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A trace formula for the vector Sturm-Liouville operator with a constant delay

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Abstract. In this work, the vector differential operator with a delay variable is studied and the regularized trace formula of the operator is obtained.

§1 Introduction

The earliest result for the trace formula of the Sturm-Liouville differential equation was obtained by Gelfand and Levitan in [7]. Since then, research on the trace formula of differential operators has been carried out in a variety of fields. For instance, the Sturm-Liouville issues on a graph were studied in [6,11,12] and the problem of differential operators with discontinuity was investigated in [2,10,21,23]. The increasing research on the trace formula of matrix differential operators is also attributed to its wide applications in physics, such as the propagation of electromagnetic waves. In addition, more relevant studies on the theory of traces can be found in [4,5,20]. Recently, differential equations with retarded arguments have attracted a lot of attention (see [1,9,14,16,17]) due to their widespread applications in the theory of selfoscillatory systems and other places. Pikula investigated Sturm-Liouville problem with retarded argument and general separation boundary conditions, and obtained the trace formula in [13]. A discontinuous boundary value problem with a delay variable and transmission conditions at the points of discontinuity, was studied in [18]. Then a Sturm-Liouville problem with constant delays on a star graph was considered in [19].

In this work, we consider the vector differential problem with a retarded argument

$$-y''(x)+q(x)y(x-h)=\lambda y(x),\ x\in(0,\pi), \tag{1}$$

and boundary conditions

$$y'(0)=\vartheta y(\pi)+\sigma y(0),\ y'(\pi)=\psi y(\pi)+\eta y(0), \tag{2}$$

where $\vartheta,\sigma,\psi,\eta$ are $m\times m$ symmetric scalar matrices, $h\in(0,\pi)$, $q(x)$ is a $m\times m$ symmetric matrix-valued function in $W_2^1(h,\pi)$ and $q(x)=0$ *a.e.* on $(0,h)$.

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The corresponding scalar case ($m = 1,\ h = 0$) was considered by the authors in [3], and Yang studied the boundary conditions of problems (1) and (2), which did not contain the constant delay h in [22]. As a generalization of the scalar Sturm-Liouville equation, the matrix case is more difficult to investigate, see details in [15]. The remainder of this paper is organized as follows. Firstly, we obtain the asymptotic expression of the characteristic function. Then, we get the regularized trace formula of the differential operator by using the contour integral method.

In this work, we will find a regularized trace for the problem (1) and (2).

Theorem 1. *If $\lambda_n^{(k)}, n \geq 0, k = 1, 2, \cdots, m$, is the spectrum of the problem (1)-(2), then the trace formula has the following expression:*

$$\sum_{n=0}^{\infty}\left\{\sum_{k=1}^{m}\left(\lambda_n^{(k)} - n^2\right) - \frac{\cos(nh)}{\pi} tr \int_h^{\pi} q(\xi)d\xi - \frac{2}{\pi} tr\Big(\sigma - \psi + (-1)^n(\vartheta - \eta)\Big)\right\}$$
$$= -\frac{1}{\pi} tr\left[\sigma - \psi + \vartheta - \eta + \frac{1}{2}\int_h^{\pi} q(\xi)d\xi\right] - \frac{1}{2} tr[\sigma^2 + \psi^2] + tr(\eta\vartheta).$$

§2 Proof

Step 1: Calculate the characteristic function of the problem

Suppose $C(x, \lambda)$ and $S(x, \lambda)$ are the solutions of equation (1) with the initial conditions $C(0, \lambda) = I_m,\ C'(0, \lambda) = 0$ and $S(0, \lambda) = 0,\ S'(0, \lambda) = I_m$, respectively, where I_m is a $m \times m$ identity matrix. The functions $C(x, \lambda)$ and $S(x, \lambda)$ satisfy the integral equations

$$C(x, \lambda) = \cos(\rho x) I_m + \int_0^x \frac{\sin[\rho(x - t)]}{\rho} q(t) C(t - h, \lambda) dt,$$

and

$$S(x, \lambda) = \frac{\sin(\rho x)}{\rho} I_m + \int_0^x \frac{\sin[\rho(x - t)]}{\rho} q(t) S(t - h, \lambda) dt,$$

respectively, where $\lambda = \rho^2$. Using integral by parts, it then follows from [8] that for $h \in (0, \frac{\pi}{2})$,

$$\begin{aligned} C(x, \lambda) = & \cos(\rho x) I_m + \frac{\sin[\rho(x - h)]}{2\rho} \int_h^x q(\xi) d\xi \\ & + \frac{\cos[\rho(x - h)]}{4\rho^2} [q(x) - q(h)] \\ & - \frac{1}{4\rho^2} \int_h^x \cos[\rho(x - 2\xi + h)] q'(\xi) d\xi \\ & - \frac{\cos[\rho(x - 2h)]}{4\rho^2} \int_{2h}^x q(\xi) \int_h^{\xi - h} q(\varsigma) d\varsigma d\xi + o\left(\frac{e^{|\Im\rho|x}}{\rho^2}\right), \end{aligned} \tag{3}$$

$$\begin{aligned} S(x, \lambda) = & \frac{\sin(\rho x)}{\rho} I_m - \frac{\cos[\rho(x - h)]}{2\rho^2} \int_h^x q(\xi) d\xi \\ & + \frac{\sin[\rho(x - h)]}{4\rho^3} [q(x) + q(h)] \\ & - \frac{\sin[\rho(x - 2h)]}{4\rho^3} \int_{2h}^x q(\xi) \int_h^{\xi - h} q(\varsigma) d\varsigma d\xi \end{aligned} \tag{4}$$

$$
\begin{aligned}
&+\frac{1}{4\rho^3}\int_h^x \sin[\rho(x-2\xi+h)]q'(\xi)d\xi+o\left(\frac{e^{|\Im\rho|x}}{\rho^3}\right),\\
C'(x,\lambda)=&-\rho\sin(\rho x)I_m+\frac{\cos[\rho(x-h)]}{2}\int_h^x q(\xi)d\xi\\
&+\frac{\sin[\rho(x-h)]}{4\rho}[q(x)-q(h)] \qquad (5)\\
&+\frac{1}{4\rho}\int_h^x \sin[\rho(x-2\xi+h)]q'(\xi)d\xi\\
&+\frac{\sin[\rho(x-2h)]}{4\rho}\int_{2h}^x q(\xi)\int_h^{\xi-h}q(\varsigma)d\varsigma d\xi+o\left(\frac{e^{|\Im\rho|x}}{\rho}\right),\\
S'(x,\lambda)=&\cos(\rho x)I_m+\frac{\sin[\rho(x-h)]}{2\rho}\int_h^x q(\xi)d\xi\\
&-\frac{\cos[\rho(x-h)]}{4\rho^2}[q(x)-q(h)] \qquad (6)\\
&-\frac{\cos[\rho(x-2h)]}{4\rho^2}\int_{2h}^x q(\xi)\int_h^{\xi-h}q(\varsigma)d\varsigma d\xi\\
&+\frac{1}{4\rho^2}\int_h^x \cos[\rho(x-2\xi+h)]q'(\xi)d\xi+o\left(\frac{e^{|\Im\rho|x}}{\rho^2}\right),
\end{aligned}
$$

and for $h\in[\frac{\pi}{2},\pi)$, we have

$$
\begin{aligned}
C(x,\lambda)=&\cos(\rho x)I_m+\frac{\sin[\rho(x-h)]}{2\rho}\int_h^x q(\xi)d\xi\\
&+\frac{\cos[\rho(x-h)]}{4\rho^2}[q(x)-q(h)] \qquad (7)\\
&-\frac{1}{4\rho^2}\int_h^x \cos[\rho(x-2\xi+h)]q'(\xi)d\xi+o\left(\frac{e^{|\Im\rho|x}}{\rho^2}\right),\\
S(x,\lambda)=&\frac{\sin(\rho x)}{\rho}I_m-\frac{\cos[\rho(x-h)]}{2\rho^2}\int_h^x q(\xi)d\xi\\
&+\frac{\sin[\rho(x-h)]}{4\rho^3}[q(x)+q(h)] \qquad (8)\\
&+\frac{1}{4\rho^3}\int_h^x \sin[\rho(x-2\xi+h)]q'(\xi)d\xi+o\left(\frac{e^{|\Im\rho|x}}{\rho^3}\right),\\
C'(x,\lambda)=&-\rho\sin(\rho x)I_m+\frac{\cos[\rho(x-h)]}{2}\int_h^x q(\xi)d\xi\\
&+\frac{\sin[\rho(x-h)]}{4\rho}[q(x)-q(h)] \qquad (9)\\
&+\frac{1}{4\rho}\int_h^x \sin[\rho(x-2\xi+h)]q'(\xi)d\xi+o\left(\frac{e^{|\Im\rho|x}}{\rho}\right),\\
S'(x,\lambda)=&\cos(\rho x)I_m+\frac{\sin[\rho(x-h)]}{2\rho}\int_h^x q(\xi)d\xi\\
&-\frac{\cos[\rho(x-h)]}{4\rho^2}[q(x)-q(h)] \qquad (10)
\end{aligned}
$$

$$+\frac{1}{4\rho^2}\int_h^x \cos[\rho(x-2\xi+h)]q'(\xi)d\xi + o\left(\frac{e^{|\Im\rho|x}}{\rho^2}\right).$$

We see that the eigenvalues of the problem (1) and (2) are determined by whether the matrix-valued function (Ref. [22])

$$\varpi(\lambda) = C'(\pi,\lambda) + \sigma S'(\pi,\lambda) - \psi C(\pi,\lambda) + (\eta\vartheta - \psi\sigma)S(\pi,\lambda) + \vartheta - \eta$$

is singular or not. Then, we can get the following characteristic equation

$$\begin{aligned}\varpi(\lambda) =& -\sqrt{\lambda}\sin(\sqrt{\lambda}\pi)I_m + \cos(\sqrt{\lambda}\pi)(\sigma-\psi)\\ &+\cos[\sqrt{\lambda}(\pi-h)]R(\pi) + \vartheta - \eta + \frac{\sin(\sqrt{\lambda}\pi)}{\sqrt{\lambda}}(\eta\vartheta-\psi\sigma)\\ &+\frac{\sin[\sqrt{\lambda}(\pi-h)]}{\sqrt{\lambda}}\left[\frac{q(\pi)+q(h)}{4} + R(\pi)(\sigma-\psi)\right]\\ &+\frac{\sin[\sqrt{\lambda}(\pi-2h)]}{\sqrt{\lambda}}B(\pi) + \frac{1}{4\sqrt{\lambda}}\int_h^\pi \sin[\sqrt{\lambda}(\pi-2\xi+h)]q'(\xi)d\xi\\ &-\frac{\cos[\sqrt{\lambda}(\pi-h)]}{\lambda}\left[\frac{q(\pi)+q(h)}{4}(\sigma+\psi) + (\eta\vartheta-\psi\sigma)R(\pi)\right]\\ &+\frac{\psi-\sigma}{\lambda}\cos[\sqrt{\lambda}(\pi-2h)]B(\pi)\\ &+\frac{(\sigma+\psi)}{4\lambda}\int_h^\pi \cos[\sqrt{\lambda}(\pi-2\xi+h)]q'(\xi)d\xi + o\left(\frac{e^{|\Im\rho|\pi}}{\rho^3}\right),\end{aligned} \tag{11}$$

where

$$B(\pi) = \begin{cases} \frac{1}{4}\int_{2h}^\pi q(\xi)\int_h^{\xi-h} q(\varsigma)d\varsigma d\xi, & for\ 0<h<\pi/2,\\ 0, & for\ \pi/2 \le h < \pi,\end{cases}$$

$$R(\pi) = \frac{1}{2}\int_h^\pi q(\xi)d\xi.$$

Remark 3.1 Since $q(x)$ is a $m\times m$ symmetric matrix-valued function in $W_2^1(h,\pi)$, there is $\psi\sigma C(\pi,\lambda)S(\pi,\lambda) = \psi\sigma S(\pi,\lambda)C(\pi,\lambda)$.

Step 2: Expand characteristic determinant $\det\varpi(\lambda)$

Let

$$\varpi_0(\lambda) = \big(-\sqrt{\lambda}\sin(\sqrt{\lambda}\pi)\big)^m.$$

Obviously, the zeros of $\varpi_0(\lambda)$ are

$$\mu_n^{(k)} = n^2,\ n\ge 0,\ k=1,2,\cdots,m.$$

Consider the counterclockwise square contour Γ_N with the vertices

$$E = (N+\frac{1}{2})^2 + (N+\frac{1}{2})^2 i,\ F = -(N+\frac{1}{2})^2 + (N+\frac{1}{2})^2 i,$$
$$G = -(N+\frac{1}{2})^2 - (N+\frac{1}{2})^2 i,\ H = (N+\frac{1}{2})^2 - (N+\frac{1}{2})^2 i.$$

From (11), on the contour Γ_N, we obtain

$$\frac{\det\varpi(\lambda)}{\varpi_0(\lambda)} = \det\left[I_m - \frac{\cot(\sqrt{\lambda}\pi)}{\sqrt{\lambda}}(\sigma-\psi) - \frac{\cos[\sqrt{\lambda}(\pi-h)]}{\sqrt{\lambda}\sin(\sqrt{\lambda}\pi)}R(\pi)\right.$$

$$
\begin{aligned}
&-\frac{1}{\sqrt{\lambda}\sin(\sqrt{\lambda}\pi)}(\vartheta-\eta)-\frac{1}{\lambda}(\eta\vartheta-\psi\sigma)\\
&-\frac{\sin[\sqrt{\lambda}(\pi-h)]}{\sqrt{\lambda}\sin(\sqrt{\lambda}\pi)}\left(\frac{q(\pi)+q(h)}{4}+R(\pi)(\sigma-\psi)\right)\\
&-\frac{\sin[\sqrt{\lambda}(\pi-2h)]}{\sqrt{\lambda}\sin(\sqrt{\lambda}\pi)}B(\pi)+o\left(\frac{1}{\lambda\sqrt{\lambda}}\right)\Bigg]\\
=&\prod_{i=1}^{N}\Bigg[1-\frac{\cot(\sqrt{\lambda}\pi)}{\sqrt{\lambda}}(\sigma-\psi)_{jj}-\frac{\cos[\sqrt{\lambda}(\pi-h)]}{\sqrt{\lambda}\sin(\sqrt{\lambda}\pi)}(R(\pi))_{jj}\\
&-\frac{1}{\sqrt{\lambda}\sin(\sqrt{\lambda}\pi)}(\vartheta-\eta)_{jj}-\frac{1}{\lambda}(\eta\vartheta-\psi\sigma)_{jj}\\
&-\frac{\sin[\sqrt{\lambda}(\pi-h)]}{\lambda\sin(\sqrt{\lambda}\pi)}\left(\frac{q(\pi)+q(h)}{4}+R(\pi)(\sigma-\psi)\right)_{jj}\\
&-\frac{\sin[\sqrt{\lambda}(\pi-2h)]}{\lambda\sin(\sqrt{\lambda}\pi)}(B(\pi))_{jj}+o\left(\frac{1}{\lambda\sqrt{\lambda}}\right)\Bigg]\\
&+\frac{b}{\lambda\sin^2(\sqrt{\lambda}\pi)}+o\left(\frac{1}{\lambda\sqrt{\lambda}}\right),
\end{aligned}
\tag{12}
$$

where f_{jk} is determined by the number of rows j and columns k of matrix f, $j,k=1,2,\cdots,m$ and

$$
\begin{aligned}
b=&-\cos^2[\sqrt{\lambda}(\pi-h)]\sum_{j=1}^{m}\sum_{j<k}(R(\pi))_{jk}(R(\pi))_{kj}\\
&-\cos^2(\sqrt{\lambda}\pi)\sum_{j=1}^{m}\sum_{j<k}(\sigma-\psi)_{jk}(\sigma-\psi)_{kj}-\sum_{j=1}^{m}\sum_{j<k}(\vartheta-\eta)_{jk}(\vartheta-\eta)_{kj}\\
&-\cos(\sqrt{\lambda}\pi)\cos[\sqrt{\lambda}(\pi-h)]\sum_{j=1}^{m}\sum_{j<k}\big[(\sigma-\psi)_{jk}(R(\pi))_{kj}\\
&+(\sigma-\psi)_{kj}(R(\pi))_{jk}\big]\\
&-\cos(\sqrt{\lambda}\pi)\sum_{j=1}^{m}\sum_{j<k}[(\sigma-\psi)_{jk}(\vartheta-\eta)_{kj}+(\sigma-\psi)_{kj}(\vartheta-\eta)_{jk}]\\
&-\cos[\sqrt{\lambda}(\pi-h)]\sum_{j=1}^{m}\sum_{j<k}[(R(\pi))_{jk}(\vartheta-\eta)_{kj}+(R(\pi))_{kj}(\vartheta-\eta)_{jk}].
\end{aligned}
$$

Consequently, we have

$$
\begin{aligned}
\frac{\det\varpi(\lambda)}{\varpi_0(\lambda)}=&1-\frac{\cot(\sqrt{\lambda}\pi)}{\sqrt{\lambda}}\sum_{j=1}^{m}(\sigma-\psi)_{jj}-\frac{\cos[\sqrt{\lambda}(\pi-h)]}{\sqrt{\lambda}\sin(\sqrt{\lambda}\pi)}\sum_{j=1}^{m}(R(\pi))_{jj}\\
&-\frac{1}{\sqrt{\lambda}\sin(\sqrt{\lambda}\pi)}\sum_{j=1}^{m}(\vartheta-\eta)_{jj}-\frac{1}{\lambda}\sum_{j=1}^{m}(\eta\vartheta-\psi\sigma)_{jj}\\
&-\frac{\sin[\sqrt{\lambda}(\pi-h)]}{\lambda\sin(\sqrt{\lambda}\pi)}\sum_{j=1}^{m}\left[\frac{q(\pi)+q(h)}{4}+R(\pi)(\sigma-\psi)\right]_{jj}
\end{aligned}
$$

$$
\begin{aligned}
&-\frac{\sin[\sqrt{\lambda}(\pi-2h)]}{\lambda\sin(\sqrt{\lambda}\pi)}\sum_{j=1}^{m}(B(\pi))_{jj}\\
&+\frac{\cos^2(\sqrt{\lambda}\pi)}{\lambda\sin^2(\sqrt{\lambda}\pi)}\sum_{j=1}^{m}\sum_{j<k}(\sigma-\psi)_{jj}(\sigma-\psi)_{kk}\\
&+\frac{\cos^2[\sqrt{\lambda}(\pi-h)]}{\lambda\sin^2(\sqrt{\lambda}\pi)}\sum_{j=1}^{m}\sum_{j<k}(R(\pi))_{jj}(R(\pi))_{kk}\\
&+\frac{1}{\lambda\sin^2(\sqrt{\lambda}\pi)}\sum_{j=1}^{m}\sum_{j<k}(\vartheta-\eta)_{jj}(\vartheta-\eta)_{kk}\\
&+\frac{\cos(\sqrt{\lambda}\pi)\cos[\sqrt{\lambda}(\pi-h)]}{\lambda\sin^2(\sqrt{\lambda}\pi)}\sum_{j=1}^{m}\sum_{j<k}\left[(\sigma-\psi)_{jj}(R(\pi))_{kk}\right.\\
&\left.+(\sigma-\psi)_{kk}(R(\pi))_{jj}\right]\\
&+\frac{\cos(\sqrt{\lambda}\pi)}{\lambda\sin^2(\sqrt{\lambda}\pi)}\sum_{j=1}^{m}\sum_{j<k}\left[(\sigma-\psi)_{jj}(\vartheta-\eta)_{kk}+(\sigma-\psi)_{kk}(\vartheta-\eta)_{jj}\right]\\
&+\frac{\cos[\sqrt{\lambda}(\pi-h)]}{\lambda\sin^2(\sqrt{\lambda}\pi)}\sum_{j=1}^{m}\sum_{j<k}\left[(R(\pi))_{jj}(\vartheta-\eta)_{kk}+(R(\pi))_{kk}(\vartheta-\eta)_{jj}\right]\\
&+\frac{b}{\lambda\sin^2(\sqrt{\lambda}\pi)}+o\left(\frac{1}{\lambda\sqrt{\lambda}}\right). \qquad (13)
\end{aligned}
$$

Using Maclaurin formula to expand $\ln\frac{\det\varpi(\lambda)}{\varpi_0(\lambda)}$, we obtain

$$
\begin{aligned}
\ln\frac{\det\varpi(\lambda)}{\varpi_0(\lambda)}=&-\frac{\cot(\sqrt{\lambda}\pi)}{\sqrt{\lambda}}\sum_{j=1}^{m}(\sigma-\psi)_{jj}-\frac{\cos[\sqrt{\lambda}(\pi-h)]}{\sqrt{\lambda}\sin(\sqrt{\lambda}\pi)}\sum_{j=1}^{m}(R(\pi))_{jj}\\
&-\frac{1}{\sqrt{\lambda}\sin(\sqrt{\lambda}\pi)}\sum_{j=1}^{m}(\vartheta-\eta)_{jj}-\frac{1}{\lambda}\sum_{j=1}^{m}(\eta\vartheta-\psi\sigma)_{jj}\\
&-\frac{\sin[\sqrt{\lambda}(\pi-h)]}{\lambda\sin(\sqrt{\lambda}\pi)}\sum_{j=1}^{m}\left[\frac{q(\pi)+q(h)}{4}+R(\pi)(\sigma-\psi)\right]_{jj}\\
&-\frac{\sin[\sqrt{\lambda}(\pi-2h)]}{\lambda\sin(\sqrt{\lambda}\pi)}\sum_{j=1}^{m}(B(\pi))_{jj}-\frac{\cos^2(\sqrt{\lambda}\pi)}{2\lambda\sin^2(\sqrt{\lambda}\pi)}\sum_{j=1}^{m}(\sigma-\psi)_{jj}^2\\
&-\frac{\cos^2[\sqrt{\lambda}(\pi-h)]}{2\lambda\sin^2(\sqrt{\lambda}\pi)}\sum_{j=1}^{m}(R(\pi))_{jj}^2-\frac{1}{2\lambda\sin^2(\sqrt{\lambda}\pi)}\sum_{j=1}^{m}(\vartheta-\eta)_{jj}^2\\
&+\frac{\cos(\sqrt{\lambda}\pi)\cos[\sqrt{\lambda}(\pi-h)]}{\lambda\sin^2(\sqrt{\lambda}\pi)}\sum_{j=1}^{m}\sum_{j<k}\left[(\sigma-\psi)_{jj}(R(\pi))_{kk}\right.\\
&\left.+(\sigma-\psi)_{kk}(R(\pi))_{jj}\right]\\
&+\frac{\cos(\sqrt{\lambda}\pi)}{\lambda\sin^2(\sqrt{\lambda}\pi)}\sum_{j=1}^{m}\sum_{j<k}\left[(\sigma-\psi)_{jj}(\vartheta-\eta)_{kk}+(\sigma-\psi)_{kk}(\vartheta-\eta)_{jj}\right]
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{\cos[\sqrt{\lambda}(\pi-h)]}{\lambda\sin^2(\sqrt{\lambda}\pi)}\sum_{j=1}^{m}\sum_{j<k}[(R(\pi))_{jj}(\vartheta-\eta)_{kk}+(R(\pi))_{kk}(\vartheta-\eta)_{jj}]\\
&-\frac{\cos(\sqrt{\lambda}\pi)\cos[\sqrt{\lambda}(\pi-h)]}{\lambda\sin^2(\sqrt{\lambda}\pi)}\sum_{j=1}^{m}(\sigma-\psi)_{jj}\times\sum_{j=1}^{m}(R(\pi))_{jj}\\
&-\frac{\cos(\sqrt{\lambda}\pi)}{\lambda\sin^2(\sqrt{\lambda}\pi)}\sum_{j=1}^{m}(\sigma-\psi)_{jj}\times\sum_{j=1}^{m}(\vartheta-\eta)_{jj}\\
&-\frac{\cos[\sqrt{\lambda}(\pi-h)]}{\lambda\sin^2(\sqrt{\lambda}\pi)}\sum_{j=1}^{m}(R(\pi))_{jj}\times\sum_{j=1}^{m}(\vartheta-\eta)_{jj}\\
&+\frac{b}{\lambda\sin^2(\sqrt{\lambda}\pi)}+o\left(\frac{1}{\lambda\sqrt{\lambda}}\right). \qquad (14)
\end{aligned}
$$

Step 3: Derivation of trace formula

Using Rouché's theorem, the number of zeros of $\det\varpi(\lambda)$ and $\varpi_0(\lambda)$ are the same inside the contour Γ_N, then we can get the following formula

$$
\begin{aligned}
\sum_{\Gamma_N}(\lambda_n^{(k)}-\mu_n^{(k)}) &= \frac{1}{2\pi i}\oint_{\Gamma_N}\lambda\left[\frac{\det(\varpi(\lambda))'}{\det\varpi(\lambda)}-\frac{(\varpi_0(\lambda))'}{\varpi_0(\lambda)}\right]d\lambda\\
&=\frac{1}{2\pi i}\oint_{\Gamma_N}\lambda d\ \ln\frac{\det\varpi(\lambda)}{\varpi_0(\lambda)} \qquad (15)\\
&=-\frac{1}{2\pi i}\oint_{\Gamma_N}\ln\frac{\det\varpi(\lambda)}{\varpi_0(\lambda)}d\lambda.
\end{aligned}
$$

By directly calculating residues, it follows that

$$
\begin{aligned}
\frac{1}{2\pi i}\oint_{\Gamma_N}\frac{\cot(\sqrt{\lambda}\pi)}{\sqrt{\lambda}}d\lambda &=\frac{1}{\pi}+\frac{2}{\pi}\sum_{n=1}^{N}1,\\
\frac{1}{2\pi i}\oint_{\Gamma_N}\frac{\cos[\sqrt{\lambda}(\pi-h)]}{\sqrt{\lambda}\sin(\sqrt{\lambda}\pi)}d\lambda &=\frac{1}{\pi}+\frac{2}{\pi}\sum_{n=1}^{N}\cos(nh),\\
\frac{1}{2\pi i}\oint_{\Gamma_N}\frac{1}{\sqrt{\lambda}\sin(\sqrt{\lambda}\pi)}d\lambda &=\frac{1}{\pi}+\frac{2}{\pi}\sum_{n=1}^{N}(-1)^n,\\
\frac{1}{2\pi i}\oint_{\Gamma_N}\frac{\sin[\sqrt{\lambda}(\pi-h)]}{\lambda\sin(\sqrt{\lambda}\pi)}d\lambda &=A_1(h,n),\\
\frac{1}{2\pi i}\oint_{\Gamma_N}\frac{\sin[\sqrt{\lambda}(\pi-2h)]}{\lambda\sin(\sqrt{\lambda}\pi)}d\lambda &=B_1(h,n),\\
\frac{1}{2\pi i}\oint_{\Gamma_N}\frac{\cot^2(\sqrt{\lambda}\pi)}{\lambda}d\lambda &=-1+O\left(\frac{1}{N}\right), \qquad (16)\\
\frac{1}{2\pi i}\oint_{\Gamma_N}\frac{\cos(\sqrt{\lambda}\pi)\cos[\sqrt{\lambda}(\pi-h)]}{\lambda\sin^2(\sqrt{\lambda}\pi)}d\lambda &=C_1(h,n)+O\left(\frac{1}{N^2}\right),\\
\frac{1}{2\pi i}\oint_{\Gamma_N}\frac{\cos^2[\sqrt{\lambda}(\pi-h)]}{\lambda\sin^2(\sqrt{\lambda}\pi)}d\lambda &=D_1(h,n)+O\left(\frac{1}{N^2}\right),
\end{aligned}
$$

$$\frac{1}{2\pi i}\oint_{\Gamma_N}\frac{1}{\lambda\sin^2(\sqrt{\lambda}\pi)}d\lambda = O\left(\frac{1}{N}\right),$$

$$\frac{1}{2\pi i}\oint_{\Gamma_N}\frac{\cos(\sqrt{\lambda}\pi)}{\lambda\sin^2(\sqrt{\lambda}\pi)}d\lambda = O\left(\frac{1}{N^{3/2}}\right),$$

$$\frac{1}{2\pi i}\oint_{\Gamma_N}\frac{\cos[\sqrt{\lambda}(\pi-h)]}{\lambda\sin^2(\sqrt{\lambda}\pi)}d\lambda = O\left(\frac{1}{N^{3/2}}\right),$$

where

$$A_1(h,n) = 1-\frac{a}{\pi}-\sum_{n=1}^{N}\frac{2\sin(na)}{n\pi},$$

$$B_1(h,n) = 1-\frac{2a}{\pi}-\sum_{n=1}^{N}\frac{2\sin(2na)}{n\pi},$$

$$C_1(h,n) = \frac{1}{3}-\frac{\pi^2+(\pi-a)^2}{2\pi^2}+\frac{2}{\pi^2}\sum_{n=1}^{N}\left[\frac{(\pi-a)\sin(na)}{n}-\frac{\cos(na)}{n^2}\right],$$

$$D_1(h,n) = -\frac{(\pi-a)^2}{\pi^2}+\sum_{n=1}^{N}\left[\frac{1}{n\pi^2}-\frac{\cos(2na)}{n^2\pi^2}+\frac{2(\pi-a)\sin(2na)}{n\pi^2}\right].$$

Putting the formulas (14), (16) into (15), we obtain

$$\begin{aligned}
&\sum_{n=0}^{N}\sum_{k=1}^{m}\left(\lambda_n^{(k)}-n^2\right)-\frac{2}{\pi}\sum_{n=0}^{N}tr\Big(\sigma-\psi+\cos(nh)R(\pi)+(-1)^n(\vartheta-\eta)\Big)\\
&=-\frac{1}{\pi}tr\Big[\sigma-\psi+R(\pi)+\vartheta-\eta\Big]+tr[\eta\vartheta-\psi\sigma]\\
&\quad+A_1(h,n)tr\left[\frac{q(\pi)+q(h)}{4}+R(\pi)(\sigma-\psi)\right]\\
&\quad+B_1(h,n)tr(B(\pi))-\frac{1}{2}tr(\sigma-\psi)^2+\frac{1}{2}D_1(h,n)tr(R(\pi))^2\\
&\quad-C_1(h,n)\sum_{j=1}^{m}\sum_{j<k}\Big[(\sigma-\psi)_{jj}(R(\pi))_{kk}+(\sigma-\psi)_{kk}(R(\pi))_{jj}\\
&\quad-(\sigma-\psi)_{jk}(R(\pi))_{kj}-(\sigma-\psi)_{kj}(R(\pi))_{jk}\Big]\\
&\quad+C_1(h,n)tr(\sigma-\psi)\times tr(R(\pi))+O\left(\frac{1}{N}\right).
\end{aligned}\tag{17}$$

Taking the limit as $N\to\infty$ in (17), and applying the Fourier series expansions

$$\sum_{n=1}^{\infty}\frac{\sin nh}{n}=\frac{\pi-h}{2},\ \sum_{n=1}^{\infty}\frac{\cos nh}{n^2}=\frac{\pi^2}{6}-\frac{\pi h}{2}+\frac{h^2}{4},\ 0<h<\pi$$

we obtain

$$\begin{aligned}
&\sum_{n=0}^{\infty}\sum_{k=1}^{m}\left(\lambda_n^{(k)}-n^2\right)-\frac{2}{\pi}\sum_{n=0}^{\infty}tr\Big(\sigma-\psi+\cos(nh)R(\pi)+(-1)^n(\vartheta-\eta)\Big)\\
&=-\frac{1}{\pi}tr\Big[\sigma-\psi+R(\pi)+\vartheta-\eta\Big]+tr[\eta\vartheta-\psi\sigma]-\frac{1}{2}tr(\sigma-\psi)^2.
\end{aligned}$$

Simplifying the above formula, the proof is finished.

§3 Conclusion

This paper mainly studies the trace formula of vector differential operator with a delay variable. Specifically, we firstly calculate the asymptotic estimation formula of eigenvalue by the successive approximation method and Laplace expansion of determinant, and on the basis, we obtain the trace formula by the Maclaurin formula and residue theorem.

Our work is a new generalization of the trace formula of vector differential operator with coupled boundary conditions studied by Yang [22]. Under the assumption of the retardation $h = 0$ in (1) and $m = 1$, we obtain the formula of the first regularized trace for the classical Sturm-Liouville operator (see [3]). The trace identity of differential operators profoundly reveals the spectral structure of differential operators and has important applications in the numerical calculation of eigenvalues, soliton theory and integrable system theory. Especially, it is an important tool for inverse spectral problems.

Declarations

Conflict of interest The authors declare no conflict of interest.

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