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On approximation of Bernstein-Stancu operators in movable interval

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Abstract. In the present paper, we obtain the converse results of approximation of a newly introduced genuine Bernstein-Durrmeyer operators in movable interval. We also get the moments properties of an auxiliary operator which has its own independent values. The moments of the auxiliary operators play important roles in establishing the main result (Theorem 4).

§1 Introduction

For any given $f \in C_{[0,1]}$, define

$$U_n(f,x) := f(0)p_{n,0}(x) + f(1)p_{n,n}(x) + (n-1)\sum_{k=1}^{n-1} p_{n,k}(x)\int_0^1 p_{n-2,k-1}(t)f(t)dt, \qquad (1.1)$$

where $p_{n,k}(x) = \binom{n}{k}x^k(1-x)^{n-k}, k = 0, 1, \cdots, n$, are the Bernstein fundamental polynomials. As we know, $U_n(f,x)$ are the so-called genuine Bernstein-Durrmeyer operators, which first appeared in the papers of Goodman and Sharma [15] and Chen [7]. The operators $U_n(f)$ are limits of the Bernstein-Durrmeyer operators with Jacobi weights. One of the advantages of $U_n(f,x)$ compared to the usual Bernstein-Durrmeyer operators is that $U_n(f,x)$ can reproduce the linear functions. Lots of authors have done many excellent works on the degree of approximation, Voronovskaja's asymptotic estimate, the eigenstructure of the $U_n(f,x)$ and its generalizations (see [2]-[4], [11]-[14], [16]-[25], [29], [30]).

Recently, we [27] introduced a new type of genuine Bernstein-Durrmeyer operators in movable interval $A_n := \left[\frac{\alpha}{n+\beta}, \frac{n+\alpha}{n+\beta}\right]$. Our new operators are defined as follows:

$$U_n^{(\alpha,\beta)}(f,x) := \left(\frac{n+\beta}{n}\right)^n f\left(\frac{\alpha}{n+\beta}\right) q_{n,0}(x) + \left(\frac{n+\beta}{n}\right)^n f\left(\frac{n+\alpha}{n+\beta}\right) q_{n,n}(x)$$

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$$+\left(\frac{n+\beta}{n}\right)^n \lambda_{n-2}^{-1}\sum_{k=1}^{n-1} q_{n,k}(x)\int_{A_{n-2}} q_{n-2,k-1}(t) f\left(\frac{n(n-2+\beta)}{(n-2)(n+\beta)}\left(t-\frac{2\alpha}{n(n-2+\beta)}\right)\right)dt, \tag{1.2}$$

where $0 \le \alpha \le \beta$ are fixed numbers,

$$\lambda_{n-2} := \left(\frac{n-2}{n-2+\beta}\right)^{n-1}\frac{1}{n-1},$$

$$q_{n,k}(x) := \binom{n}{k}\left(x-\frac{\alpha}{n+\beta}\right)^k\left(\frac{n+\alpha}{n+\beta}-x\right)^{n-k}, \ k=0,1,\cdots,n.$$

For convenience, write $\bar{x} := x - \frac{\alpha}{n+\beta}$. Then, $q_{n,k}(x)$ can be rewritten as follows:

$$q_{n,k}(x) = \binom{n}{k}\bar{x}^k\left(\frac{n}{n+\beta}-\bar{x}\right)^{n-k}, \ k=0,1,\cdots,n.$$

When $t \in A_{n-2}$, we have

$$\frac{n(n-2+\beta)}{(n-2)(n+\beta)}\left(t-\frac{2\alpha}{n(n-2+\beta)}\right) \in A_n.$$

So, $U_n^{(\alpha,\beta)}(f,x)$ is well defined in A_n. We obtained the approximation rate for continuous functions and Voronovskaja's asymptotic estimate of the new operators in [27].

For the approximation rate of $U_n^{(\alpha,\beta)}(f,x)$ for $f \in C(A_n)$, we have

Theorem 1. Let $0 \le \lambda \le 1$ be a fixed number. For any $f \in C(A_n)$, there is a positive constant only depending on λ, α and β such that

$$\left|U_n^{(\alpha,\beta)}(f,x) - f(x)\right| \le C\omega^2_{\varphi_n^\lambda}\left(f, \frac{\varphi_n^{1-\lambda}(x)}{\sqrt{n}}\right), \tag{1.3}$$

where $\varphi_n(x) := \sqrt{\bar{x}\left(\frac{n}{n+\beta}-\bar{x}\right)}$.

We also have the following Voronovskaja's asymptotic estimate of $U_n^{(\alpha,\beta)}(f,x)$:

Theorem 2. Let $f \in C(A_n)$. If f'' exists at a point $x \in A_n$, then

$$\lim_{n\to\infty} n\left(U_n^{(\alpha,\beta)}(f,x) - f(x)\right) = \frac{\varphi_n^2(x)}{2}f''(x).$$

The main purpose of the present paper is to establish the converse result of approximation by $U_n^{(\alpha,\beta)}(f,x)$ (see Theorem 4, section 3). To prove the main result, we need the moments properties of the following auxiliarly operators:

$$S_n^{(\alpha,\beta)}(f,x) := \left(\frac{n+\beta}{n}\right)^n\sum_{k=0}^n f\left(\frac{k+\alpha}{n+\beta}\right)q_{n,k}(x), \tag{1.4}$$

where $0 \le \alpha \le \beta$ are fixed numbers. $S_n^{(\alpha,\beta)}(f,x)$ is a special case of the following generalized Bernstein-Stancu operators with four parameters which were introduced by Gadjiev and Ghorbanalizadeh [10]:

$$B_{n,\alpha,\beta}(f;x) = \left(\frac{n+\beta_2}{n}\right)^n\sum_{k=0}^n f\left(\frac{k+\alpha_1}{n+\beta_1}\right)Q_{n,k}(x), \tag{1.5}$$

where $x \in \left[\frac{\alpha_2}{n+\beta_2}, \frac{n+\alpha_2}{n+\beta_2}\right]$, and $Q_{n,k}(x) = \binom{n}{k}\left(x-\frac{\alpha_2}{n+\beta_2}\right)^k\left(\frac{n+\alpha_2}{n+\beta_2}-x\right)^{n-k}$, $k=0,1,\cdots,n$ with $\alpha_k, \beta_k, k=1,2$ are positive real numbers satisfying $0 \le \alpha_1 \le \beta_1$, $0 \le \alpha_2 \le \beta_2$. When $\alpha_2 = \beta_2 = 0$, $B_{n,\alpha,\beta}(f;x)$ reduces to the well known Bernstein-Stancu operators, when $\alpha_1 = \alpha_2 = 0$,

$\beta_1 = \beta_2 = 0$, $B_{n,\alpha,\beta}(f;x)$ reduces to the classical Bernstein operators. Many authors have investigated the approximation properties of $B_{n,\alpha,\beta}(f;x)$ and its generalizations (see [1], [5], [9], [17], [21], [26], [28]).

§2 Some properties of the operators $S_n^{(\alpha,\beta)}(f;x)$

In this section, we give the moments properties and the approximation rate of $S_n^{(\alpha,\beta)}(f;x)$ for $f \in C(A_n)$.

Lemma 1. It holds that

$$\sum_{k=1}^{n} \frac{k}{n} q_{n,k}(x) = \left(\frac{n}{n+\beta}\right)^{n-1} \bar{x}; \tag{2.1}$$

$$\sum_{k=1}^{n} \frac{k^2}{n^2} q_{n,k}(x) = \frac{n-1}{n}\left(\frac{n}{n+\beta}\right)^{n-2} \bar{x}^2 + \frac{1}{n}\left(\frac{n}{n+\beta}\right)^{n-1} \bar{x}; \tag{2.2}$$

$$\begin{aligned}\sum_{k=1}^{n} \frac{k^3}{n^3} q_{n,k}(x) &= \frac{(n-1)(n-2)}{n^2}\left(\frac{n}{n+\beta}\right)^{n-3} \bar{x}^3 \\ &+ \frac{3(n-1)}{n^2}\left(\frac{n}{n+\beta}\right)^{n-2} \bar{x}^2 + \frac{1}{n^2}\left(\frac{n}{n+\beta}\right)^{n-1} \bar{x};\end{aligned} \tag{2.3}$$

$$\begin{aligned}\sum_{k=1}^{n} \frac{k^4}{n^4} q_{n,k}(x) = {}& \frac{(n-1)(n-2)(n-3)}{n^3}\left(\frac{n}{n+\beta}\right)^{n-4} \bar{x}^4 + \frac{6(n-1)(n-2)}{n^3}\left(\frac{n}{n+\beta}\right)^{n-3} \bar{x}^3 \\ &+ \frac{7(n-1)}{n^3}\left(\frac{n}{n+\beta}\right)^{n-2} \bar{x}^2 + \frac{1}{n^3}\left(\frac{n}{n+\beta}\right)^{n-1} \bar{x}.\end{aligned} \tag{2.4}$$

Proof. Direct calculations yield that

$$\begin{aligned}\sum_{k=1}^{n} \frac{k}{n} q_{nk}(x) &= \sum_{k=1}^{n} \binom{n-1}{k-1} \bar{x}^k \left(\frac{n}{n+\beta} - \bar{x}\right)^{n-k} \\ &= \bar{x} \sum_{k=0}^{n-1} \binom{n-1}{k} \bar{x}^k \left(\frac{n}{n+\beta} - \bar{x}\right)^{n-1-k} = \left(\frac{n}{n+\beta}\right)^{n-1} \bar{x},\end{aligned}$$

and

$$\begin{aligned}&\sum_{k=1}^{n} \frac{k^2}{n^2} q_{nk}(x) \\ = {}& \sum_{k=1}^{n} \frac{k}{n} \binom{n-1}{k-1} \bar{x}^k \left(\frac{n}{n+\beta} - \bar{x}\right)^{n-k} \\ = {}& \frac{n-1}{n} \sum_{k=1}^{n} \frac{k-1}{n-1} \binom{n-1}{k-1} \bar{x}^k \left(\frac{n}{n+\beta} - \bar{x}\right)^{n-k} + \frac{1}{n} \sum_{k=1}^{n} \binom{n-1}{k-1} \bar{x}^k \left(\frac{n}{n+\beta} - \bar{x}\right)^{n-k} \\ = {}& \frac{n-1}{n} \bar{x}^2 \sum_{k=0}^{n-2} \binom{n-2}{k} \bar{x}^{k-2} \left(\frac{n}{n+\beta} - \bar{x}\right)^{n-k} + \frac{\bar{x}}{n} \sum_{k=0}^{n-1} \binom{n-1}{k} \bar{x}^k \left(\frac{n}{n+\beta} - \bar{x}\right)^{n-k}\end{aligned}$$

$$= \frac{n-1}{n}\left(\frac{n}{n+\beta}\right)^{n-2}\bar{x}^2+\frac{1}{n}\left(\frac{n}{n+\beta}\right)^{n-1}\bar{x}.$$

Similarly,

$$\sum_{k=1}^{n}\frac{k^3}{n^3}q_{nk}(x)=\frac{(n-1)(n-2)}{n^2}\bar{x}^3\sum_{k=0}^{n-3}\binom{n-3}{k}\bar{x}^{k-3}\left(\frac{n}{n+\beta}-\bar{x}\right)^{n-k}$$
$$+\frac{3(n-1)}{n^2}\bar{x}^2\sum_{k=0}^{n-2}\binom{n-2}{k}\bar{x}^{k-2}\left(\frac{n}{n+\beta}-\bar{x}\right)^{n-k}+\frac{\bar{x}}{n^2}\sum_{k=0}^{n-1}\binom{n-1}{k}\bar{x}^{k}\left(\frac{n}{n+\beta}-\bar{x}\right)^{n-k}$$
$$= \frac{(n-1)(n-2)}{n^2}\left(\frac{n}{n+\beta}\right)^{n-3}\bar{x}^3+\frac{3(n-1)}{n^2}\left(\frac{n}{n+\beta}\right)^{n-2}\bar{x}^2+\frac{1}{n^2}\left(\frac{n}{n+\beta}\right)^{n-1}\bar{x},$$

and

$$\sum_{k=1}^{n}\frac{k^4}{n^4}q_{nk}(x)=\frac{(n-1)(n-2)(n-3)}{n^3}\bar{x}^4\sum_{k=0}^{n-4}\binom{n-4}{k}\bar{x}^{k-4}\left(\frac{n}{n+\beta}-\bar{x}\right)^{n-k}$$
$$+\frac{6(n-1)(n-2)}{n^3}\bar{x}^3\sum_{k=0}^{n-3}\binom{n-3}{k}\bar{x}^{k-3}\left(\frac{n}{n+\beta}-\bar{x}\right)^{n-k}$$
$$+\frac{7(n-1)}{n^3}\bar{x}^2\sum_{k=0}^{n-2}\binom{n-2}{k}\bar{x}^{k-2}\left(\frac{n}{n+\beta}-\bar{x}\right)^{n-k}+\frac{\bar{x}}{n^3}\sum_{k=0}^{n-1}\binom{n-1}{k}\bar{x}^{k}\left(\frac{n}{n+\beta}-\bar{x}\right)^{n-k}$$
$$= \frac{(n-1)(n-2)(n-3)}{n^3}\left(\frac{n}{n+\beta}\right)^{n-4}\bar{x}^4+\frac{6(n-1)(n-2)}{n^3}\left(\frac{n}{n+\beta}\right)^{n-3}\bar{x}^3$$
$$+\frac{7(n-1)}{n^3}\left(\frac{n}{n+\beta}\right)^{n-2}\bar{x}^2+\frac{1}{n^3}\left(\frac{n}{n+\beta}\right)^{n-1}\bar{x}.$$

□

Lemma 2. It holds that

(i) $S_n^{(\alpha,\beta)}(1,x)=1$;

(ii) $S_n^{(\alpha,\beta)}(t,x)=x$;

(iii) $S_n^{(\alpha,\beta)}(t^2,x)=\frac{n-1}{n}\bar{x}^2+\frac{2\alpha+1}{n+\beta}\bar{x}+\left(\frac{\alpha}{n+\beta}\right)^2$;

(iv)

$$S_n^{(\alpha,\beta)}(t^3,x)=\frac{(n-1)(n-2)}{n^2}\bar{x}^3+\frac{3(n-1)(1+\alpha)}{n(n+\beta)}\bar{x}^2+\frac{1+3\alpha+3\alpha^2}{(n+\beta)^2}\bar{x}+\left(\frac{\alpha}{n+\beta}\right)^3;$$

(v)

$$S_n^{(\alpha,\beta)}(t^4,x)=\frac{(n-1)(n-2)(n-3)}{n^3}\bar{x}^4+\frac{(n-1)(n-2)\left(6+4\alpha\right)}{n^2(n+\beta)}\bar{x}^3$$
$$+\frac{(n-1)\left(7+12\alpha+6\alpha^2\right)}{n(n+\beta)^2}\bar{x}^2+\frac{1+4\alpha+6\alpha^2+4\alpha^3}{(n+\beta)^3}\bar{x}+\left(\frac{\alpha}{n+\beta}\right)^4.$$

Proof. (i) is obvious. It follows from (2.1) that

$$S_n^{(\alpha,\beta)}(t,x) = \left(\frac{n+\beta}{n}\right)^{n-1}\sum_{k=0}^{n}\frac{k+\alpha}{n}q_{n,k}(x)$$

$$= \left(\frac{n+\beta}{n}\right)^{n-1}\left(\left(\frac{n}{n+\beta}\right)^{n-1}\bar{x}+\frac{\alpha}{n}\left(\frac{n}{n+\beta}\right)^{n}\right)$$
$$= \bar{x}+\frac{\alpha}{n+\beta}=x,$$

which implies (ii).

By (2.1) and (2.2), we have

$$\begin{aligned}
S_n^{(\alpha,\beta)}(t^2,x) &= \left(\frac{n+\beta}{n}\right)^{n-2}\sum_{k=0}^{n}\left(\frac{k+\alpha}{n}\right)^2 q_{n,k}(x)\\
&= \left(\frac{n+\beta}{n}\right)^{n-2}\left(\frac{n-1}{n}\left(\frac{n}{n+\beta}\right)^{n-2}\bar{x}^2+\frac{1}{n}\left(\frac{n}{n+\beta}\right)^{n-1}\bar{x}\right.\\
&\left.+2\frac{\alpha}{n}\left(\frac{n}{n+\beta}\right)^{n-1}\bar{x}+\frac{\alpha^2}{n^2}\left(\frac{n}{n+\beta}\right)^{n}\right)\\
&= \frac{n-1}{n}\bar{x}^2+\frac{2\alpha+1}{n+\beta}\bar{x}+\frac{\alpha^2}{(n+\beta)^2}.
\end{aligned}$$

Thus, (iii) is proved.

By (2.1)-(2.3) , we have

$$\begin{aligned}
S_n^{(\alpha,\beta)}(t^3,x) &= \left(\frac{n+\beta}{n}\right)^{n-3}\sum_{k=0}^{n}\left(\frac{k+\alpha}{n}\right)^3 q_{n,k}(x)\\
&= \left(\frac{n+\beta}{n}\right)^{n-3}\left(\frac{(n-1)(n-2)}{n^2}\left(\frac{n}{n+\beta}\right)^{n-3}\bar{x}^3+\frac{3(n-1)}{n^2}\left(\frac{n}{n+\beta}\right)^{n-2}\bar{x}^2\right.\\
&+\frac{1}{n^2}\left(\frac{n}{n+\beta}\right)^{n-1}\bar{x}+3\cdot\frac{\alpha}{n}\left(\frac{n-1}{n}\left(\frac{n}{n+\beta}\right)^{n-2}\bar{x}^2+\frac{1}{n}\left(\frac{n}{n+\beta}\right)^{n-1}\bar{x}\right)\\
&\left.+3\cdot\frac{\alpha^2}{n^2}\left(\frac{n}{n+\beta}\right)^{n-1}\bar{x}+\frac{\alpha^3}{n^3}\left(\frac{n}{n+\beta}\right)^{n}\right)\\
&= \frac{(n-1)(n-2)}{n^2}\bar{x}^3+\frac{3(n-1)(1+\alpha)}{n(n+\beta)}\bar{x}^2+\frac{1+3\alpha+3\alpha^2}{(n+\beta)^2}\bar{x}+\left(\frac{\alpha}{n+\beta}\right)^3.
\end{aligned}$$

By (2.1)-(2.4) , we have

$$\begin{aligned}
&S_n^{(\alpha,\beta)}(t^4,x)\\
&= \left(\frac{n+\beta}{n}\right)^{n-4}\sum_{k=0}^{n}\left(\frac{k+\alpha}{n}\right)^4 q_{n,k}(x)\\
&= \left(\frac{n+\beta}{n}\right)^{n-4}\left(\frac{(n-1)(n-2)(n-3)}{n^3}\left(\frac{n}{n+\beta}\right)^{n-4}\bar{x}^4+\frac{6(n-1)(n-2)}{n^3}\right.\\
&\left(\frac{n}{n+\beta}\right)^{n-3}\bar{x}^3+\frac{7(n-1)}{n^3}\left(\frac{n}{n+\beta}\right)^{n-2}\bar{x}^2+\frac{1}{n^3}\left(\frac{n}{n+\beta}\right)^{n-1}\bar{x}\\
&+4\frac{\alpha}{n}\left(\frac{(n-1)(n-2)}{n^2}\left(\frac{n}{n+\beta}\right)^{n-3}\bar{x}^3+\frac{3(n-1)}{n^2}\left(\frac{n}{n+\beta}\right)^{n-2}\bar{x}^2\right.
\end{aligned}$$

$$+\frac{1}{n^2}\left(\frac{n}{n+\beta}\right)^{n-1}\bar{x}\right)+6\left(\frac{\alpha}{n}\right)^2\left(\frac{n-1}{n}\left(\frac{n}{n+\beta}\right)^{n-2}\bar{x}^2+\frac{1}{n}\left(\frac{n}{n+\beta}\right)^{n-1}\bar{x}\right)$$

$$+4\left(\frac{\alpha}{n}\right)^3\left(\frac{n}{n+\beta}\right)^{n-1}\bar{x}+\left(\frac{\alpha}{n}\right)^4\left(\frac{n}{n+\beta}\right)^{n}\right)$$

$$= \frac{(n-1)(n-2)(n-3)}{n^3}\bar{x}^4+\frac{(n-1)(n-2)\left(6+4\alpha\right)}{n^2(n+\beta)}\bar{x}^3$$

$$+\frac{(n-1)\left(7+12\alpha+6\alpha^2\right)}{n(n+\beta)^2}\bar{x}^2+\frac{1+4\alpha+6\alpha^2+4\alpha^3}{(n+\beta)^3}\bar{x}+\left(\frac{\alpha}{n+\beta}\right)^4.$$

□

Lemma 3. It holds that

$$S_n^{(\alpha,\beta)}(t-x,x)=0; \tag{2.5}$$

$$S_n^{(\alpha,\beta)}((t-x)^2,x)=\frac{\varphi_n^2(x)}{n}; \tag{2.6}$$

$$S_n^{(\alpha,\beta)}((t-x)^3,x)=\frac{\varphi_n^2(x)\left(\frac{n}{n+\beta}-2\bar{x}\right)}{n^2}; \tag{2.7}$$

$$S_n^{(\alpha,\beta)}((t-x)^4,x)=\frac{\left(\frac{n}{n+\beta}\right)^2\varphi_n^2(x)+3(n-2)\varphi_n^4(x)}{n^3}, \tag{2.8}$$

where $\varphi_n(x):=\sqrt{\bar{x}\left(\frac{n}{n+\beta}-\bar{x}\right)}$ defined as in Theorem 1.

Proof. It is obvious that (2.5) is valid by (ii) of Lemma 2.

By using (i)-(iii) of Lemma 2, we have

$$\begin{aligned}
S_n^{(\alpha,\beta)}((t-x)^2,x) &= S_n^{(\alpha,\beta)}(t^2,x)-2xS_n^{(\alpha,\beta)}(t,x)+x^2\\
&= \frac{n-1}{n}\bar{x}^2+\frac{2\alpha+1}{n+\beta}\bar{x}+\left(\frac{\alpha}{n+\beta}\right)^2-2x^2+x^2\\
&= \left(\bar{x}^2+2\frac{\alpha}{n+\beta}\bar{x}+\left(\frac{\alpha}{n+\beta}\right)^2\right)+\frac{1}{n}\bar{x}\left(\frac{n}{n+\beta}-\bar{x}\right)-x^2\\
&= \frac{1}{n}\bar{x}\left(\frac{n}{n+\beta}-\bar{x}\right)=\frac{\varphi_n^2(x)}{n},
\end{aligned}$$

which proves (2.6).

Similarly, by (i)-(iv) of lemma 2, we can prove (2.7) by the following procedure:

$$\begin{aligned}
&S_n^{(\alpha,\beta)}((t-x)^3,x)\\
=\;& S_n^{(\alpha,\beta)}(t^3,x)-3xS_n^{(\alpha,\beta)}(t^2,x)+3x^2S_n^{(\alpha,\beta)}(t,x)-x^3\\
=\;& \frac{(n-1)(n-2)}{n^2}\bar{x}^3+\frac{3(n-1)(1+\alpha)}{n(n+\beta)}\bar{x}^2+\frac{1+3\alpha+3\alpha^2}{(n+\beta)^2}\bar{x}+\left(\frac{\alpha}{n+\beta}\right)^3\\
&-3x\left(\frac{n-1}{n}\bar{x}^2+\frac{2\alpha+1}{n+\beta}\bar{x}+\left(\frac{\alpha}{n+\beta}\right)^2\right)+3x^3-x^3\\
=\;& \frac{(n-1)(n-2)}{n^2}\bar{x}^3+\frac{3(n-1)}{n}\left(\frac{1}{n+\beta}-\left(x-\frac{\alpha}{n+\beta}\right)\right)\bar{x}^2+\left(\left(\frac{1}{n+\beta}\right)^2\right.
\end{aligned}$$

$$
\begin{aligned}
&+3\left(\frac{\alpha}{n+\beta}\right)^2+\frac{3\alpha}{(n+\beta)^2}-\frac{6x\alpha}{n+\beta}-\frac{3x}{n+\beta}\Big)\bar{x}-\bar{x}^3+3x^2\bar{x}\\
=\;&\left(\frac{(n-1)(n-2)}{n^2}-1-\frac{3(n-1)}{n}\right)\bar{x}^3+\left(3x^2-6x\frac{\alpha}{n+\beta}+3\left(\frac{\alpha}{n+\beta}\right)^2\right)\bar{x}\\
&+\left(\frac{-3x}{n+\beta}+\left(\frac{1}{n+\beta}\right)^2+\frac{3\alpha}{(n+\beta)^2}\right)\bar{x}+\frac{3(n-1)}{n}\frac{1}{n+\beta}\bar{x}^2\\
=\;&\frac{1}{n^2}\left(n^2-3n+2-n^2-3n(n-1)+3n^2\right)\bar{x}^3+\frac{1}{n^2}\Big(3(n-1)\frac{n}{n+\beta}\\
&-3n\frac{n}{n+\beta}\Big)\bar{x}^2+\frac{1}{n^2}\left(\frac{n}{n+\beta}\right)^2\bar{x}\\
=\;&\frac{1}{n^2}\left(2\bar{x}^3-3\frac{n}{n+\beta}\bar{x}^2+\left(\frac{n}{n+\beta}\right)^2\bar{x}\right)\\
=\;&\frac{\varphi_n^2(x)\left(\frac{n}{n+\beta}-2\bar{x}\right)}{n^2}.
\end{aligned}
$$

The proof of (2.8) is similar, but is rather complicated. By Lemma 2, we deduce that

$$
\begin{aligned}
&S_n^{(\alpha,\beta)}((t-x)^4,x)=S_n^{(\alpha,\beta)}(t^4,x)-4xS_n^{(\alpha,\beta)}(t^3,x)+6x^2S_n^{(\alpha,\beta)}(t^2,x)-4x^4+x^4\\
=\;&\frac{(n-1)(n-2)(n-3)}{n^3}\bar{x}^4+\frac{(n-1)(n-2)\,(6+4\alpha)}{n^2(n+\beta)}\bar{x}^3+\frac{n-1}{n(n+\beta)^2}\left(7+12\alpha+6\alpha^2\right)\bar{x}^2\\
&+\frac{1+4\alpha+6\alpha^2+4\alpha^3}{(n+\beta)^3}\bar{x}+\left(\frac{\alpha}{n+\beta}\right)^4-4x\Big(\frac{(n-1)(n-2)}{n^2}\bar{x}^3+\frac{3(n-1)(1+\alpha)}{n(n+\beta)}\bar{x}^2\\
&+\frac{1+3\alpha+3\alpha^2}{(n+\beta)^2}\bar{x}+\left(\frac{\alpha}{n+\beta}\right)^3\Big)+6x^2\left(\frac{n-1}{n}\bar{x}^2+\frac{2\alpha+1}{n+\beta}\bar{x}+\frac{\alpha^2}{(n+\beta)^2}\right)-4x^4+x^4\\
=\;&\frac{(n-1)(n-2)(n-3)}{n^3}\bar{x}^4+\frac{(n-1)(n-2)}{n^2}\left(\frac{6}{n+\beta}+\frac{4\alpha}{n+\beta}-4x\right)\bar{x}^3\\
&+\frac{n-1}{n}\left(6\left(\frac{\alpha}{n+\beta}\right)^2-\frac{12x\alpha}{n+\beta}+6x^2-\frac{12x}{n+\beta}+\frac{12\alpha}{(n+\beta)^2}+7\left(\frac{1}{n+\beta}\right)^2\right)\bar{x}^2\\
&+\Big(\left(\frac{1}{n+\beta}\right)^3+4\left(\frac{1}{n+\beta}\right)^3\alpha+6\left(\frac{1}{n+\beta}\right)^3\alpha^2+4\left(\frac{\alpha}{n+\beta}\right)^3-4x\left(\frac{1}{n+\beta}\right)^2\\
&-12x\frac{1}{n+\beta}\frac{\alpha}{n+\beta}-12x\left(\frac{\alpha}{n+\beta}\right)^2+6x^2\frac{1}{n+\beta}+12x^2\frac{\alpha}{n+\beta}\Big)\bar{x}+\bar{x}^4-4x^3\bar{x}\\
=\;&\left(\frac{(n-1)(n-2)(n-3)}{n^3}+1\right)\bar{x}^4+\frac{(n-1)(n-2)}{n^2}\left(\frac{6}{n+\beta}-4\bar{x}\right)\bar{x}^3+\frac{n-1}{n}\Big(6\bar{x}^2\\
&-\frac{12}{n+\beta}\bar{x}+7\left(\frac{1}{n+\beta}\right)^2\Big)\bar{x}^2+\left(-4\bar{x}^3+\frac{6}{n+\beta}\bar{x}^2-4\left(\frac{1}{n+\beta}\right)^2\bar{x}+\left(\frac{1}{n+\beta}\right)^3\right)\bar{x}\\
=\;&\frac{1}{n^3}\Big[\left((n-1)(n-2)(n-3)+n^3-4n(n-1)(n-2)+6n^2(n-1)-4n^3\right)\bar{x}^4+\\
&\left(6(n-1)(n-2)\frac{n}{n+\beta}-12n(n-1)\frac{n}{n+\beta}+6n^2\frac{n}{n+\beta}\right)\bar{x}^3+\Big(7(n-1)\left(\frac{n}{n+\beta}\right)^2
\end{aligned}
$$

$$
\begin{aligned}
&\quad -4n\left(\frac{n}{n+\beta}\right)^2\Big)\bar{x}^2+\left(\frac{n}{n+\beta}\right)^3\bar{x}\Big]\\
&= \frac{1}{n^3}\left[3n\bar{x}^2\left(\frac{n}{n+\beta}-\bar{x}\right)^2-6\bar{x}^2\left(\frac{n}{n+\beta}-\bar{x}\right)^2-\left(\frac{n}{n+\beta}\right)^2\bar{x}^2+\left(\frac{n}{n+\beta}\right)^3\bar{x}\right]\\
&= \frac{\left(\frac{n}{n+\beta}\right)^2\varphi_n^2(x)+3(n-2)\varphi_n^4(x)}{n^3}.
\end{aligned}
$$

□

Lemma 4. For any $t,x\in\left(\frac{\alpha}{n+\beta},\frac{n+\alpha}{n+\beta}\right)$, $0\le\lambda\le 1$, it holds that

$$
\left|\int_x^t\frac{1}{\varphi_n^\lambda(u)}du\right|\le\frac{|t-x|}{\varphi_n^\lambda(x)}. \tag{2.9}
$$

Proof. Direct calculations yield that

$$
\begin{aligned}
&\left|\int_x^t\frac{1}{\varphi_n^\lambda(u)}du\right|\\
\le\ &\left|\int_x^t\frac{1}{\varphi_n(u)}du\right|^\lambda|t-x|^{1-\lambda}\\
\le\ &\left(\frac{n+\beta}{n}\right)^\lambda|t-x|^{1-\lambda}\left|\int_x^t\left(\frac{1}{\sqrt{u-\frac{\alpha}{n+\beta}}}+\frac{1}{\sqrt{\frac{n+\alpha}{n+\beta}-u}}\right)du\right|^\lambda\\
\le\ &\left(\frac{n+\beta}{n}\right)^\lambda|t-x|^{1-\lambda}\Bigg(2\left|\sqrt{t-\frac{\alpha}{n+\beta}}-\sqrt{x-\frac{\alpha}{n+\beta}}\right|\\
&+2\left|\sqrt{\frac{n+\alpha}{n+\beta}-t}-\sqrt{\frac{n+\alpha}{n+\beta}-x}\right|\Bigg)^\lambda\\
\le\ &2^\lambda\left(\frac{n+\beta}{n}\right)^\lambda|t-x|\left(\frac{1}{\sqrt{t-\frac{\alpha}{n+\beta}}+\sqrt{x-\frac{\alpha}{n+\beta}}}+\frac{1}{\sqrt{\frac{n+\alpha}{n+\beta}-t}+\sqrt{\frac{n+\alpha}{n+\beta}-x}}\right)^\lambda\\
\le\ &2^\lambda\left(\frac{n+\beta}{n}\right)^\lambda|t-x|\left(\frac{1}{\sqrt{x-\frac{\alpha}{n+\beta}}}+\frac{1}{\sqrt{\frac{n+\alpha}{n+\beta}-x}}\right)^\lambda\\
\le\ &C\frac{|t-x|}{\varphi_n^\lambda(x)}.
\end{aligned}
$$

□

From Lemma 2, we see that $S_n^{(\alpha,\beta)}(f,x)$ keeps the linear functions. Generally speaking, $B_{n,\alpha,\beta}(f,x)$ defined in (1.5) has no such preserving properties. For the approximation rate of $S_n^{(\alpha,\beta)}(f,x)$ for $f\in C(A_n)$, we have the following direct results including both the pointwise estimates and global estimates:

Theorem 3. Let $0\le\lambda\le 1$ be a fixed number. For any $f\in C(A_n)$, there is a positive constant only depending on λ,α and β such that

$$\left|S_n^{(\alpha,\beta)}(f,x)-f(x)\right| \le C\omega^2_{\varphi_n^\lambda}\left(f,\frac{\varphi_n^{1-\lambda}(x)}{\sqrt{n}}\right). \tag{2.10}$$

Proof. It is obvious that

$$\|S_n^{(\alpha,\beta)}(f)\| \le \|f\|, \tag{2.11}$$

where $\|f\|$ is the uniform norm of f in A_n.

Set $D^2_\lambda := \left\{f \in C(A_n),\ f' \in A.C._{loc},\ \|\varphi_n^{2\lambda}f''\| < +\infty\right\}$. Define

$$K_{\varphi_n^\lambda}(f,t^2) = \inf_{g\in D^2_\lambda}\{\|f-g\| + t^2\|\varphi_n^{2\lambda}g''\|\}.$$

It is well known that (see [8]) $K_{\varphi_n^\lambda}(f,t^2) \sim \omega^2_{\varphi_n^\lambda}(f,t)$. Therefore, for any fixed n, λ and x, we may choose a $g_{n,x,\lambda}(t) \in D^2_\lambda$ such that

$$\|f-g\| \le C\omega^2_{\varphi_n^\lambda}\left(f,\frac{\varphi_n^{1-\lambda}(x)}{\sqrt{n}}\right), \tag{2.12}$$

$$\frac{\varphi_n^{2(1-\lambda)}(x)}{n}\|\varphi_n^{2\lambda}g''\| \le C\omega^2_{\varphi_n^\lambda}\left(f,\frac{\varphi_n^{1-\lambda}(x)}{\sqrt{n}}\right). \tag{2.13}$$

By (2.11) and (2.12), we have

$$\begin{aligned}
&|S_n^{(\alpha,\beta)}(f,x)-f(x)|\\
&\le |S_n^{(\alpha,\beta)}(f-g,x)| + |f(x)-g(x)| + |S_n^{(\alpha,\beta)}(g,x)-g(x)|\\
&\le 2\|f-g\| + |S_n^{(\alpha,\beta)}(g,x)-g(x)|\\
&\le C\omega^2_{\varphi_n^\lambda}\left(f,\frac{\varphi_n^{1-\lambda}(x)}{\sqrt{n}}\right) + |S_n^{(\alpha,\beta)}(g,x)-g(x)|.
\end{aligned} \tag{2.14}$$

By using Taylor's expansion

$$g(t) = g(x) + g'(x)(t-x) + \int_x^t (t-u)g''(u)du,$$

and the following inequality (see [9]):

$$\frac{|t-u|}{\varphi_n^{2\lambda}(u)} \le \frac{|t-x|}{\varphi_n^{2\lambda}(x)},\ \text{for any } u \text{ between } x \text{ and } t,$$

we have

$$\begin{aligned}
&|S_n^{(\alpha,\beta)}(g,x)-g(x)|\\
&= \left|S_n^{(\alpha,\beta)}\left(\int_x^t (t-u)g''(u)du, x\right)\right|\\
&\le C\|\varphi_n^{2\lambda}g''\|S_n^{(\alpha,\beta)}\left(\frac{(t-x)^2}{\varphi_n^{2\lambda}(x)}, x\right)\\
&\le C\frac{\varphi_n^{2(1-\lambda)}(x)}{n}\|\varphi_n^{2\lambda}g''\|\\
&\le C\omega^2_{\varphi_n^\lambda}\left(f,\frac{\varphi_n^{1-\lambda}(x)}{\sqrt{n}}\right),
\end{aligned} \tag{2.15}$$

where in the last inequality, (2.13) is used.

By combining (2.14) and (2.15), we obtain Theorem 3. □

§3 Converse results of approximation by $U_n^{(\alpha,\beta)}(f,x)$

Lemma 5. Let $f \in C(A_n)$. Then

$$\left|\left(U_n^{(\alpha,\beta)}(f,x)\right)''\right| \leq \frac{\sqrt{2}n}{\varphi_n^2(x)}\|f\|.$$

Proof. With the notation

$$F_0^{(n)}(f) := f\left(\frac{\alpha}{n+\beta}\right),\ F_n^{(n)}(f) := f\left(\frac{n+\alpha}{n+\beta}\right),$$

$$F_k^{(n)}(f) := \lambda_{n-2}^{-1}\int_{A_{n-2}} q_{n-2,k-1}(t) f\left(\frac{n(n-2+\beta)}{(n-2)(n+\beta)}\left(t-\frac{2\alpha}{n(n-2+\beta)}\right)\right)dt,\ 1\leq k\leq n-1,$$

we have

$$U_n^{(\alpha,\beta)}(f,x) = \left(\frac{n+\beta}{n}\right)^n \sum_{k=0}^n F_k^{(n)}(f) q_{n,k}(x).$$

It is easy to deduce that

$$q_{n,k}''(x) = q_{n,k}(x)\frac{\left(\frac{n(k+\alpha)}{n+\beta}-nx\right)^2 - n\varphi_n^2(x) - \left(\frac{n(k+\alpha)}{n+\beta}-nx\right)\left(\frac{n}{n+\beta}-2\left(x-\frac{\alpha}{n+\beta}\right)\right)}{\varphi_n^4(x)}.$$

Thus,

$$\begin{aligned}
&\left|\left(U_n^{(\alpha,\beta)}(f,x)\right)''\right| \\
&= \left|\left(\frac{n+\beta}{n}\right)^n \sum_{k=0}^n F_k^{(n)}(f) q_{n,k}''(x)\right| \\
&\leq \frac{n\|f\|}{\varphi_n^2(x)}\left(\frac{n+\beta}{n}\right)^n \sum_{k=0}^n q_{n,k}(x)\left|\frac{n\left(\frac{k+\alpha}{n+\beta}-x\right)^2}{\varphi_n^2(x)} - 1 - \frac{\left(\frac{k+\alpha}{n+\beta}-x\right)\left(\frac{n}{n+\beta}-2\left(x-\frac{\alpha}{n+\beta}\right)\right)}{\varphi_n^2(x)}\right|.
\end{aligned}$$

By Cauchy-Schwarz's inequality, we have

$$\begin{aligned}
&\left|\left(U_n^{(\alpha,\beta)}(f,x)\right)''\right| \\
\leq\ & \frac{n\|f\|}{\varphi_n^2(x)}\left[S_n^{(\alpha,\beta)}\left(\left(\frac{n\left(\frac{k+\alpha}{n+\beta}-x\right)^2}{\varphi_n^2(x)} - 1 - \frac{\left(\frac{k+\alpha}{n+\beta}-x\right)\left(\frac{n}{n+\beta}-2\bar{x}\right)}{\varphi_n^2(x)}\right)^2;x\right) S_n^{(\alpha,\beta)}(1;x)\right]^{\frac{1}{2}} \\
=\ & \frac{n\|f\|}{\varphi_n^2(x)}\left[\frac{n^2}{\varphi_n^4(x)} S_n^{(\alpha,\beta)}((t-x)^4,x) + 1 + \frac{\left(\frac{n}{n+\beta}-2\left(x-\frac{\alpha}{n+\beta}\right)\right)^2}{\varphi_n^4(x)} S_n^{(\alpha,\beta)}((t-x)^2,x)\right. \\
& -\frac{2n}{\varphi_n^2(x)} S_n^{(\alpha,\beta)}((t-x)^2,x) - \frac{2n\left(\frac{n}{n+\beta}-2\left(x-\frac{\alpha}{n+\beta}\right)\right)}{\varphi_n^4(x)} S_n^{(\alpha,\beta)}((t-x)^3,x) \\
& \left.+\frac{2\left(\frac{n}{n+\beta}-2\left(x-\frac{\alpha}{n+\beta}\right)\right)}{\varphi_n^2(x)} S_n^{(\alpha,\beta)}(t-x,x)\right]^{\frac{1}{2}}.
\end{aligned}$$

Therefore, by Lemma 3,

$$\begin{aligned}
&\left|\left(U_n^{(\alpha,\beta)}(f,x)\right)''\right| \\
\leq\ & \frac{n\|f\|}{\varphi_n^2(x)}\left[\frac{n^2}{\varphi_n^4(x)}\frac{\left(\frac{n}{n+\beta}\right)^2\varphi_n^2(x)+3(n-2)\varphi_n^4(x)}{n^3}\right. \\
&+1+\frac{\left(\frac{n}{n+\beta}-2\left(x-\frac{\alpha}{n+\beta}\right)\right)^2}{\varphi_n^4(x)}\frac{\varphi_n^2(x)}{n}-\frac{2n}{\varphi_n^2(x)}\frac{\varphi_n^2(x)}{n} \\
&\left.-\frac{2n\left(\frac{n}{n+\beta}-2\left(x-\frac{\alpha}{n+\beta}\right)\right)}{\varphi_n^4(x)}\frac{\varphi_n^2(x)\left(\frac{n}{n+\beta}-2\bar{x}\right)}{n^2}\right]^{\frac{1}{2}} \\
=\ & \frac{n\|f\|}{\varphi_n^2(x)}\sqrt{2-\frac{2}{n}},
\end{aligned}$$

which means that

$$\left|\left(U_n^{(\alpha,\beta)}(f,x)\right)''\right|\leq\frac{\sqrt{2}n}{\varphi_n^2(x)}\|f\|.$$

□

Lemma 6. Let $f\in D_\lambda^2$ with $\lambda\in[0,1]$. Then,

$$\left\|\varphi_n^{2\lambda}\left(U_n^{(\alpha,\beta)}(f)\right)''\right\|\leq(1+\beta)^2\left\|\varphi_n^{2\lambda}f''\right\|. \tag{3.1}$$

Proof. For convenience, we use the following notations:

$$t^*:=\frac{n(n-2+\beta)}{(n-2)(n+\beta)}\left(t-\frac{2\alpha}{n(n-2+\beta)}\right),$$

$$\Delta^1F_k^{(n)}(f)=F_{k+1}^{(n)}(f)-F_k^{(n)}(f),\ \Delta^2F_k^{(n)}(f)=\Delta^1(\Delta^1F_k^{(n)}(f)).$$

Obviously,

$$\begin{aligned}
\left(U_n^{(\alpha,\beta)}(f,x)\right)'&=\left(\frac{n+\beta}{n}\right)^n\sum_{k=0}^{n}F_k^{(n)}(f)\binom{n}{k}\left(k\left(x-\frac{\alpha}{n+\beta}\right)^{k-1}\left(\frac{n+\alpha}{n+\beta}-x\right)^{n-k}\right. \\
&\quad\left.-(n-k)\left(x-\frac{\alpha}{n+\beta}\right)^k\left(\frac{n+\alpha}{n+\beta}-x\right)^{n-k-1}\right) \\
&=n\left(\frac{n+\beta}{n}\right)^n\sum_{k=1}^{n}\binom{n-1}{k-1}F_k^{(n)}(f)\left(x-\frac{\alpha}{n+\beta}\right)^{k-1}\left(\frac{n+\alpha}{n+\beta}-x\right)^{n-k}-n\left(\frac{n+\beta}{n}\right)^n \\
&\quad\sum_{k=0}^{n-1}\binom{n-1}{k}F_k^{(n)}(f)\left(x-\frac{\alpha}{n+\beta}\right)^k\left(\frac{n+\alpha}{n+\beta}-x\right)^{n-k-1} \\
&=n\left(\frac{n+\beta}{n}\right)^n\sum_{k=0}^{n-1}\Delta^1F_k^{(n)}(f)\binom{n-1}{k}\left(x-\frac{\alpha}{n+\beta}\right)^k\left(\frac{n+\alpha}{n+\beta}-x\right)^{n-k-1}.
\end{aligned}$$

From the above representation, we see that

$$\left(U_n^{(\alpha,\beta)}(f,x)\right)'' = (n^2-n)\left(\frac{n+\beta}{n}\right)^n \times \sum_{k=0}^{n-2} \Delta^2 F_k^{(n)}(f)\binom{n-2}{k}\bar{x}^k\left(\frac{n+\alpha}{n+\beta}-x\right)^{n-k-2}. \tag{3.2}$$

Now we shall prove that

$$\Delta^2 F_k^{(n)}(f) = \lambda_{n-2}^{-1}\left(\frac{n(n-2+\beta)}{(n-2)(n+\beta)}\right)^2 \frac{1}{n(n-1)} \times \int_{A_{n-2}}\binom{n}{k+1}\left(t-\frac{\alpha}{n-2+\beta}\right)^{k+1}\left(\frac{n-2+\alpha}{n-2+\beta}-t\right)^{n-k-1} f''(t^*)dt, \tag{3.3}$$

for $0 \le k \le n-2$. When $1 \le k \le n-3$, noting that

$$\left(\binom{n}{k+1}\left(t-\frac{\alpha}{n-2+\beta}\right)^{k+1}\left(\frac{n-2+\alpha}{n-2+\beta}-t\right)^{n-k-1}\right)'' = n(n-1)\left(q_{n-2,k+1}(t)-2q_{n-2,k}(t)+q_{n-2,k-1}(t)\right),$$

and by using integration by parts, we obtain

$$\begin{aligned}\Delta^2 F_k^{(n)}(f) &= \lambda_{n-2}^{-1}\int_{A_{n-2}}\left(q_{n-2,k+1}(t)-2q_{n-2,k}(t)+q_{n-2,k-1}(t)\right)f(t^*)dt \\ &= \lambda_{n-2}^{-1}\int_{A_{n-2}}\frac{1}{n(n-1)}\left(\binom{n}{k+1}\left(t-\frac{\alpha}{n-2+\beta}\right)^{k+1}\left(\frac{n-2+\alpha}{n-2+\beta}-t\right)^{n-k-1}\right)'' f(t^*)dt \\ &= \lambda_{n-2}^{-1}\left(\frac{n(n-2+\beta)}{(n-2)(n+\beta)}\right)^2\frac{1}{n(n-1)} \\ &\quad \times \int_{A_{n-2}}\binom{n}{k+1}\left(t-\frac{\alpha}{n-2+\beta}\right)^{k+1}\left(\frac{n-2+\alpha}{n-2+\beta}-t\right)^{n-k-1} f''(t^*)dt,\end{aligned}$$

which proves (3.3) in this case. In the case when $k=0$, we have

$$\begin{aligned}\Delta^2 F_0^{(n)}(f) &= \lambda_{n-2}^{-1}\int_{A_{n-2}}\left(q_{n-2,1}(t)-2q_{n-2,0}(t)\right)f(t^*)dt + f\left(\frac{\alpha}{n+\beta}\right) \\ &= \lambda_{n-2}^{-1}\int_{A_{n-2}}\frac{1}{n(n-1)}\left(\binom{n}{1}\left(t-\frac{\alpha}{n-2+\beta}\right)\left(\frac{n-2+\alpha}{n-2+\beta}-t\right)^{n-1}\right)'' f(t^*)dt \\ &\quad + f\left(\frac{\alpha}{n+\beta}\right) \\ &= \lambda_{n-2}^{-1}\left(\frac{n(n-2+\beta)}{(n-2)(n+\beta)}\right)^2\frac{1}{n(n-1)}\int_{A_{n-2}}\binom{n}{1}\left(t-\frac{\alpha}{n-2+\beta}\right) \\ &\quad \left(\frac{n-2+\alpha}{n-2+\beta}-t\right)^{n-1} f''(t^*)dt.\end{aligned}$$

Thus, (3.3) also holds when $k=0$. The case when $k=n-2$ can be proved in a way similar to the case $k=0$.

By (3.2) and (3.3), we have the representation

$$\left(U_n^{(\alpha,\beta)}(f,x)\right)'' = \left(\frac{n+\beta}{n}\right)^n \left(\frac{n(n-2+\beta)}{(n-2)(n+\beta)}\right)^2 \sum_{k=0}^{n-2} \binom{n-2}{k} \bar{x}^k \left(\frac{n+\alpha}{n+\beta}-x\right)^{n-k-2}$$
$$\times \lambda_{n-2}^{-1} \int_{A_{n-2}} \binom{n}{k+1} \left(t-\frac{\alpha}{n-2+\beta}\right)^{k+1} \left(\frac{n-2+\alpha}{n-2+\beta}-t\right)^{n-k-1} f''(t^*)dt. \tag{3.4}$$

Direct calculation yields that

$$\varphi^2(t^*) = \left(\frac{n(n-2+\beta)}{(n-2)(n+\beta)}\right)^2 \varphi_{n-2}^2(t). \tag{3.5}$$

For any $\lambda \in (0,1)$, by using Hölder's inequality, we get

$$\begin{aligned} & \Delta_{n,k}(x) \\ := \; & \lambda_{n-2}^{-1} \int_{A_{n-2}} \binom{n}{k+1} \left(t-\frac{\alpha}{n-2+\beta}\right)^{k+1} \left(\frac{n-2+\alpha}{n-2+\beta}-t\right)^{n-k-1} \varphi_{n-2}^{-2\lambda}(t)dt \\ \leq \; & \left(\lambda_{n-2}^{-1} \int_{A_{n-2}} \binom{n}{k+1} \left(t-\frac{\alpha}{n-2+\beta}\right)^{k+1} \left(\frac{n-2+\alpha}{n-2+\beta}-t\right)^{n-k-1} dt\right)^{1-\lambda} \\ & \times \left(\lambda_{n-2}^{-1} \int_{A_{n-2}} \binom{n}{k+1} \left(t-\frac{\alpha}{n-2+\beta}\right)^{k+1} \left(\frac{n-2+\alpha}{n-2+\beta}-t\right)^{n-k-1} \varphi_{n-2}^{-2}(t)dt\right)^{\lambda} \\ = \; & \left(\lambda_{n-2}^{-1}\binom{n}{k+1} \left(\frac{n-2}{n-2+\beta}\right)^{n+1} \int_0^1 u^{k+1}(1-u)^{n-k-1}dt\right)^{1-\lambda} \\ & \times \left(\lambda_{n-2}^{-1}\binom{n}{k+1} \left(\frac{n-2}{n-2+\beta}\right)^{n-1} \int_0^1 u^{k}(1-u)^{n-k-2}dt\right)^{\lambda}, \end{aligned}$$

where in the last inequality, we used the transformation $u = \frac{n-2+\beta}{n-2}t - \frac{\alpha}{n-2}$. Therefore,

$$\begin{aligned} \Delta_{n,k}(x) \leq & \left(\lambda_{n-2}^{-1}\binom{n}{k+1} \left(\frac{n-2}{n-2+\beta}\right)^{n+1} \frac{\Gamma(k+2)\Gamma(n-k)}{\Gamma(n+2)}\right)^{1-\lambda} \\ & \times \left(\lambda_{n-2}^{-1}\binom{n}{k+1} \left(\frac{n-2}{n-2+\beta}\right)^{n-1} \frac{\Gamma(k+1)\Gamma(n-k-1)}{\Gamma(n)}\right)^{\lambda} \\ = & \left(\left(\frac{n-2}{n-2+\beta}\right)^2 \frac{n-1}{n+1}\right)^{1-\lambda} \left(\frac{n(n-1)}{(k+1)(n-k-1)}\right)^{\lambda} \\ \leq & \left(\frac{n(n-1)}{(k+1)(n-k-1)}\right)^{\lambda}. \end{aligned} \tag{3.6}$$

Now, we are in a position to prove (3.1). We only prove the case when $\lambda \in (0,1)$, the cases when $\lambda = 0$ and $\lambda = 1$ can be proved similarly and more simpler. From (3.2)-(3.6), by using Hölder's inequality, we get

$$\left|\varphi_n^{2\lambda}(x)\left(U_n^{(\alpha,\beta)}(f,x)\right)''\right|$$

$$\begin{aligned}
\leq\ & \left(\frac{n+\beta}{n}\right)^n \left(\frac{n(n-2+\beta)}{(n-2)(n+\beta)}\right)^2 \varphi_n^{2\lambda}(x)\sum_{k=0}^{n-2}\binom{n-2}{k}\bar{x}^k\left(\frac{n+\alpha}{n+\beta}-x\right)^{n-k-2}\lambda_{n-2}^{-1} \\
& \times\int_{A_{n-2}}\binom{n}{k+1}\left(t-\frac{\alpha}{n-2+\beta}\right)^{k+1}\left(\frac{n-2+\alpha}{n-2+\beta}-t\right)^{n-k-1}\varphi_n^{-2\lambda}(t^*)|\varphi_n^{2\lambda}(t^*)f''(t^*)|dt \\
\leq\ & \|\varphi^{2\lambda}f''\|\left(\frac{n+\beta}{n}\right)^n\varphi_n^{2\lambda}(x)\sum_{k=0}^{n-2}\binom{n-2}{k}\bar{x}^k\left(\frac{n+\alpha}{n+\beta}-x\right)^{n-k-2}\Delta_{n,k}(x) \\
\leq\ & \|\varphi_n^{2\lambda}f''\|\left(\frac{n+\beta}{n}\right)^n\varphi_n^{2\lambda}(x)\sum_{k=0}^{n-2}\binom{n-2}{k}\bar{x}^k\left(\frac{n+\alpha}{n+\beta}-x\right)^{n-k-2}\left(\frac{n(n-1)}{(k+1)(n-k-1)}\right)^\lambda \\
\leq\ & \|\varphi_n^{2\lambda}f''\|\left(\frac{n+\beta}{n}\right)^n\varphi_n^{2\lambda}(x)\left(\sum_{k=0}^{n-2}\binom{n-2}{k}\bar{x}^k\left(\frac{n+\alpha}{n+\beta}-x\right)^{n-k-2}\right)^{1-\lambda} \\
& \times\left(\sum_{k=0}^{n-2}\binom{n-2}{k}\bar{x}^k\left(\frac{n+\alpha}{n+\beta}-x\right)^{n-k-2}\frac{n(n-1)}{(k+1)(n-k-1)}\right)^\lambda \\
\leq\ & \|\varphi_n^{2\lambda}f''\|\left(\frac{n+\beta}{n}\right)^{n-(n-2)(1-\lambda)}\left(\sum_{k=0}^{n-2}\binom{n}{k+1}\bar{x}^k\left(\frac{n+\alpha}{n+\beta}-x\right)^{n-k-2}\varphi_n^2(x)\right)^\lambda \\
=\ & \|\varphi_n^{2\lambda}f''\|\left(\frac{n+\beta}{n}\right)^{n-(n-2)(1-\lambda)}\left(\sum_{k=0}^{n-2}\binom{n}{k+1}\bar{x}^{k+1}\left(\frac{n+\alpha}{n+\beta}-x\right)^{n-k-1}\right)^\lambda \\
=\ & \|\varphi_n^{2\lambda}f''\|\left(\frac{n+\beta}{n}\right)^{n-(n-2)(1-\lambda)}\left(\sum_{k=1}^{n-1}\binom{n}{k}\left(x-\frac{\alpha}{n+\beta}\right)^k\left(\frac{n+\alpha}{n+\beta}-x\right)^{n-k}\right)^\lambda \\
\leq\ & \|\varphi_n^{2\lambda}f''\|\left(\frac{n+\beta}{n}\right)^{n-(n-2)(1-\lambda)-(n-1)\lambda} = \|\varphi_n^{2\lambda}f''\|\left(\frac{n+\beta}{n}\right)^2 \\
\leq\ & (1+\beta)^2\|\varphi_n^{2\lambda}f''\|.
\end{aligned}$$

Lemma 6 is proved. □

Lemma 7. Let $\phi : [a,b] \to R,\ \phi \not\equiv 0$, be a function with ϕ^2 concave. Then for all $x \in [a,b],\ h > 0$, with $x \pm h \in [a,b]$, the inequality

$$\int_{-\frac{h}{2}}^{\frac{h}{2}}\int_{-\frac{h}{2}}^{\frac{h}{2}}\frac{dsdt}{\phi^2(x+s+t)} \leq C\frac{h^2}{\phi^2(x)}, \tag{3.7}$$

holds true.

Theorem 4. Let $f \in C(A_n)$ and $0 < \theta < 2$. then

$$\left|U_n^{(\alpha,\beta)}(f,x)-f(x)\right| \leq C\left(\frac{\varphi_n^{1-\lambda}(x)}{\sqrt{n}}\right)^\theta \Leftrightarrow \omega_{\varphi_n^\lambda}^2(f,t) \leq Ct^\theta, t \geq 0.$$

Proof. The proof of sufficiency is easily obtained by the direct theorem (Theorem 1). Now prove the necessity. Let $x,h\in A_n$, such that $x\pm h\in A_n$. Obviously,

$$\left|\left(\Delta_h^2 f\right)(x)\right|\le\left|\Delta_h^2\left(f(x)-U_n^{(\alpha,\beta)}(f,x)\right)\right|+\left|\Delta_h^2\left(U_n^{(\alpha,\beta)}(f,x)\right)\right|.$$

We estimate the values of the two terms on the right of the above inequality respectively. Use the convexity of $\varphi_n^{2(1-\lambda)}(x)$, we have

$$\varphi_n^{2(1-\lambda)}(x-h)\le 2\varphi_n^{2(1-\lambda)}(x),\ \varphi_n^{2(1-\lambda)}(x+h)\le 2\varphi_n^{2(1-\lambda)}(x).$$

By the assumption of the theorem, we have

$$\begin{aligned}\left|\Delta_h^2\left(f(x)-U_n^{(\alpha,\beta)}(f,x)\right)\right| &\le Cn^{-\theta/2}\left(\varphi_n^{\theta(1-\lambda)}(x-h)+2\varphi_n^{\theta(1-\lambda)}(x)+\varphi_n^{\theta(1-\lambda)}(x+h)\right)\\ &\le C\left(\frac{\varphi_n^{1-\lambda}(x)}{\sqrt{n}}\right)^{\theta}.\end{aligned}$$

By using $K_{\varphi_n^\lambda}(f,t^2)\sim\omega^2_{\varphi_n^\lambda}(f,t)$ again, for any fixed δ, we can choose a $g_\delta\in A.C._{loc}$ such that

$$\|f-g_\delta\|\le C\omega^2_{\varphi_n^\lambda}(f,\delta),\ \|\varphi^{2\lambda}g_\delta''\|\le C\delta^{-2}\omega^2_{\varphi_n^\lambda}(f,\delta).$$

By Lemma 5 and Lemma 6, we have

$$\begin{aligned}\left|\left(U_n^{(\alpha,\beta)}(f,x)\right)''\right| &\le \left|\left(U_n^{(\alpha,\beta)}(f-g,x)\right)''\right|+\left|\left(U_n^{(\alpha,\beta)}(g,x)\right)''\right|\\ &\le \frac{\sqrt{2}n}{\varphi_n^2(x)}\|f-g\|+\frac{(1+\beta)^2}{\varphi_n^{2\lambda}(x)}\|\varphi_n^{2\lambda}g''\|\\ &\le C\left(\frac{\sqrt{2}n}{\varphi_n^2(x)}+\frac{1}{\delta^2\varphi_n^{2\lambda}(x)}\right)\omega^2_{\varphi_n^\lambda}(f,\delta).\end{aligned}$$

By Lemma 7, we have

$$\begin{aligned}\left|\Delta_h^2\left(U_n^{(\alpha,\beta)}(f,x)\right)\right| &= \int_{-\frac{h}{2}}^{\frac{h}{2}}\int_{-\frac{h}{2}}^{\frac{h}{2}}\left(U_n^{(\alpha,\beta)}(f,x+s+t)\right)''dsdt\\ &\le C\omega^2_{\varphi_n^\lambda}(f,\delta)\int_{-\frac{h}{2}}^{\frac{h}{2}}\int_{-\frac{h}{2}}^{\frac{h}{2}}\left(\frac{\sqrt{2}n}{\varphi_n^2(x+s+t)}+\frac{1}{\delta^2\varphi_n^{2\lambda}(x+s+t)}\right)dsdt\\ &\le C\left(\frac{\sqrt{2}nh^2}{\varphi_n^2(x)}+\frac{h^2}{\delta^2\varphi_n^{2\lambda}(x)}\right)\omega^2_{\varphi_n^\lambda}(f,\delta).\end{aligned}$$

Replacing h with $h\varphi_n^\lambda(x)$,

$$\left|\Delta^2_{h\varphi_n^\lambda}\left(U_n^{(\alpha,\beta)}(f,x)\right)\right|\le C\left(\frac{\sqrt{2}nh^2\varphi_n^{2\lambda}(x)}{\varphi_n^2(x)}+\frac{h^2}{\delta^2}\right)\omega^2_{\varphi_n^\lambda}(f,\delta).$$

Thus,

$$\left|\Delta^2_{h\varphi_n^\lambda}(f,x)\right|\le\left(\left(\frac{\varphi_n^{1-\lambda}(x)}{\sqrt{n}}\right)^{\alpha}+\left(\sqrt{2}nh^2\varphi_n^{2(\lambda-1)}(x)+\frac{h^2}{\delta^2}\right)\right)\omega^2_{\varphi_n^\lambda}(f,\delta).$$

choosing n such that

$$\frac{\varphi_n^{1-\lambda}(x)}{\sqrt{n}}\le\delta\le\sqrt{2}\frac{\varphi_n^{1-\lambda}(x)}{\sqrt{n}}.$$

Then

$$\left|\Delta^2_{h\varphi_n^\lambda}(f,x)\right|\le C\left(\delta^\alpha+\frac{h^2}{\delta^2}\right)\omega^2_{\varphi_n^\lambda}(f,\delta),$$

which implies that

$$\omega^2_{\varphi_n^\lambda}(f,t) \le C\left(\delta^\alpha + \frac{h^2}{\delta^2}\omega^2_{\varphi_n^\lambda}(f,\delta)\right),\ 0<t\le\delta.$$

Therefore, the assertion of the theorem is proved by the well-known Berens and Lorentz Lemma ([6]). □

Declarations

Conflict of interest The authors declare no conflict of interest.

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