Appl. Math. J. Chinese Univ.
2025, 40(1): 78-84

Periodic solution of parabolic equations and stochastic process

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Abstract. In this short paper, we first establish the existence of periodic solutions to parabolic equation in the whole space by using the probability method. Then, the periodicity of some function of stochastic process is also studied.

§1 Introduction

From the point of partial differential equations, the long time behavior of the solutions is an important issue. Maybe the solutions converge to a constant, or a function which is the solution to corresponding elliptic equations. But there is another important case that the solutions are time-periodic functions, which have clearly long time behavior. Therefore, the research of periodic solutions can help us understand the long time behavior of the solutions.

In this short paper, we first consider the existence of periodic solutions of parabolic equations in the whole space by using the probability method. There is a lot of work about the periodic solutions of ordinary differential equations and parabolic equations. Here we only recall some results of parabolic equations. In the book [10], Hess considered the periodic-parabolic boundary value problems. He first obtained the existence of eigenvalue for linear case, and then established the relationship between periodic-parabolic eigenvalue problem and the eigenvalue problem of the corresponding elliptic operator. Lastly, by defining the Poincaré map and using the super-lower solution method, he obtained the existence of periodic solution of the periodic-parabolic boundary value problems. See [2] for the related work.

In [9], the author considered the following periodicity problem

$$\begin{cases} u_t - \Delta u = f(t,x,u,\nabla u), & t>0, \quad x \in \mathbb{R}^d, \\ u(t,x) = u(t+T,x), & t \geq 0, \quad x \in \mathbb{R}^d, \end{cases} \tag{1}$$

where $d \geq 2$, $f \in C(\mathbb{R},\mathbb{R}^d,\mathbb{R},\mathbb{R}^d)$, $\nabla u = (u_{x_1}, u_{x_2}, \cdots, u_{x_d})$, f is T-periodic function with respect to the time variable t, the period $T>0$ is arbitrary chosen and fixed. They obtained

Received: 2021-10-19. Revised: 2023-11-27.
MR Subject Classification: 35K20, 60H15, 60H40.
Keywords: periodic solutions, Itô's formula, stochastic process.
Digital Object Identifier(DOI): https://doi.org/10.1007/s11766-025-4593-2.
Supported by the National Natural Science Foundation of China(12171247).

the following result.

Proposition 1.1. *Let $d \geq 2$, $n \in \mathbb{N}$ be fixed, $T > 0$ be fixed, $f \in C(\mathbb{R}, \mathbb{R}^d, \mathbb{R}, \mathbb{R}^d)$. f is T-periodic with respect to the time variable t. Also let $0 \leq c_i, l_i, m_i, p_i, q_i, l_i < \infty$, $i = 1, 2, \cdots, n$, be fixed constants, $0 \leq k_i, n_i < \infty$, $i = 1, 2, \cdot, d$, be fixed constants, $b_i(t) \in C(\mathbb{R}_+)$, $g_i(x) \in C(\mathbb{R}^d)$, $\sup_{\mathbb{R}_+} |b_i(t)| < \infty$, $\sup_{\mathbb{R}^d} |g_i(x)| < \infty$, $i = 1, 2, \cdots, n$,*

$$|f(t, x, u, u_x)| \leq \sum_{i=1}^{n} \left(c_i |b_i(t)|^{p_i} + l_i |u|^{q_i} + m_i |g_i(x)|^{l_i}\right) + \sum_{i=1}^{d} k_i |u_{x_i}|^{n_i}$$

for every $(t, x, u, u_x) \in (\mathbb{R}, \mathbb{R}^d, \mathbb{R}, \mathbb{R}^d)$. Then the problem (1) has a solution $u \in \mathcal{C}^1(\mathbb{R}_+, \mathcal{C}^2(\mathbb{R}^d))$ ($\mathcal{C}^1(\mathbb{R}_+, \mathcal{C}^2(\mathbb{R}^d))$ will be defined later).

The proof of Proposition 1.1 is complicated. The assumptions of Proposition 1.1 looks very general but it is hard to generalize it. In this short paper, we will give some interesting examples which were not covered by Proposition 1.1.

The second aim of the short paper is to consider the periodicity of some function of a stochastic process. Zhao-Zheng [14] gave a definition of pathwise random periodic solutions for C^1-cocycles. Recently, Feng et al. did some work about the periodic solutions, see [7,8]. Chen et al. [4] considered the periodic solutions of Fokker-Planck equation, also see [11]. Liu et al. did a series of work about the almost periodic solution of stochastic differential equations, see [3,13]. Lin et al. [12] studied the nontrivial periodic solution of a stochastic epidemic model. About the periodic phenomenon, also see [1]. In this paper, we consider another phenomenon. For example, the function $u(t) = t$ is clearly not a periodic function, but $\sin(u(t)) = \sin t$ is a 2π-periodic function. When the function X_t becomes a stochastic process, one can use the results of [3,7,8,13,14] to study the periodicity of function of X_t, that is to say, the periodicity of $\psi(X_t)$ can be studied clearly by the earlier results, where ψ is some smooth function. However, the periodicity of $\mathbb{E}\psi(X_t)$ and $\mathbb{E}\psi(t, X_t)$ is not considered. That is to say, we want to study that under what assumptions on X_t and ψ, it holds that

$$\mathbb{E}\psi(X_{t+T}) = \mathbb{E}\psi(X_t), \quad \mathbb{E}\psi(t, X_{t+T}) = \mathbb{E}\psi(t, X_t).$$

The first question is what is T ? Generally speaking, the periodic T is related to the density of X_t. Since we consider a stochastic process which satisfies a stochastic differential equation (SDE), the the periodic T is related to the coefficient of SDE. Note that if the stochastic process has a density $p(t, x)$, then

$$\mathbb{E}[\psi(X_{t+T}) - \mathbb{E}\psi(X_t)] = \int_{\mathbb{R}} \psi(x)[p(t+T, x) - p(t, x)]dx,$$

which implies that $p(t+T, x) = p(t, x)$ for any $x \in \mathbb{R}$. And thus it need not study the periodicity of $\mathbb{E}\psi(X_t)$ because if $p(t+T, x) = p(t, x)$, then for any bounded continuous function ψ, it holds that $\mathbb{E}\psi(X_{t+T}) = \mathbb{E}\psi(X_t)$. But for $\mathbb{E}\psi(t, X_t)$, we have

$$\mathbb{E}\psi(t, X_{t+T}) - \mathbb{E}\psi(t, X_t) = \int_{\mathbb{R}} [\psi(t+T, x)p(t+T, x) - \psi(t, x)p(t, x)]dx.$$

Hence if $p(t+T, x) \neq p(t, x)$, we can choose special function ψ such that $\mathbb{E}\psi(t+T, X_{t+T}) = \mathbb{E}\psi(t, X_t)$. We only study the periodicity of $\mathbb{E}\psi(t, X_t)$.

In the next section, we will state the main results and give some examples to illustrate them.

§2 Main Results

For simplicity, we only consider the one-dimensional case. It is well known that the solutions of the following equations

$$\begin{cases} \frac{\partial u}{\partial t}+b(t,x)\frac{\partial u}{\partial x}+\frac{1}{2}\sigma^2(t,x)\frac{\partial^2 u}{\partial x^2}-v(t,x)u+f(t,x)=0,\ 0<t<T,\ x\in\mathbb{R}, \\ u(T,x)=\psi(x), \quad x\in\mathbb{R}, \end{cases} \tag{1}$$

can be represented as

$$u(t,x)=\mathbb{E}^Q\left[\int_t^T e^{-\int_t^r v(\tau,X_\tau)d\tau} f(r,X_r)dr+e^{-\int_t^T v(\tau,X_\tau)d\tau}\psi(X_T)|X_t=x\right],$$

where X_t is a stochastic process and satisfies

$$\begin{cases} dX_s=b(s,X_s)ds+\sigma(s,X_s)dW_s, \quad t<s<T, \\ X_t=x. \end{cases}$$

Here we need to assume that $\sigma(t,x)>0$ for all $t>0$, $x\in\mathbb{R}$, see [6]. We aim to get

$$u(0,x)=u(T,x).$$

That is to say, we will prove $u(0,x)=\psi(x)$ for some ψ. Meanwhile, we note that

$$u(0,x)=\mathbb{E}^Q\left[\int_0^T e^{-\int_0^r v(\tau,X_\tau)d\tau} f(r,X_r)dr+e^{-\int_0^T v(\tau,X_\tau)d\tau}\psi(X_T)|X_0=x\right].$$

We first consider a special case, i.e., $v=0$. In this case, we have

$$u(0,x)=\mathbb{E}^Q\left[\int_0^T f(r,X_r)dr+\psi(X_T)|X_0=x\right].$$

By Itô formula, we have

$$\begin{aligned}\psi(X_T) &= \psi(x)+\int_0^T \psi'(X_t)b(t,X_t)dt \\ &\quad +\frac{1}{2}\int_0^T \psi''(X_t)\sigma(t,X_t)\sigma^T(t,X_t)dt+\int_0^T \psi'(X_t)\sigma(t,X_t)dW_t,\end{aligned}$$

which implies that

$$\mathbb{E}\psi(X_T)=\psi(x)+\mathbb{E}\int_0^T\left(\psi'(X_t)b(t,X_t)+\frac{1}{2}\psi''(X_t)\sigma(t,X_t)\sigma^T(t,X_t)\right)dt.$$

Therefore, if we assume that

$$\mathbb{E}\int_0^T\left(f(t,X_t)+\psi'(X_t)b(t,X_t)+\frac{1}{2}\psi''(X_t)\sigma(t,X_t)\sigma^T(t,X_t)\right)dt=0,$$

then we will get the desired result

$$u(T,x)=\psi(x)=u(0,x).$$

A strong sufficient condition is

$$f(t,x)+\psi'(x)b(t,x)+\frac{1}{2}\psi''(x)\sigma(t,x)\sigma^T(t,x)=0, \quad \forall x\in\mathbb{R},\forall t\in[0,T]. \tag{2}$$

Note that equation (2) is an ordinary differential equation with parameter $t\in[0,T]$, and

$\sigma^2(t,x) > 0$, equation (2) is equivalent to

$$\phi'(x) + \frac{2b(t,x)}{\sigma^2(t,x)}\phi(x) + \frac{2f(t,x)}{\sigma^2(t,x)} = 0,\ \phi = \psi',\ \forall x \in \mathbb{R}, \forall t \in [0,T].$$

By using the results of ODEs we get

$$\phi(x) = e^{\int P_t(x)dx}\left(\int Q_t(x)e^{-\int P_t(x)dx}dx + C\right), \tag{3}$$

where C is a constant and

$$P_t(x) = \frac{2b(t,x)}{\sigma^2(t,x)},\ Q_t(x) = \frac{2f(t,x)}{\sigma^2(t,x)}.$$

Consequently,

$$\psi(x) = \int \phi(x)dx = \int \left[e^{\int P_t(x)dx}\left(\int Q_t(x)e^{-\int P_t(x)dx}dx + C\right)\right]dx. \tag{4}$$

It is easy to check that if we take $\psi(x)$ as in (4), then it holds that

$$f(t,x) + \psi'(x)b(t,x) + \frac{1}{2}\psi''(x)\sigma(t,x)\sigma^T(t,x) = 0, \quad \forall x \in \mathbb{R}, \forall t \in [0,T].$$

However, on the left-hand side of (4) does not depend on the variable t, but on the right-hand side of (4) maybe depend on the variable t. Thus in order to assure that (4) makes sense, we need to give some assumptions on $b(t,x), \sigma(t,x)$ and $f(t,x)$.

Summing up the above discussions, we arrive at the main result.

Theorem 2.1. *Assume that the functions $b(t,x), \sigma(t,x)$ and $f(t,x)$ are T-periodic functions in time satisfying*

$$\frac{2b(t,x)}{\sigma^2(t,x)} = a(t) + b_1(x),\ \ \frac{2f(t,x)}{\sigma^2(t,x)} = f_1(x), \tag{5}$$

where $a(t+T) = a(t)$, $b_1(x)$ and $f_1(x)$ are continuous functions. Then equation (1) has a periodic solution with periodic T and $v = 0$.

Proof. The existence of periodic solution of equation (1) is equivalent to the existence of ψ. It suffices to prove the existence of ψ. Using the assumptions (5) and (4), we have

$$\begin{aligned}\psi(x) &= \int \left(e^{\int P_t(x)dx}\int Q_t(x)e^{-\int P_t(x)dx}dx\right)dx \\ &= \int \left(e^{\int [a(t)+b_1(x)]dx}\int [f_1(x)]e^{-\int [a(t)+b_1(x)]dx}dx\right)dx \\ &= \int \left(e^{\int b_1(x)dx}\int [f_1(x)]e^{-\int b_1(x)dx}dx\right)dx,\end{aligned}$$

where we take $C = 0$ in (4). Due to the continuous of b_1 and f_1, the existence of ψ is obtained. The proof is complete. □

In Theorem 2.1, we consider the case that $v = 0$. In the following, we consider another case: $v(t+T) = v(t)$. Assume that $\beta(t) = \int v(t)dt$, then (2) will be written as

$$e^{\beta(T)-\beta(t)}f(t,x) + \psi'(x)b(t,x) + \frac{1}{2}\psi''(x)\sigma(t,x)\sigma^T(t,x) = 0, \quad \forall x \in \mathbb{R}, \forall t \in [0,T].$$

Then one can give similar assumptions to those of Theorem 2.1 to assure the existence of periodic solution of equation (1).

Example 1: Let $v(t,x) = 0$, $f(t,x) = (2+\sin t) \times x^2$, $b(t,x) = (2+\sin t) \times x$, $\sigma^2(t,x) =$

$(2+\sin t)$. We have

$$\frac{2b(t,x)}{\sigma^2(t,x)} = 2x = a(t) + b_1(x), \quad \frac{2f(t,x)}{\sigma^2(t,x)} = 2x^2 = f_1(x),$$

and

$$\psi(x) = \int \left(e^{\int b_1(x)dx} \int [f_1(x)] e^{-\int b_1(x)dx} dx \right) dx = -\frac{x^3}{3} - \frac{x^2}{2} + \frac{x}{2}.$$

Theorem 2.1 shows that equation (1) with the above f, b, σ admits a periodic solution. It is remarked that f does not satisfy the assumption of Proposition 1.1. More precisely, in Proposition 1.1, the function $g(x)$ is a bounded function but in Example 1, $g(x) = x^2$.

Example 2: Let $v(t,x) = \frac{\cos t}{2+\sin t}$, then $\beta(t) = \int v(t)dt = \ln(2+\sin t)$. Assume that

$$\frac{2b(t,x)}{\sigma^2(t,x)} = a(t) + b_1(x), \quad \frac{2f(t,x)(2+\sin T)}{\sigma^2(t,x)(2+\sin t)} = f_1(x),$$

where b_1 and f_1 are continuous functions. Then we can take

$$\psi(x) = \int \left(e^{\int b_1(x)dx} \int [f_1(x)] e^{-\int b_1(x)dx} dx \right) dx.$$

Theorem 2.1 shows that equation (1) with the above f, b, σ admits a periodic solution. It is remarked that f does not satisfy the assumption of Proposition 1.1.

By Itô formula, we have

$$\begin{aligned} \psi(T, X_T) &= \psi(0,x) + \int_0^T [\partial_t \psi(t, X_t) + \psi'(t, X_t) b(t, X_t)] dt \\ &\quad + \frac{1}{2} \int_0^T \psi''(t, X_t) \sigma(t, X_t) \sigma^T(t, X_t) dt + \int_0^T \psi'(t, X_t) \sigma(t, X_t) dW_t, \end{aligned}$$

which implies that

$$\mathbb{E}\psi(T, X_T) = \psi(0,x) + \mathbb{E} \int_0^T \left(\partial_t \psi(t, X_t) + \psi'(t, X_t) b(t, X_t) + \frac{1}{2} \psi''(t, X_t) \sigma(t, X_t) \sigma^T(t, X_t) \right) dt.$$

Thus if

$$\mathbb{E} \int_0^T \left(\partial_t \psi(t, X_t) + \psi'(t, X_t) b(t, X_t) + \frac{1}{2} \psi''(t, X_t) \sigma(t, X_t) \sigma^T(t, X_t) \right) dt = 0, \tag{6}$$

we have $\mathbb{E}\psi(T, X_T) = \psi(0,x)$. It is easy to see that if

$$\mathbb{E} \left(\partial_t \psi(t, X_t) + \psi'(t, X_t) b(t, X_t) + \frac{1}{2} \psi''(t, X_t) \sigma(t, X_t) \sigma^T(t, X_t) \right) = \alpha(t)$$

with $\alpha(t) = \beta'(t)$ and $\beta(t+T) = \beta(t)$, then we have

$$\begin{aligned} &\mathbb{E} \int_0^T \left(\partial_t \psi(t, X_t) + \psi'(t, X_t) b(t, X_t) + \frac{1}{2} \psi''(t, X_t) \sigma(t, X_t) \sigma^T(t, X_t) \right) dt \\ &= \int_0^T \alpha(t) dt = \beta(T) - \beta(0) = 0. \end{aligned}$$

Next, we give another sufficient condition to ensure that (6) holds. If the following second order different equation

$$\begin{cases} \partial_t \psi(t,x) + b(t,x)\psi'(t,x) + \frac{\sigma^2(t,x)}{2} \psi''(t,x) = 0, & t \in [0,T], \ x \in \mathbb{R}, \\ \psi(T,x) = \psi_T(x), & x \in \mathbb{R} \end{cases} \tag{7}$$

admits a continuous bounded solution on $[0,T] \times \mathbb{R}$, then (6) holds.

Theorem 2.2. *Assume that (7) admits a continuous bounded solution on* $[0,T]\times\mathbb{R}$*, then* $\mathbb{E}[\psi(t,X_t)]$ *is a* T*-periodic function.*

Note that the time periodic T is related to the solvability of (7). We give a special case, where T is any positive constant. Assume that $b(t,x)=\beta(t)b_1(x)$, $\sigma^2(t,x)=\beta(t)\sigma_1(x)$. Set $\psi(t,x)=e^{\int\beta(s)ds}\phi(x)$. If

$$\frac{\sigma_1(x)}{2}\phi''(x)+b_1(x)\phi'(x)+\phi(x)=0 \tag{8}$$

admit a continuous bounded solution, then (6) holds, where T is any positive constant. In particular, if $\sigma_1(x)=2,\ b_1(x)=-2$, we can solve (8). That is to say, we can take $\phi(x)=e^x$.

Declarations

Conflict of interest The authors declare no conflict of interest.

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