)}(1+x)\ e^{-\lambda x}\left[1-\left(1+\frac{\lambda x}{\lambda+1}\right)\ e^{-\lambda x}\right]^{a\alpha-1} \cdot\left\{1-\left[1-\left(1+\frac{\lambda x}{\lambda+1}\right)\ e^{-\lambda x}\right]^{\alpha}\right\}^{b-1},\ x,a,b,\alpha,\beta,\lambda>0.$$

- The beta-exponentiated power Lindley (BEPL) distribution, introduced by Pararai et al. [32], with given pdf as

$$f_{BEPL}(x) = \frac{\alpha\beta^2 w}{B(a,b)\ (1+\beta)}\ (1+x^{\alpha})\ x^{\alpha-1}\ e^{-\beta x^{\alpha}}\left[1-\left(1+\frac{\beta x^{\alpha}}{\beta+1}\right)\ e^{-\beta x^{\alpha}}\right]^{aw-1} \cdot\left\{1-\left[1-\left(1+\frac{\beta x^{\alpha}}{\beta+1}\right)\ e^{-\beta x^{\alpha}}\right]^{w}\right\}^{b-1},\ x,a,b,\alpha,\beta,w>0.$$

- The generalized linear failure rate-geometric (GLFRG) distribution, introduced by Nadarajah et al. [40], with pdf given by

$$f_{GLFRG}(x) = \frac{\alpha\ (aa+bb\ x)(1-\theta)e^{-aa\ x-1/2\ bb\ x^2}\left(1-e^{-aa\ x-1/2\ bb\ x^2}\right)^{\alpha-1}}{\left(1-\theta\ \left(1-\left(1-e^{-aa\ x-1/2\ bb\ x^2}\right)^{\alpha}\right)\right)^2},\ x,\alpha,\theta,a,b>0.$$

- The McLomax distribution, introduced by Lemonte and Cordeiro [41], with pdf given by

$$f_{McLomax}(x) = \frac{c\alpha\beta^{\alpha}\ (\beta+x)^{-(\alpha+1)}}{B\ (ac^{-1},\eta+1)}\left\{1-\left(\frac{\beta}{\beta+x}\right)^{\alpha}\right\}^{a-1}\left[1-\left\{1-\left(\frac{\beta}{\beta+x}\right)^{\alpha}\right\}^{c}\right]^{\eta},$$

where $x,\alpha,\beta,a,c>0$ and $0\le\eta$.

- The Weibull (W) distribution, introduced by Weibull [42], with given pdf as

$$f_W(x) = \alpha\gamma x^{\gamma-1}e^{-\alpha x^{\gamma}},\ x,\alpha,\gamma>0.$$

- The exponentiated Weibull (EW) distribution, introduced by Mudholkar and Srivastava [43] and Mudholkar [44], with pdf given by

$$f_{EW}(x) = \alpha\gamma\beta^{\gamma}x^{\gamma-1}e^{-(\beta x)^{\gamma}}\left[1-e^{-(\beta x)^{\gamma}}\right]^{\alpha-1},\ x,\alpha,\beta,\gamma>0.$$

- The modified Weibull (MW) distribution, introduced by Mudholkar et al. [45], with pdf given by

$$f_{MW}(x) = \alpha x^{\gamma-1}(\gamma+\lambda x)e^{\lambda x}e^{-\alpha x^{\gamma}e^{\lambda x}},\ x,\alpha,\beta,\gamma>0.$$

Let Lik denote the value of the likelihood function evaluated at the parameter estimates, n the number of observations, and p the number of estimated parameters. The estimates of the parameters of the fitted distribution, Akaike Information Criterion (AIC $=2p-2\ln(Lik)$) value, Bayesian Information Criterion (BIC $=p\ln(n)-2\ln(Lik)$) value and $-2\ln(Lik)$ are given in: Table 1 for the life time of 50 devices data; Table 2 for the glass fibres data; and Table 3 for guinea pigs data. For more discussion, values of Kolmogorov-Smirnov (KS) statistic, Anderson-Darling statistic (AD) and Cramer-von Mises statistic (CM) for the fitted distributions are given in Table 4. The smaller the values of these statistics the better the fit. For more details of these statistics, see Chen and Balakrishnan [46].

6.1 Data set 1 - Lifetime of 50 devices data

Consider the data set presented by Aarset [47]. The data describe lifetimes of 50 industrial devices put on life test at time zero. The data from Aarset [47] are presented as: 0.1, 0.2, 1, 1, 1, 1, 1, 2, 3, 6, 7, 11, 12, 18, 18, 18, 18, 18, 21, 32, 36, 40, 45, 46, 47, 50, 55, 60, 63, 63, 67, 67, 67, 67, 72, 75, 79, 82, 82, 83, 84, 84, 84, 85, 85, 85, 85, 85, 86, 86.

Table 1. Estimates of distributions fitted to the Aarset data.

	Parameter estimates						Model selection criteria		
Distribution	$\widehat{\alpha}$	$\widehat{\beta}$	$\widehat{\lambda}$	$\widehat{\theta}$	$\widehat{\gamma}$	$\widehat{c}$	AIC	BIC	$-2\ln(Lik)$
EPLP	0.118	1e-04	2.222	1.800			472.75	480.40	464.75
EPLG	0.074	0.001	2.225	0.846			468.62	476.27	460.62
EPLL	0.150	0.224	1.013	1			457.46	465.11	449.46
W	0.027				0.949		486.00	489.83	482.00
EW	0.148	0.011			5.210		463.59	469.33	457.59
MW	0.062		0.023		0.354		460.31	466.05	454.31
L			0.042				504.86	757.60	502.86
GL	0.528		0.026		0.053		479.92	485.65	473.92
NGL	0.454		0.027				481.98	485.81	477.98
WL			0.025			0.249	482.83	486.66	478.83
PL	0.663	0.161					488.17	492.00	484.17
EPL	0.173	1e-04	2.183				475.04	480.78	469.04
LP			0.016	4.617			514.62	518.44	510.62
LG			0.039	0.225			507.46	511.28	503.46

						Statistic		
Distribution	Estimates					AIC	BIC	$-2\ln(Lik)$
BGL	$\widehat{\alpha} = 1.001$	$\widehat{\lambda} =0.029$	$\widehat{a} =0.455$	$\widehat{b} =0.949$		486.04	493.69	478.04
BEPL	$\widehat{\alpha} =0.335$	$\widehat{\beta} =1.678$	$\widehat{a} =14.108$	$\widehat{b} =0.453$	$\widehat{w} =0.349$	516.86	526.42	506.86
GLFRG	$\widehat{\alpha} =0.801$	$\widehat{\theta} =0.406$	$\widehat{a} =0.006$	$\widehat{b} =0.0002$		480.71	488.36	472.71
McLomax	$\widehat{\alpha} =74.356$	$\widehat{\beta} =12234.085$	$\widehat{c} =0.744$	$\widehat{a} =12864.284$	$\widehat{\eta} =12.191$	467.04	476.60	457.04

According to the statistics in Tables 1 and 4, the EPLPS distribution fits better than the others for the Aarset data.

6.2 Data set 2 - Glass fibres data

The data set represents the strengths of 1.5 cm glass fibers, measured at the National Physical Laboratory, England. Unfortunately, the units of measurement are not known. It was analyzed by Barreto-Souza et al. [48]. The data are: 0.55, 0.93, 1.25, 1.36, 1.49, 1.52, 1.58, 1.61, 1.64, 1.68, 1.73, 1.81, 2.00, 0.74, 1.04, 1.27, 1.39, 1.49, 1.53, 1.59, 1.61, 1.66, 1.68, 1.76, 1.82, 2.01, 0.77, 1.11, 1.28, 1.42, 1.50, 1.54, 1.60, 1.62, 1.66, 1.69, 1.76, 1.84, 2.24, 0.81, 1.13, 1.29, 1.48, 1.50, 1.55, 1.61, 1.62, 1.66, 1.70, 1.77, 1.84, 0.84, 1.24, 1.30, 1.48, 1.51, 1.55, 1.61, 1.63, 1.67, 1.70, 1.78, 1.89.

The statistics in Tables 2 and 4 are smaller for the EPLPS distribution and so the EPLPS distribution gives better fit than other distributions for the glass fibres data.

6.3 Data set 3 - Guinea pigs data

The third data set represents the survival times (in days) of 72 guinea pigs infected with virulent tubercle bacilli, observed and reported by Bjerkedal et al. [49]: 10, 33, 44, 56, 59, 72,

Table 2. Estimates of distributions fitted to glass fibres data.

	Parameter estimates						Model selection criteria		
Distribution	$\widehat{\alpha}$	$\widehat{\beta}$	$\widehat{\lambda}$	$\widehat{\theta}$	$\widehat{\gamma}$	$\widehat{c}$	AIC	BIC	$-2\ln(Lik)$
EPLP	0.563	0.250	4.520	2.508			33.62	42.19	25.62
EPLG	0.699	0.917	3.087	0.942			31.88	40.45	23.88
EPLL	0.784	0.349	4.265	0.909			34.54	43.11	26.54
W	0.059				5.780		34.41	38.70	30.41
EW	0.671	0.582			7.284		35.35	41.78	29.35
MW	0.008		2.160		2.402		34.71	41.14	28.71
L			0.996				164.56	172.58	162.56
GL	17.439		11.573		0.004		53.90	60.33	47.90
NGL	26.171		2.990				65.24	69.53	61.24
WL			11.738			17.095	51.78	56.06	47.78
PL	4.458	0.222					33.38	37.67	29.38
EPL	0.567	0.079	5.843				33.71	40.14	27.71
LP			0.023	709.759			141.96	146.25	137.96
LG			0.951	3e-05			166.79	171.07	162.79

						Statistic		
Distribution	Estimates					AIC	BIC	$-2\ln(Lik)$
BGL	$\widehat{\alpha}=1.173$	$\widehat{\lambda}=0.899$	$\widehat{a}=11.541$	$\widehat{b}=11.668$		54.56	63.13	46.56
BEPL	$\widehat{\alpha}=0.475$	$\widehat{\beta}=2.735$	$\widehat{a}=9.210$	$\widehat{b}=8.994$	$\widehat{w}=8.653$	64.76	75.48	54.76
GLFRG	$\widehat{\alpha}=6.001$	$\widehat{\theta}=0.852$	$\widehat{a}=0.149$	$\widehat{b}=0.888$		64.14	72.71	56.14
McLomax	$\widehat{\alpha}=133.130$	$\widehat{\beta}=196.294$	$\widehat{c}=7.453$	$\widehat{a}=463.679$	$\widehat{\eta}=18.15068$	40.10	50.81	30.10

74, 77, 92, 93, 96, 100, 100, 102, 105, 107, 107, 108, 108, 108, 109, 112, 113, 115, 116, 120, 121, 122, 122, 124, 130, 134, 136, 139, 144, 146, 153, 159, 160, 163, 163, 168, 171, 172, 176, 183, 195, 196, 197, 202, 213, 215, 216, 222, 230, 231, 240, 245, 251, 253, 254, 254, 278, 293, 327, 342, 347, 361, 402, 432, 458, 555.

Table 3. Estimates of distributions fitted to guinea pigs data.

	Parameter estimates						Model selection criteria		
Distribution	$\widehat{\alpha}$	$\widehat{\beta}$	$\widehat{\lambda}$	$\widehat{\theta}$	$\widehat{\gamma}$	$\widehat{c}$	AIC	BIC	$-2\ln(Lik)$
EPLP	2.090	0.180	0.627	5.325			856.75	865.86	848.75
EPLG	6.738	4.343	0.231	0.999			857.25	866.35	849.25
EPLL	2.218	0.029	0.887	1e-05			859.53	868.63	851.53
W	6e-05				1.825		858.72	863.28	854.72
EW	2.653	0.008			1.160		857.31	864.14	851.31
MW	6e-05		4e-05		1.825		860.72	867.55	854.72
L			0.011				860.55	877.55	858.55
GL	2.084		0.017		44.469		857.60	864.43	851.60
NGL	1.695		0.014				855.32	859.88	851.32
WL			0.017			2.10518	855.55	860.10	851.55
PL	1.260	0.002					855.99	860.55	851.99
EPL	1.654	0.013	1.011				857.32	864.15	851.32
LP			0.001	21.030			856.98	861.54	852.98
LG			0.011	1e-04			862.55	867.11	858.55

						Statistic		
Distribution	Estimates					AIC	BIC	$-2\ln(Lik)$
BGL	$\widehat{\alpha}=0.814$	$\widehat{\lambda}=0.001$	$\widehat{a}=1.463$	$\widehat{b}=18.607$		860.18	869.29	852.18
BEPL	$\widehat{\alpha}=0.744$	$\widehat{\beta}=0.046$	$\widehat{a}=1.328$	$\widehat{b}=2.112$	$\widehat{w}=1.98428$	861.49	872.87	851.49
GLFRG	$\widehat{\alpha}=3.149$	$\widehat{\theta}=5e\text{-}04$	$\widehat{a}=0.009$	$\widehat{b}=1e\text{-}05$		859.52	868.63	851.52
McLomax	$\widehat{\alpha}=0.084$	$\widehat{\beta}=243.567$	$\widehat{c}=2.979$	$\widehat{a}=661.497$	$\widehat{\eta}=1.985$	861.08	872.46	851.08

The statistics in Tables 3 and 4 are smaller for the EPLPS distribution and indicate that the EPLPS distribution gives better fit than other distributions for the guinea pigs data.

Table 4. Goodness of fit tests.

Aarset data				Glass fibres data				Guinea pigs data			
Distribution	AD	CM	K-S	Distribution	AD	CM	K-S	Distribution	AD	CM	K-S
EPLP	2.397	0.365	0.170	EPLP	0.625	0.102	0.110	EPLP	0.380	0.063	0.075
EPLG	1.939	0.270	0.132	EPLG	0.484	0.074	0.096	EPLG	0.396	0.066	0.076
EPLL	2.373	0.322	0.153	EPLL	0.772	0.142	0.131	EPLL	0.514	0.079	0.092
W	3.544	0.534	0.192	W	1.256	0.216	0.152	W	1.018	0.169	0.104
EW	3.180	0.453	0.185	EW	1.100	0.199	0.146	EW	0.544	0.086	0.089
MW	1.833	0.266	0.133	MW	0.956	0.171	0.135	MW	1.018	0.169	0.104
L	7.901	0.678	0.199	L	16.447	3.358	0.386	L	1.926	0.317	0.170
GL	2.577	0.433	0.179	GL	3.125	0.570	0.216	GL	0.585	0.094	0.090
NGL	3.080	0.521	0.193	NGL	4.257	0.778	0.226	NGL	0.534	0.084	0.089
WL	2.796	0.455	0.184	WL	3.115	0.568	0.216	WL	0.585	0.094	0.090
PL	3.446	0.539	0.195	PL	1.132	0.191	0.144	PL	0.691	0.114	0.093
EPL	3.082	0.507	0.198	EPL	0.858	0.151	0.131	EPL	0.538	0.085	0.089
LP	8.523	0.663	0.186	LP	15.396	3.150	0.382	LP	0.866	0.135	0.115
LG	7.730	0.695	0.206	LG	15.510	3.115	0.364	LG	1.926	0.317	0.170
BGL	3.092	0.524	0.194	BGL	45.961	0.545	0.808	BGL	0.723	0.120	0.091
BEPL	5.278	1.015	0.250	BEPL	3.675	0.663	0.227	BEPL	0.497	0.076	0.089
GRFRG	3.243	0.473	0.187	GRFRG	3.410	0.532	0.182	GRFRG	0.576	0.092	0.093
McLomax	3.970	0.678	476.602	McLomax	5.129	0.241	50.816	McLomax	6.542	0.082	872.462

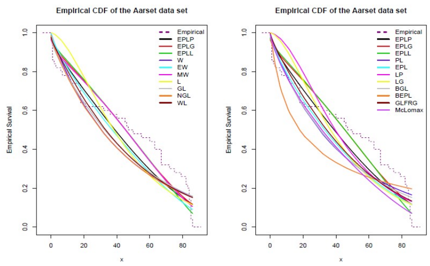

Figure 5. Estimated survival functions and the empirical survival function for the Aarset data.

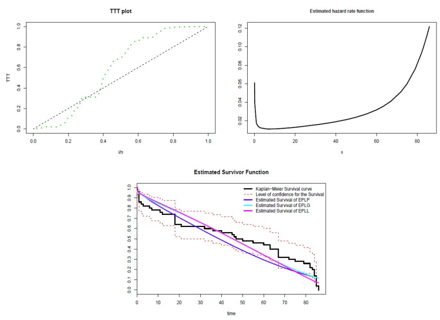

Figure 6. Empirical TTT-plot (top left), estimated hazard rate functions (top right) and estimated survival functions (bottom) of the fitted distributions to lifetime devices data.

Figures 5 and 6 show the estimated survival functions, TTT plot and the Kaplan-Meier curve for the fitted distributions to the Aarset data. The figures show that the EPLPS distribution fits the data better than the other distributions.

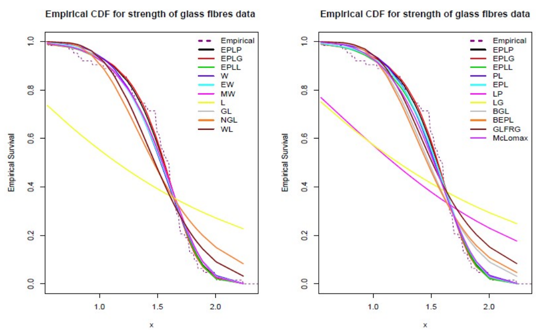

Figure 7. Estimated survival functions and the empirical survival function for the glass fibres data.

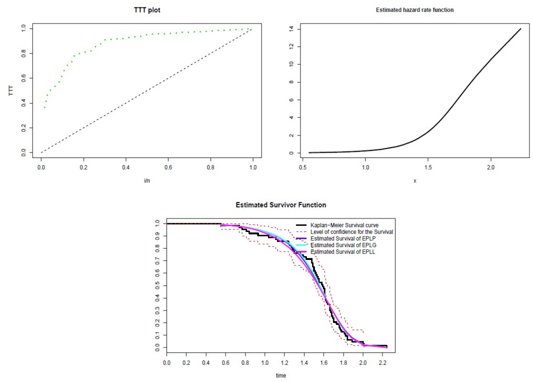

Figure 8. Empirical TTT plot (top left), estimated hazard rate functions (top right) and estimated survival functions (bottom) of the fitted distributions for the glass fibres data.

Figures 7 and 8 show the estimated survival functions, TTT plot and the Kaplan-Meier curve for the fitted distributions to the glass fibres data. The figures show that the EPLPS distribution fits better than the other distributions.

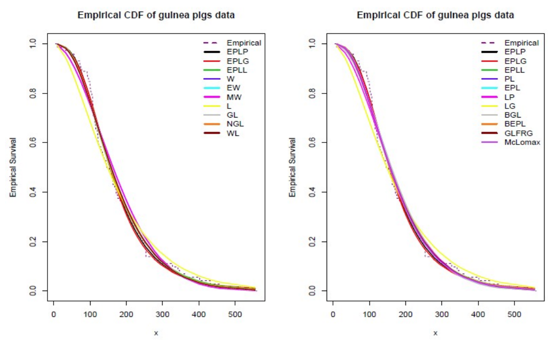

Figure 9. Estimated survival functions and the empirical survival function for the guinea pigs data.

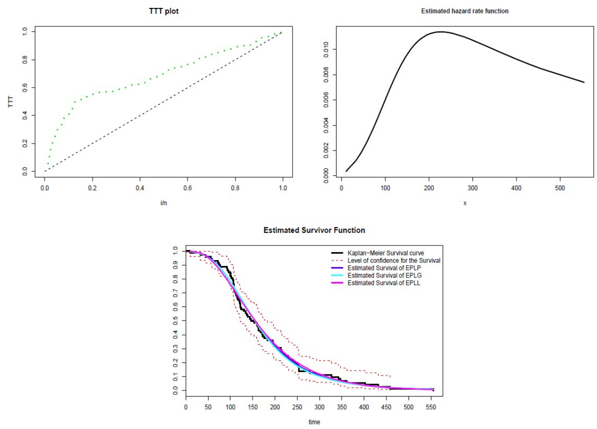

Figure 10. Empirical TTT-plot (top left), estimated hazard rate functions (top right) and estimated survival functions (bottom) of the fitted distributions to the guinea pigs data.

Figures 9 and 10 show the estimated survival functions, TTT plot and the Kaplan-Meier curve for the fitted distributions to the guinea pigs data. The figures show that the EPLPS distribution fits better than the other distributions.

§7 Conclusions

We have introduced a lifetime distribution called the exponentiated power Lindley power series (EPLPS) distribution. This distribution was obtained by mixing the exponentiated power Lindley and power series distributions. The EPLPS distribution is flexible in modelling various types of failure data with bathtub shaped hazard rate functions. It is more flexible than Weibull, exponentiated Weibull, modified Weibull, Lindley, generalized Lindley, new generalized Lindley, weighted Lindley, power Lindley, exponentiated power Lindley, Lindley Poisson, Lindley geometric, beta generalized Lindley, beta exponentiated power Lindley, generalized linear failure rate geometric and McLomax distributions.

We have derived various statistical properties of the EPLPS distribution. Finally, a simulation study was performed and three real data sets were analyzed to show the potential of the newly proposed distribution. We hope that the proposed distribution may attract wider applications in survival analysis and reliability studies.

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Declarations

Conflict of interest The authors declare no conflict of interest.

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