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On the monotonicity of limit wave speed to a perturbed gKdV equation

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Abstract. This paper deals with the monotonicity of limit wave speed $c_0(h)$ to a perturbed gKdV equation. We show the decrease of $c_0(h)$ by combining the analytic method and the numerical technique. Our results solve a special case of the open question presented by Yan et al., and the method potentially provides a way to study the monotonicity of $c_0(h)$ for general $m \in \mathbb{N}^+$.

§1 Introduction

Many shallow water wave models have been found in the field of water wave dynamics, which play an important role in describing natural phenomena [1–4]. In recent years, there exist more and more works concerning the perturbed models and their solutions due to their realistic nature [5–13], among which, the perturbed KdV equation and its variants have gained considerate concern since they are significant in physics and waves contexts.

In 2014, Yan et al. [14] showed the persistence of solitary waves and periodic waves to the perturbed generalized KdV (gKdV) equation

$$v_t + v^m v_x + v_{xxx} + \sigma(v_{xx} + v_{xxxx}) = 0, \tag{1}$$

for general positive integer m and sufficiently small $\sigma > 0$. When $m = 1$, we have the perturbed KdV equation

$$v_t + vv_x + v_{xxx} + \sigma(v_{xx} + v_{xxxx}) = 0, \tag{2}$$

which was investigated by Owaga [15]. Owaga [15] derived the existence of the above two types of waves to (2), and showed that $c_0(h)$ is decreasing.

Naturally, one may wonder how about the monotonicity of $c_0(h)$ to Eq.(1) with general $m \in \mathbb{N}^+$, which was presented as an open question in [14]. In fact, the monotonicity of $c_0(h)$ for $m = 1, 2, 3, 4$ had been shown [15–17]. However, as suggested in [17], for $m \geq 5$, it is difficult

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to show $(J'_m/J'_0)' > 0$ analytically (J_m is the Abelian integral defined later), which is a crucial result for later analysis, due to the complexity of the Ricatti equations. To find a way to address the monotonicity of $c_0(h)$ for odd m, in this paper, we first focus on a special case when $m = 5$.

By the transformation $\zeta = x - ct$ with the wave speed $c > 0$, Eq.(1) with $m = 5$ can be converted into

$$-cv'(\zeta) + v^5(\zeta)v'(\zeta) + v'''(\zeta) + \sigma(v''(\zeta) + v''''(\zeta)) = 0, \tag{3}$$

and it follows that

$$-cv(\zeta) + \frac{1}{6}v^6(\zeta) + v''(\zeta) + \sigma(v'(\zeta) + v'''(\zeta)) = 0, \tag{4}$$

by integrating.

Introducing $\eta = \sqrt{c}\zeta$, $v = \sqrt[5]{c}X$, we obtain

$$-X(\eta) + \frac{1}{6}X^6(\eta) + X''(\eta) + \sigma\left(\frac{1}{\sqrt{c}}X'(\eta) + \sqrt{c}X'''(\eta)\right) = 0. \tag{5}$$

If we set $\sigma = 0$, then (5) becomes the unperturbed system

$$X'' + \frac{1}{6}X^6 - X = 0, \tag{6}$$

which can be rewritten as a planar system

$$\begin{cases} \frac{\mathrm{d}X}{\mathrm{d}\eta} = Y, \\ \frac{\mathrm{d}Y}{\mathrm{d}\eta} = X - \frac{1}{6}X^6, \end{cases} \tag{7}$$

whose first integral is

$$H(X,Y) = -Y^2 + X^2 - \frac{1}{21}X^7 = h. \tag{8}$$

Note that system (7) has one saddle $(0,0)$ and one center $(\sqrt[5]{6}, 0)$, and its phase portrait is shown in Figure 1. Besides, the traveling waves of the system (7) can be parameterized through h. Exploiting this parametrization, we give the monotonicity of $c_0(h)$ in Theorem 1.

Theorem 1. *For $h \in [0, \frac{5\sqrt[5]{36}}{7})$, the perturbed gKdV equation (1) with $m = 5$ and sufficiently small $\sigma > 0$ has a traveling wave $v = \sqrt[5]{c}X(\sigma, h, c, \eta)$, where $X(\sigma, h, c, \eta)$ is the solution of (5), and $c = c(\sigma, h)$ depends on σ and h smoothly, with the limit $c_0(h)$ as $\sigma \to 0$. Furthermore, $c_0(h)$ is a smooth decreasing function for $h \in [0, \frac{5\sqrt[5]{36}}{7})$.*

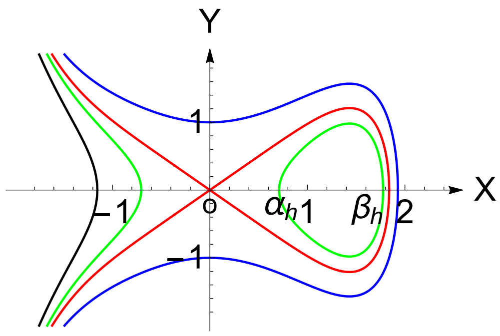

Figure 1. The phase portrait of system (7).

§2 The theoretic derivations of the monotonicity of $c_0(h)$

In this section, we focus on theoretic derivations of the monotonicity of $c_0(h)$ by applying the Abelian integral theory and numerical technique, see [14] for other results. Suppose that $X(\eta)$ is the solution of the system (7), and we define Q and R as

$$Q = \frac{1}{2}\int (X'')^2 \mathrm{d}\eta,\ R = \frac{1}{2}\int (X')^2 \mathrm{d}\eta,$$

where the integrals are performed along the orbits of system (7). Then from [14], it is known that $c_0(h)$ can be expressed as

$$c_0(h) = \frac{1}{Z(h)},$$

where

$$Z(h) = \frac{Q}{R}.$$

Assume that $\alpha(h)$ and $\beta(h)$ $(0 \le \alpha(h) < \beta(h))$ are the two roots of $X^2 - \frac{1}{21}X^7 = h$, where $0 \le h < \frac{5\sqrt[5]{36}}{7}$. Therefore, Q and R can be rewritten as

$$Q = \int_{\alpha(h)}^{\beta(h)} \frac{\left(X - \frac{1}{6}X^6\right)^2}{E(X)} \mathrm{d}X,\ R = \int_{\alpha(h)}^{\beta(h)} E(X)\mathrm{d}X, \tag{9}$$

with $E(X) = \sqrt{X^2 - \frac{1}{21}X^7 - h}$.

Introducing the Abelian integrals:

$$J_n(h) = \int_{\alpha(h)}^{\beta(h)} X^n E(X)\mathrm{d}X,\ n = 0, 1, 2, \cdots, \tag{10}$$

which satisfy

$$\int_{\alpha(h)}^{\beta(h)} \frac{X^n}{E(X)}\mathrm{d}X = -2J_n'(h), \tag{11}$$

by direct computation.

To prove that $c_0(h)$ is decreasing, we only need to show that $Z(h)$ is increasing. Here we need Lemmas 1 and 2 from [14].

Lemma 1. *[14]. We have $J = A(h)J'$, where $J = (J_0, J_1, J_2, J_3, J_4, J_5)^T$ and*

$$A(h) = \begin{pmatrix} \frac{14}{9}h & 0 & -\frac{10}{9} & 0 & 0 & 0 \\ 0 & \frac{14}{11}h & 0 & -\frac{10}{11} & 0 & 0 \\ 0 & 0 & \frac{14}{13}h & 0 & -\frac{10}{13} & 0 \\ 0 & 0 & 0 & \frac{14}{15}h & 0 & -\frac{10}{15} \\ 0 & -\frac{60}{17} & 0 & 0 & \frac{14}{17}h & 0 \\ \frac{140}{57}h & 0 & -\frac{280}{57} & 0 & 0 & \frac{14}{19}h \end{pmatrix}.$$

Lemma 2. *[14]. We have $Q = J_5 - J_0, R = J_0$ and $Z(h) = \frac{J_5}{J_0} - 1$.*

From Lemma 1, we can derive the inverse of A,

$$A^{-1}=\frac{1}{\triangle}\times \begin{pmatrix} 9\left((14h)^5-56\cdot 10^5\right) & 66\cdot 10^3(14h)^2 & 130(14h)^4 & 9\cdot 10^5\cdot 14h & 17\cdot 10^2(14h)^3 & 114\cdot 10^5 \\ -3\cdot 10^3(14h)^3 & 11(14h)^5 & 78\cdot 10^3(14h)^2 & 150(14h)^4 & 102\cdot 10^4\cdot 14h & 19\cdot 10^2(14h)^3 \\ -18\cdot 10^5\cdot 14h & 66\cdot 10^2(14h)^3 & 13(14h)^5 & 9\cdot 10^4(14h)^2 & 170(14h)^4 & 114\cdot 10^4\cdot 14h \\ -3\cdot 10^2(14h)^4 & 396\cdot 10^4 14h & 78\cdot 10^2(14h)^3 & 15(14h)^5 & 102\cdot 10^3(14h)^2 & 190(14h)^4 \\ -18\cdot 10^4(14h)^2 & 660(14h)^4 & 468\cdot 10^4 14h & 9\cdot 10^3(14h)^3 & 17(14h)^5 & 114\cdot 10^3(14h)^2 \\ -30(14h)^5 & 396\cdot 10^3(14h)^2 & 780(14h)^4 & 54\cdot 10^5\cdot 14h & 102\cdot 10^2(14h)^3 & 19(14h)^5 \end{pmatrix}, \tag{12}$$

with $\triangle = 14h\left((14h)^5-36\times 10^5\right)$.

With these preparations, now we will prove that $Z(h)$ is increasing. Exploiting Lemma 1 and (12), we derive the following Ricatti equation

$$\begin{aligned} Z'(h) &= \left(\frac{J_5}{J_0}\right)' = \frac{J_5'}{J_0}-\frac{J_0'J_5}{(J_0)^2} \\ &= \frac{1}{\triangle J_0}\left(-30(14h)^5J_0+396\cdot 10^3(14h)^2J_1+780(14h)^4J_2+54\cdot 10^5\cdot 14hJ_3+102\cdot 10^2(14h)^3J_4\right. \\ &\left.+19(14h)^5J_5\right)-\frac{J_5}{\triangle(J_0)^2}\left(9\left((14h)^5-56\cdot 10^5\right)J_0+66\cdot 10^3(14h)^2J_1+130(14h)^4J_2\right. \\ &\left.+9\cdot 10^5\cdot 14hJ_3+17\cdot 10^2(14h)^3J_4+114\cdot 10^5J_5\right) \\ &= \frac{1}{\triangle}\left(-30(14h)^5+396\cdot 10^3(14h)^2\frac{J_1}{J_0}+780(14h)^4\frac{J_2}{J_0}+54\cdot 10^5\cdot 14h\frac{J_3}{J_0}+102\cdot 10^2(14h)^3\frac{J_4}{J_0}\right. \\ &+\left(10(14h)^5+504\cdot 10^5\right)\frac{J_5}{J_0}-66\cdot 10^3(14h)^2\frac{J_1}{J_0}\frac{J_5}{J_0}+130(14h)^4\frac{J_2}{J_0}\frac{J_5}{J_0}-9\cdot 10^5\cdot 14h\frac{J_3}{J_0}\frac{J_5}{J_0} \\ &\left.-17\cdot 10^2(14h)^3\frac{J_4}{J_0}\frac{J_5}{J_0}-114\cdot 10^5\left(\frac{J_5}{J_0}\right)^2\right) \\ &= \frac{1}{\triangle}F(h), \end{aligned} \tag{13}$$

where

$$\begin{aligned} F(h) &= -114\cdot 10^5\left(\frac{J_5}{J_0}\right)^2+\left(10(14h)^5+504\cdot 10^5\right)\frac{J_5}{J_0}-30(14h)^5 \\ &+\frac{140h}{J_0}\left(6-\frac{J_5}{J_0}\right)\left(6600\cdot 14hJ_1+13(14h)^3J_2+9\cdot 10^4J_3+170(14h)^2J_4\right). \end{aligned} \tag{14}$$

Since $\triangle$ is negative, if we can prove $F(h)<0$, then $Z'(h)>0$, and it follows that $c_0(h)$ is decreasing. However, it is not easy to directly show that $F(h)<0$ analytically. Therefore, here we develop a numerical technique to show it. Specifically, let h change from 0.0001 to 1.4626 (note that $\frac{5\sqrt[5]{36}}{7}\approx 1.4626$) with step 0.0001, then we can numerically evaluate $\alpha(h),\beta(h)$ and $J_i, i=0,1,\cdots,5$ through (10), and the values of $F(h)$ follow. The profile of $F(h)$ with respect to h is shown in Figure 2 and the maximum of $F(h)$ is -4.2866×10^3 at $h=1.4626$, which indicates that $F(h)<0$, for $0<h<\frac{5\sqrt[5]{36}}{7}$.

Remark 1. *As implied in [17], when $m\geq 5$, it is very difficult to prove the monotonicity of $c_0(h)$ analytically. Therefore, we turn to the numerical technique and solve the open question*

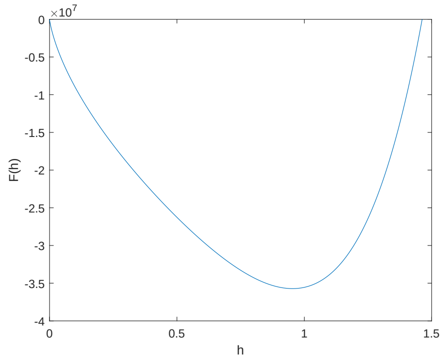

Figure 2. The profile of $F(h)$ with respect to h.

for the special case when $m = 5$.

§3 Conclusions

In this paper, we develop the numerical technique to derive the monotonicity of $c_0(h)$ for a special perturbed gKdV equation (1) when $m = 5$, which partially solves the open question presented by [14]. More importantly and interestingly, it potentially provides a way to answer the open question for general $m \in \mathbb{N}^+$ completely, which is still under consideration. The main difficulties lie in two aspects, the complexity of A^{-1} and $Z'(h)$, and the algorithm and program.

Declarations

Conflict of interest The authors declare no conflict of interest.

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