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The boundedness in a chemotaxis-haptotaxis system with ECM-dependent sensitivity

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Abstract. This paper deals with a chemotaxis-haptotaxis system with ECM-dependent sensitivity under the Neumann boundary conditions in a smooth bounded domain. It is shown that the system possesses a globally bounded solution under some conditions.

§1 Introduction

In this paper, we mainly consider the boundedness of classical solutions in a parabolic-parabolic-ODE chemotaxis-haptotaxis system with ECM-dependent sensitivity

$$\begin{cases} u_t = \Delta u - \chi \nabla \cdot (u \nabla v) - \nabla \cdot (\xi(w) u \nabla w) + \mu u(1-u-w), & x \in \Omega,\, t>0, \\ v_t = \Delta v - v + u, & x \in \Omega,\, t>0, \\ w_t = -vw, & x \in \Omega,\, t>0, \\ \dfrac{\partial u}{\partial \nu} - \chi u \dfrac{\partial v}{\partial \nu} - \xi(w) u \dfrac{\partial w}{\partial \nu} = \dfrac{\partial v}{\partial \nu} = 0, & x \in \partial\Omega,\, t>0, \\ u(x,0)=u_0(x), \quad v(x,0)=v_0(x), \quad w(x,0)=w_0(x), & x \in \Omega, \end{cases} \tag{1.1}$$

where Ω is a bounded domain in $\mathbb{R}^n (n \geq 3)$ with smooth boundary $\partial\Omega$, ν is the outward normal vector to $\partial\Omega$, χ, μ represent the chemotactic coefficient and the growth rate, respectively, which are all assumed to be positive constants. Throughout this paper, we assume that with some $\alpha \in (0,1)$, the initial data (u_0, v_0, w_0) satisfy the following conditions

$$u_0 \in C^0(\bar{\Omega}) \text{ with } u_0 \geq 0 \text{ in } \Omega \text{ and } u_0 \not\equiv 0, \tag{1.2}$$

$$v_0 \in W^{1,\infty}(\bar{\Omega}) \text{ with } v_0 \geq 0 \text{ in } \Omega, \tag{1.3}$$

$$w_0 \in C^{2+\alpha}(\bar{\Omega}) \text{ with } w_0 > 0 \text{ in } \Omega \text{ and } \frac{\partial w_0}{\partial \nu} = 0 \text{ on } \partial\Omega. \tag{1.4}$$

The function u denotes the density of cancer cells, v is the concentration of matrix-degrading enzyme(MDE) and w represents the density of extracellular matrix(ECM). $\xi(w)$ is haptotactic

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sensitivity which depends on the density of ECM. In the biomathematical model, to the best of our knowledge, except for the self-diffusion of cells, the invasion of cancer cells into healthy cells is also associated with two biological mechanisms: one is the movement towards its own secretion of diffusible chemicals, which is called chemotactic migration; the other is the movement towards the non-diffusable ECM, which is called haptotactic migration.

In order to better understand the system (1.1), let's mention some previous contributions in this direction. Many scholars have studied the following classical Keller-Segel chemotaxis system with the logistic source [1]

$$\begin{cases} u_t = \Delta u - \chi\nabla\cdot(u\nabla v) + \mu u(1-u), & x\in\Omega,\ t>0,\\ \tau v_t = \Delta v - v + u, & x\in\Omega,\ t>0,\\ \dfrac{\partial u}{\partial\nu} = \dfrac{\partial v}{\partial\nu} = 0, & x\in\partial\Omega,\ t>0,\\ u(x,0)=u_0(x),\quad v(x,0)=v_0(x), & x\in\Omega, \end{cases} \tag{1.5}$$

where $\chi,\ \mu>0,\ \tau\in\{0,1\}$. For example, if $\tau=1$, it was known that the global existence and boundedness of solutions were proved by Osaki for all $\chi,\ \mu>0$ as $n=1,2$ (see [2]). Tello and Winkler [3] obtained that the system (1.5) owns a unique global classical solution which is bounded with $\tau=0$ and $\mu>\frac{(n-2)_+}{n}\chi$. Other related results could be found in papers [4-9] and the references therein.

Next we recall chemotaxis-haptotaxis system with constant sensitivities and logistic source

$$\begin{cases} u_t = \Delta u - \chi\nabla\cdot(u\nabla v) - \xi\nabla\cdot(u\nabla w) + \mu u(1-u-w), & x\in\Omega,\ t>0,\\ v_t = \Delta v - v + u, & x\in\Omega,\ t>0,\\ w_t = -vw, & x\in\Omega,\ t>0,\\ \dfrac{\partial u}{\partial\nu} = \dfrac{\partial v}{\partial\nu} = \dfrac{\partial w}{\partial\nu} = 0, & x\in\partial\Omega,\ t>0,\\ u(x,0)=u_0(x),\quad v(x,0)=v_0(x),\quad w(x,0)=w_0(x) & x\in\Omega, \end{cases} \tag{1.6}$$

where $\chi,\ \xi,\ \mu>0$. Tao proved that the system (1.6) has a global bounded classical solution for $n=1$ in [10]. In the case of $n=2,3$, the unique globally bounded classical solution was studied by Tao and Wang in [11] for any small $\frac{\chi}{\mu}>0$; later, Tao obtained the same result for any $\mu>0$ in [12]. Moreover, Li in [13] researched that the system (1.6) possesses a unique classical solution for $\mu\geq(\frac{11}{2}+\xi^2\|w_0\|^2_{L^\infty(\Omega)})\chi^2+\frac{37}{2}+4\xi^2\|w_0\|^2_{L^\infty(\Omega)}$ in three space dimensions. When $3\leq n\leq 8$, Wang and Ke in [14] proved that (1.6) has a unique global classical solution which is uniformly bounded in time for sufficiently large μ.

Based on the biological background, the chemotactic sensitivity may depend on chemical signals. Recently, Mizukami and Otsuka in [15] considered the following chemotaxis-haptotaxis system with signal-dependent sensitivity

$$\begin{cases} u_t = \Delta u - \nabla\cdot(\chi(v)u\nabla v) - \xi\nabla\cdot(u\nabla w) + \mu u(1-u-w), & x\in\Omega,\, t>0,\\ v_t = \Delta v - v + u, & x\in\Omega,\, t>0,\\ w_t = -vw, & x\in\Omega,\, t>0,\\ \dfrac{\partial u}{\partial\nu} - \chi(v)u\dfrac{\partial v}{\partial\nu} - \xi u\dfrac{\partial w}{\partial\nu} = \dfrac{\partial v}{\partial\nu} = 0, & x\in\partial\Omega,\, t>0,\\ u(x,0)=u_0(x),\quad v(x,0)=v_0(x),\quad w(x,0)=w_0(x), & x\in\Omega. \end{cases} \tag{1.7}$$

Mizukami and Otsuka showed that the system (1.7) possesses a globally bounded classical solution when $n\ge 3$ under the conditions that $\chi\in C^{1+\theta}([0,\infty))\cap L^1(0,\infty),\ (0<\exists\theta<1),\ \chi>0;\ \exists\, C_1>0,\ \chi(s)s\le C_1$ for all $s\ge 0$ and $\exists\, p_0\in(n,n+1),\ \chi'(s)+\alpha_{p_0}|\chi(s)|^2\le 0$ for some $\alpha_{p_0}>0$.

Inspired by the literature above, we mainly consider the boundedness of the solution in the model (1.1), since the system with ECM-dependent sensitivity is important in the mathematical and biological points of view. The aim of this paper is to analyze the effect of $\xi(w)$ on the boundedness of the classical solution in the system (1.1).

The results of this paper are based on the arbitrary constant $\chi>0$ and the function ξ satisfies the following conditions:

$$\xi\in C^{1+\theta}([0,\infty))\cap L^1(0,\infty), (0<\exists\ \theta<1),\quad \xi>0, \tag{1.8}$$

$$\exists\ \xi_0>0,\quad \xi(s)s\le\xi_0\quad \text{for}\quad \text{all}\quad s\ge 0. \tag{1.9}$$

Theorem 1.1 *Let $\Omega\in\mathbb{R}^n(n\ge 3)$ be a bounded domain with smooth boundary and $\chi>0$. Assume that ξ satisfies (1.8)-(1.9) and μ is suitably large. Then for any (u_0,v_0,w_0) fulfilling (1.2)-(1.4), the model (1.1) possesses a unique global classical solution*

$$\begin{aligned} &u\in C^0(\bar\Omega\times[0,\infty))\cap C^{2,1}(\bar\Omega\times(0,\infty)),\\ &v\in C^0(\bar\Omega\times[0,\infty))\cap C^{2,1}(\bar\Omega\times(0,\infty))\cap L^\infty_{loc}([0,\infty);W^{1,q}(\Omega)),\\ &w\in C^0(\bar\Omega\times[0,\infty))\cap C^{2,1}(\bar\Omega\times(0,\infty)) \end{aligned}$$

for some $q>n$. Moreover, the solution (u,v,w) is bounded uniformly-in-time:

$$\|u(\cdot,t)\|_{L^\infty(\Omega)} + \|v(\cdot,t)\|_{W^{1,q}(\Omega)} + \|w(\cdot,t)\|_{L^\infty(\Omega)} \le C$$

for all $t>0$ with some $C>0$.

The method to prove Theorem 1.1 is to build the L^p-estimate for u with some $p>n$. The first job of this paper is to introduce the energy estimate for $\int_\Omega u^p f(v,w)$ with some smooth function $f(v,w)$ to establish the desired estimate. Then the L^p-boundedness of u can be obtained directly through the ordinary differential comparison principle, finally, our results can be obtained through the well-known Moser-Alikakos iteration in [16,17].

The arrangement of this paper is as follows. In Section 2, we mainly give the basic lemmas which are needed for the subsequent proof. Section 3 is devoted to proving the boundedness of global classical solutions.

§2 Preliminaries

In the first place, we give the local existence of the classical solution, which is derived from the standard regularity theory of parabolic equations.

Lemma 2.1 *Let $n \in N$, $\chi > 0$, $\mu > 0$ and ξ satisfy (1.8)-(1.9). Then for all (u_0, v_0, w_0) fulfilling (1.2)-(1.4), there exists $T_{max} \in (0, \infty]$ such that the model (1.1) possesses a unique classical solution*

$$u \in C^0(\bar{\Omega} \times [0, T_{max})) \cap C^{2,1}(\bar{\Omega} \times (0, T_{max})),$$
$$v \in C^0(\bar{\Omega} \times [0, T_{max})) \cap C^{2,1}(\bar{\Omega} \times (0, T_{max})) \cap L^{\infty}_{loc}([0, T_{max}); W^{1,q}(\Omega)) \quad (q > n),$$
$$w \in C^0(\bar{\Omega} \times [0, T_{max})) \cap C^{2,1}(\bar{\Omega} \times (0, T_{max}))$$

and

$$u > 0, \quad v \geq 0 \quad and \quad 0 < w \leq \|w_0\|_{L^\infty(\Omega)} \quad for \quad all \quad (x,t) \in \Omega \times (0, T_{max}).$$

Moreover, if $T_{max} < \infty$, then

$$\lim_{t \to T_{max}} sup(\|u(\cdot, t)\|_{L^\infty(\Omega)}) = \infty. \tag{2.1}$$

Proof Concerning the local-in-time existence of classical solutions to the problem (1.1) is based on the well-known standard contraction mapping argument which can be found in [18]. □

In the second place, we will give some basic lemmas which will be of great help to the proof of global boundedness in the third part.

Lemma 2.2 ([19,20]) *Suppose $g \in L^r((0,T)); L^r(\Omega))$ with $r \in (1, +\infty)$ and consider the following equation:*

$$\begin{cases} v_t - \triangle v + v = g, & (x,t) \in \Omega \times (0,T) \\ \dfrac{\partial v}{\partial \nu} = 0, & (x,t) \in \partial\Omega \times (0,T) \\ v(x,0) = v_0(x), & x \in \Omega. \end{cases} \tag{2.2}$$

For each $v_0 \in W^{2,r}(\Omega)$ such that $\frac{\partial v_0}{\partial \nu} = 0$ and $g \in L^r((0,T); L^r(\Omega))$, there exists a unique solution $v \in W^{1,r}((0,T); L^r(\Omega)) \cap L^r((0,T); W^{2,r}(\Omega))$. In addition, if $s_0 \in [0,T), v(\cdot, s_0) \in W^{2,r}(\Omega)(r > N)$ with $\frac{\partial v(\cdot, s_0)}{\partial \nu} = 0$, then there exists a positive constant $\lambda_0 := \lambda_0(\Omega, r, N)$ such that

$$\int_{s_0}^{T} e^{rs} \|v(\cdot, t)\|^r_{W^{2,r}(\Omega)} ds \leq \lambda_0 (\int_{s_0}^{T} e^{rs} \|g(\cdot, t)\|^r_{L^r(\Omega)} ds + e^{rs_0}(\|v_0(\cdot, s_0)\|^r_{W^{2,r}(\Omega)})).$$

Lemma 2.3 ([21,22]) *If S is defined as a form which has three variables*

$$S(x,y,z) := a_1 x^2 + a_2 y^2 + a_3 z^2 + 2a_4 xy + 2a_5 xz + 2a_6 yz,$$

where $a_1, a_2, a_3, a_4, a_5, a_6 \in \mathbb{R}$, and set A_1, A_2, A_3 be determinants

$$A_1 := \begin{vmatrix} a_1 \end{vmatrix}, \quad A_2 := \begin{vmatrix} a_1 & a_5 \\ a_5 & a_3 \end{vmatrix}, \quad A_3 := \begin{vmatrix} a_1 & a_5 & a_4 \\ a_5 & a_3 & a_6 \\ a_4 & a_6 & a_2 \end{vmatrix}.$$

Then for all $(x,y,z) \in \mathbb{R}^3$, $S(x,y,z) \le 0$ *is equivalent to*

$$A_1 < 0, \quad A_2 > 0, \quad A_3 \le 0.$$

§3 Global boundedness

In this section, we will obtain the boundedness of global classical solutions under some conditions in Theorem 1.1. We first give the following elementary estimates for u and v.

Lemma 3.1 *Let* $\chi > 0$ *and* $\mu > 0$. *Assume* (u_0, v_0, w_0) *satisfies (1.2)-(1.4) and* ξ *satisfies (1.8)-(1.9). Then there exists* $C > 0$ *such that*

$$\int_\Omega u(\cdot,t) \le C, \quad \int_\Omega v(\cdot,t) \le C \quad for \quad all \quad t \in (0, T_{max}).$$

Proof By integrating the first equation in (1.1) over Ω and using the Hölder inequality $(\int_\Omega u)^2 \le |\Omega| \int_\Omega u^2$, we obtain

$$\frac{d}{dt}\int_\Omega u = \mu \int_\Omega u - \mu \int_\Omega u^2 \le \mu \int_\Omega u - \frac{\mu}{|\Omega|}(\int_\Omega u)^2.$$

Then we get the L^1-boundedness of u. Similarly, integrating the second equation in (1.1) over Ω, we have

$$\frac{d}{dt}\int_\Omega v dx \le -\int_\Omega v + c$$

with some $c > 0$. Therefore, we get the L^1-boundedness of v. □

Since $\xi(w)$ is a function, not a constant, we can't get the L^p-estimate of u in the usual way, thus we need to introduce a test function $f(v,w) = \exp(-\delta \int_0^w \xi(s)ds - \delta\chi v)$ with parameter $\delta > 0$ to get the L^p-estimate for u.

The following lemma is the first step in proving the global boundedness of the L^p norm of u.

Lemma 3.2 *Let* χ, μ, $\delta > 0$ *and* ξ *satisfy (1.8)-(1.9). Then for all* $p > 1$, $\theta \in (0,1)$, *we have*

$$\begin{aligned}
&\frac{d}{dt}\int_\Omega u^p(f(v,w)+1) \\
\le& J_1 + J_2 + J_3 - p(p-1)(1-\theta)\int_\Omega u^{p-2}|\nabla u|^2 + p(p-1)\chi \int_\Omega u^{p-1}\nabla u \cdot \nabla v \\
&+ \mu p \int_\Omega u^p(1-u) + \mu p \int_\Omega u^p f(v,w)(1-u) + \xi_0 \delta \int_\Omega u^p f(v,w) v \\
&- \delta\chi \int_\Omega u^p f(v,w)(\triangle v - v + u)
\end{aligned}$$

for all $t \in (s_0, T_{max})$, *where*

$$J_1 := -p(p-1)\theta \int_\Omega u^{p-2}|\nabla u|^2,$$

$$J_2 := p(p-1)\int_\Omega u^{p-1}\xi(w)\nabla u \cdot \nabla w,$$

$$J_3 := p\int_\Omega u^{p-1}f(v,w)\nabla\cdot(\nabla u-\chi u\nabla v-u\xi(w)\nabla w).$$

Proof First multiplying the first equation in (1.1) by $u^{p-1}(p>1)$ and then integrating it over Ω combining with $\theta\in(0,1)$, we can calculate that

$$\begin{aligned}\frac{d}{dt}\int_\Omega u^p &= -p(p-1)\int_\Omega u^{p-2}|\nabla u|^2+p(p-1)\chi\int_\Omega u^{p-1}\nabla u\cdot\nabla v\\ &\quad+p(p-1)\int_\Omega u^{p-1}\xi(w)\nabla u\cdot\nabla w+\mu p\int_\Omega u^p(1-u-w)\\ &=-p(p-1)\theta\int_\Omega u^{p-2}|\nabla u|^2-p(p-1)(1-\theta)\int_\Omega u^{p-2}|\nabla u|^2\\ &\quad+p(p-1)\chi\int_\Omega u^{p-1}\nabla u\cdot\nabla v\\ &\quad+p(p-1)\int_\Omega u^{p-1}\xi(w)\nabla u\cdot\nabla w+\mu p\int_\Omega u^p(1-u-w)\\ &=J_1+J_2-p(p-1)(1-\theta)\int_\Omega u^{p-2}|\nabla u|^2+p(p-1)\chi\int_\Omega u^{p-1}\nabla u\cdot\nabla v\\ &\quad+\mu p\int_\Omega u^p(1-u-w).\end{aligned}\tag{3.1}$$

Since there is a function $\xi(w)$ in J_2, we can't directly estimate $\frac{d}{dt}\int_\Omega u^p$, we need to introduce an auxiliary function f to estimate $\frac{d}{dt}\int_\Omega u^p f(v,w)$. Then we can calculate

$$\begin{aligned}\frac{d}{dt}\int_\Omega u^pf(v,w)&=p\int_\Omega u^{p-1}f(v,w)\cdot u_t-\delta\int_\Omega u^pf(v,w)[(\xi(w)\cdot w_t+\chi v_t)]\\ &=J_3+\mu p\int_\Omega u^pf(v,w)(1-u-w)+\delta\int_\Omega u^pf(v,w)\xi(w)wv\\ &\quad-\delta\chi\int_\Omega u^pf(v,w)(\triangle v-v+u).\end{aligned}\tag{3.2}$$

According to (1.9), (3.1) and (3.2), we get

$$\begin{aligned}&\frac{d}{dt}\int_\Omega u^p(f(v,w)+1)\\ \le{}&J_1+J_2+J_3-p(p-1)(1-\theta)\int_\Omega u^{p-2}|\nabla u|^2+p(p-1)\chi\int_\Omega u^{p-1}\nabla u\cdot\nabla v\\ &+\mu p\int_\Omega u^p(1-u)+\mu p\int_\Omega u^pf(v,w)(1-u)+\xi_0\delta\int_\Omega u^pf(v,w)v\\ &-\delta\chi\int_\Omega u^pf(v,w)(\triangle v-v+u).\end{aligned}$$

Therefore, we have completed the proof of this lemma. □

In order to obtain a differential inequality that derives the estimate for $\int_\Omega u^p(f(v,w)+1)$, we need to obtain $J_1+J_2+J_3\le 0$.

Lemma 3.3 *Let $\theta\in(0,1)$. Then there exists $p>n$ such that*

$$J_1+J_2+J_3\le 0,$$

where J_1, J_2, J_3 are defined in Lemma 3.2.

Proof Noting the definition of $f(v,w)$, we get $f(v,w) \le 1$. According to Lemma 3.2, a direct computation yields that

$$\begin{aligned}
& J_1 + J_2 + J_3 \\
\le & -p(p-1)\theta \int_\Omega u^{p-2} f(v,w)|\nabla u|^2 + p(p-1)\int_\Omega u^{p-1} f(v,w)\xi(w) f^{-1}(v,w)\nabla u\cdot\nabla w \\
& + p\int_\Omega u^{p-1} f(v,w)\nabla\cdot(\nabla u - \chi u\nabla v - u\xi(w)\nabla w) \\
= & -p(p-1)\theta \int_\Omega u^{p-2} f(v,w)|\nabla u|^2 + p(p-1)\int_\Omega u^{p-1} f\xi(w) f^{-1}(v,w)\nabla u\cdot\nabla w \\
& - p\int_\Omega \nabla(u^{p-1} f(v,w))\cdot(\nabla u - \chi u\nabla v - u\xi(w)\nabla w) \\
= & -p(p-1)\theta \int_\Omega u^{p-2} f(v,w)|\nabla u|^2 + p(p-1)\int_\Omega u^{p-1} f(v,w)\xi(w) f^{-1}(v,w)\nabla u\cdot\nabla w \\
& - p(p-1)\int_\Omega u^{p-2} f(v,w)\nabla u\cdot(\nabla u - \chi u\nabla v - u\xi(w)\nabla w) \\
& + p\delta\int_\Omega u^{p-1} f(v,w)(\xi(w)\nabla w + \chi\nabla v)\cdot(\nabla u - \chi u\nabla v - u\xi(w)\nabla w) \\
= & -p(p-1)\theta \int_\Omega u^{p-2} f(v,w)|\nabla u|^2 + p(p-1)\int_\Omega u^{p-1} f(v,w)\xi(w) f^{-1}(v,w)\nabla u\cdot\nabla w \\
& - p(p-1)\int_\Omega u^{p-2} f(v,w)|\nabla u|^2 + \chi p(p-1)\int_\Omega u^{p-1} f(v,w)\nabla u\cdot\nabla v \\
& + p(p-1)\int_\Omega u^{p-1} f(v,w)\nabla u\xi(w)\nabla w + p\delta\int_\Omega u^{p-1} f(v,w)\xi(w)\nabla w\cdot\nabla u \\
& + \chi p\delta\int_\Omega u^{p-1} f(v,w)\nabla u\cdot\nabla v - p\delta\int_\Omega u^p f(v,w)|\xi(w)|^2\cdot|\nabla w|^2 \\
& - 2\chi p\delta\int_\Omega u^p f(v,w)\xi(w)\nabla w\cdot\nabla v - \chi^2 p\delta\int_\Omega u^p f(v,w)|\nabla v|^2 \\
= & -p(p-1)(\theta+1)\int_\Omega u^{p-2} f(v,w)|\nabla u|^2 \\
& + p((p-1)f^{-1}(v,w) + p - 1 + \delta)\int_\Omega u^{p-1} f(v,w)\xi(w)\nabla u\cdot\nabla w \\
& + \chi p(p-1+\delta)\int_\Omega u^{p-1} f(v,w)\nabla u\cdot\nabla v - p\delta\int_\Omega u^p f(v,w)|\xi(w)|^2\cdot|\nabla w|^2 \\
& - 2\chi p\delta\int_\Omega u^p f(v,w)\xi(w)\nabla w\cdot\nabla v - \chi^2 p\delta\int_\Omega u^p f(v,w)|\nabla v|^2 \\
= & \int_\Omega u^p f(v,w)(a_1x^2 + a_2xy + a_3xz + a_4y^2 + a_5yz + a_6z^2),
\end{aligned}$$

where x, y, z and a_1, a_2, a_3, a_4, a_5, a_6 are given as

$$x = u^{-1}\nabla u, \quad y = \xi(w)\nabla w, \quad z = \nabla v$$

and

$$\begin{aligned}
a_1 &= -p(p-1)(\theta+1), \\
a_2 &= p((p-1)f^{-1}(v,w) + p - 1 + \delta),
\end{aligned}$$

$$
\begin{aligned}
a_3 &= \chi p(p-1+\delta),\\
a_4 &= -p\delta,\\
a_5 &= -2\chi p\delta,\\
a_6 &= -\chi^2 p\delta.
\end{aligned}
$$

Next we need to find some $\delta > 0$ such that

$$a_1x^2 + a_2xy + a_3xz + a_4y^2 + a_5yz + a_6z^2 \le 0. \tag{3.3}$$

By applying the Sylvester criterion and Lemma 2.3 to obtain (3.3), we can see that

$$A_1 = -p(p-1)(\theta+1) < 0. \tag{3.4}$$

Then, we calculate that

$$
\begin{aligned}
A_2 &= a_1a_6 - \frac{a_3^2}{4}\\
&= p^2\chi^2\delta(p-1)(\theta+1) - \frac{p^2\chi^2}{4}(p-1+\delta)^2\\
&= p^2\chi^2[(p-1)(\theta+1)\delta - \frac{1}{4}((p-1)^2+\delta^2+2(p-1)\delta)]\\
&= p^2\chi^2[-\frac{1}{4}(\delta-(p-1)(2\theta+1))^2 + (p-1)^2(\theta+1)\theta].
\end{aligned}
$$

Choosing $\delta = (p-1)(2\theta+1)$, so we can get

$$A_2 = p^2\chi^2(p-1)^2(\theta+1)\theta > 0. \tag{3.5}$$

Next, we aim to prove

$$A_3 = \begin{vmatrix} a_1 & \frac{a_3}{2} & \frac{a_2}{2}\\ \frac{a_3}{2} & a_6 & \frac{a_5}{2}\\ \frac{a_2}{2} & \frac{a_5}{2} & a_4 \end{vmatrix} \le 0.$$

Let $d = 1+2\theta$, then $\delta = (p-1)d$. Straightforward calculations yield

$$
\begin{aligned}
A_3 &= (a_1a_6 - \frac{a_3^2}{4})a_4 + \frac{a_2a_3a_5}{4} - \frac{a_2^2a_6}{4} - \frac{a_1a_5^2}{4}\\
&= -p^3\chi^2\delta(p-1)^2(\theta+1)\theta - \frac{1}{2}p^3\chi^2\delta(p-1+\delta)((p-1)f^{-1}(v,w)+p-1+\delta)\\
&\quad + \frac{p^3\chi^2\delta}{4}((p-1)f^{-1}(v,w)+p-1+\delta)^2 + p^3\chi^2\delta^2(p-1)(\theta+1)\\
&= -p^3\chi^2(p-1)^3d(\theta^2+\theta) - \frac{p^3\chi^2}{2}(p-1)^3d(d+1)(f^{-1}(v,w)+d+1)\\
&\quad + \frac{p^3\chi^2(p-1)^3d}{4}(f^{-1}(v,w)+1+d)^2 + p^3\chi^2(p-1)^3d^2(\theta+1)\\
&= -p^3\chi^2(p-1)^3d[(\theta+1)\theta + \frac{1}{2}(f^{-1}(v,w)+1+d)(1+d)\\
&\quad - \frac{1}{4}(f^{-1}(v,w)+1+d)^2 - d(\theta+1)]\\
&\le p^3\chi^2(p-1)^3d[(\theta+1)\theta + \frac{1}{2}(f^{-1}(v,w)+1+d)(1+d)\\
&\quad - \frac{1}{4}(f^{-1}(v,w)+1+d)^2 - d(\theta+1)]
\end{aligned}
$$

$$= p^2\chi^2(p-1)^3 d(-p(\theta+1)^2 + \phi(p,\theta)),$$

where $\phi(p,\theta)$ is defined as follow

$$\phi(p,\theta) := -\frac{1}{4}p(f^{-1}(v,w)+1+d)^2 + p(\theta+1)(f^{-1}+d+1).$$

Because of the boundedness of $f(v,w) : 1 \leq f^{-1}(v,w) \leq m$, where $m = \exp\{\int_0^\infty \xi(w) + \chi\|v_0\|_{L^\infty}\} > 1$. For any $m' \in [1,m]$, the following inequality holds

$$\begin{aligned}\phi(p,\theta) &= -\frac{1}{4}p(m'+d+1)^2 + p(\theta+1)(m'+d+1)\\ &= p(1+\theta)^2 - \frac{1}{4}pm'^2\\ &< p(1+\theta)^2.\end{aligned}$$

So, we have

$$A_3 \leq p^2(p-1)^3\chi^2 d(-p(\theta+1)^2\lambda + \varphi(p,\theta)) = 0. \tag{3.6}$$

Thanks to Lemma 2.3, by (3.4)-(3.6), we have completed the proof of Lemma 3.3. □

We next show the desired L^p-estimate for u with some $p > n$.

Lemma 3.4 *Let $p > n \geq 3,\ \chi > 0,\ \xi$ satisfy (1.8)-(1.9) and $\mu > \mu_0$, where $\mu_0 = [\frac{2\lambda_0(\gamma_1+\gamma_2)}{p-1}]^{\frac{1}{p+1}}$.* Then there exists a constant $C > 0$ such that

$$\|u(\cdot,t)\|_{L^p(\Omega)} \leq C \ for\ all\ t \in (s_0, T_{max}).$$

Proof In light of Lemmas 3.2 and 3.3, we have

$$\begin{aligned}&\frac{d}{dt}\int_\Omega u^p(f(v,w)+1)\\ &\leq -p(p-1)(1-\theta)\int_\Omega u^{p-2}|\nabla u|^2 + p(p-1)\chi\int_\Omega u^{p-1}\nabla u\cdot\nabla v\\ &\quad + \mu p\int_\Omega u^p(1-u) + \mu p\int_\Omega u^p f(v,w)(1-u) + \xi_0\delta\int_\Omega u^p f(v,w)v\\ &\quad - \delta\chi\int_\Omega u^p f(v,w)\triangle v + \delta\chi\int_\Omega u^p f(v,w)v - \delta\chi\int_\Omega u^{p+1}f(v,w)\\ &\leq p(p-1)\chi\int_\Omega u^{p-1}\nabla u\cdot\nabla v + \mu p\int_\Omega u^p(1+f(v,w))\\ &\quad - \mu p\int_\Omega u^{p+1}(1+f(v,w)) + \xi_0\delta\int_\Omega u^p f(v,w)v\\ &\quad - \delta\chi\int_\Omega u^p f(v,w)\triangle v + \delta\chi\int_\Omega u^p f(v,w)v\\ &\quad - \delta\chi\int_\Omega u^{p+1}f(v,w)\\ &= p(p-1)\chi\int_\Omega u^{p-1}\nabla u\cdot\nabla v + \mu p\int_\Omega u^p(1+f(v,w))\\ &\quad - \mu p\int_\Omega u^{p+1}(1+f(v,w)) + \delta(\chi+\xi_0)\int_\Omega u^p f(v,w)v\\ &\quad - \delta\chi\int_\Omega u^p f(v,w)\triangle v - \delta\chi\int_\Omega u^{p+1}f(v,w).\end{aligned}$$

By using Young's inequality, we can get

$$\begin{aligned}
&p(p-1)\chi\int_\Omega u^{p-1}\nabla u\cdot\nabla v-\delta\chi\int_\Omega u^p f(v,w)\triangle v\\
&\le \chi(p-1)\int_\Omega u^p|\triangle v|+\delta\chi\int_\Omega u^p f(v,w)|\triangle v|\\
&\le \chi(p-1)\int_\Omega u^p|\triangle v|+\delta\chi\int_\Omega u^p|\triangle v|\\
&=\chi(\delta+p-1)\int_\Omega u^p\triangle v\\
&\le \mu\int_\Omega u^{p+1}+\gamma_1\mu^{-p}\int_\Omega|\triangle v|^{p+1},
\end{aligned}\tag{3.7}$$

where $\mu>0$, $\gamma_1=\frac{1}{p+1}(\frac{p+1}{p})^{-p}(\chi(\delta+p-1))^{p+1}$. In a same way, we can get that

$$\begin{aligned}
&\delta(\chi+\xi_0)\int_\Omega u^p f(v,w)v-\delta\chi\int_\Omega u^{p+1}f(v,w)\\
&\le \mu\int_\Omega u^{p+1}f(v,w)+\gamma_2\mu^{-p}\int_\Omega v^{p+1}-\delta\chi\int_\Omega u^{p+1}f(v,w)\\
&=(\mu-\delta\chi)\int_\Omega u^{p+1}f(v,w)+\gamma_2\mu^{-p}\int_\Omega v^{p+1},
\end{aligned}\tag{3.8}$$

where $\gamma_2=\frac{1}{p+1}(\frac{p+1}{p})^{-p}(\delta(\chi+\xi_0))^{p+1}$. By (3.7) and (3.8), we obtain

$$\begin{aligned}
&p(p-1)\chi\int_\Omega u^{p-1}\nabla u\cdot\nabla v-\delta\chi\int_\Omega u^p f(v,w)\triangle v+\delta(\chi+\xi_0)\int_\Omega u^p f(v,w)v-\delta\chi\int_\Omega u^{p+1}f(v,w)\\
&\le C_1\int_\Omega u^{p+1}(f(v,w)+1)+\gamma_1\mu^{-p}\int_\Omega|\triangle v|^{p+1}+\gamma_2\mu^{-p}\int_\Omega v^{p+1},
\end{aligned}$$

where $C_1=\max\{\mu,\mu-\delta\chi\}=\mu>0$. Then

$$\begin{aligned}
\frac{d}{dt}\int_\Omega u^p(f(v,w)+1)&\le \mu p\int_\Omega u^p(f(v,w)+1)-(\mu p-C_1)\int_\Omega u^{p+1}(f(v,w)+1)\\
&\quad+\gamma_1\mu^{-p}\int_\Omega|\triangle v|^{p+1}+\gamma_2\mu^{-p}\int_\Omega v^{p+1}\\
&\le C_2\int_\Omega u^p(f(v,w)+1)-C_3\int_\Omega u^{p+1}(f(v,w)+1)\\
&\quad+\gamma_1\mu^{-p}\int_\Omega|\triangle v|^{p+1}+\gamma_2\mu^{-p}\int_\Omega v^{p+1},
\end{aligned}\tag{3.9}$$

where $C_2=\mu p>0$ and $C_3=\mu p-C_1=\mu p-\mu>0$. Let

$$y_m(t)=\int_\Omega|u(\cdot,t)|^m(f(v(\cdot,t),w(\cdot,t))+1),$$

where $m>1$. So we can get the following inequality

$$\int_\Omega u^{p+1}\le y_{p+1}.\tag{3.10}$$

By deforming (3.9) and using the Hölder inequality with Young's inequality, we can find

$$\begin{aligned}
&\frac{d}{dt}y_p(t)\le C_2y_p(t)-C_3y_{p+1}(t)+\gamma_1\mu^{-p}\int_\Omega|\triangle v|^{p+1}+\gamma_2\mu^{-p}\int_\Omega v^{p+1}\\
&\quad=-(p+1)y_p(t)+(C_2+p+1)y_p(t)-C_3y_{p+1}(t)+\gamma_1\mu^{-p}\int_\Omega|\triangle v|^{p+1}+\gamma_2\mu^{-p}\int_\Omega v^{p+1}
\end{aligned}$$

$$\leq -(p+1)y_p(t) + \frac{C_3}{2}y_{p+1}(t) + C_4 - C_3y_{p+1}(t) + \gamma_1\mu^{-p}\int_\Omega |\triangle v|^{p+1} + \gamma_2\mu^{-p}\int_\Omega v^{p+1}$$
$$= -(p+1)y_p(t) - \frac{C_3}{2}y_{p+1}(t) + C_4 + \gamma_1\mu^{-p}\int_\Omega |\triangle v|^{p+1} + \gamma_2\mu^{-p}\int_\Omega v^{p+1},$$

where $C_4 > 0$. And then by the variation-of-constants formula, we can get

$$\begin{aligned} y_p(t) \leq\ & e^{-(p+1)(t-s_0)}y_p(s_0) - \frac{C_3}{2}\int_{s_0}^t e^{-(p+1)(t-s)}y_{p+1}ds \\ & + \gamma_1\mu^{-p}\int_{s_0}^t\int_\Omega e^{-(p+1)(t-s)}|\triangle v|^{p+1}dxds \\ & + \gamma_2\mu^{-p}\int_{s_0}^t\int_\Omega e^{-(p+1)(t-s)}v^{p+1}dxds + \frac{C_4}{p+1}. \end{aligned}$$

Next, in view of (3.10) and Lemma 2.2, we can obtain

$$\begin{aligned} y_p(t) \leq\ & e^{-(p+1)(t-s_0)}y_p(s_0) - \frac{C_3}{2}\int_{s_0}^t e^{-(p+1)(t-s)}y_{p+1}ds \\ & + \gamma_1\mu^{-p}e^{-(p+1)t}\lambda_0\Big(\int_{s_0}^t\int_\Omega e^{(p+1)s}u^{p+1}dxds + \|v_0\|^{p+1}_{W^{2,p+1}}\Big) \\ & + \gamma_2\mu^{-p}e^{-(p+1)t}\lambda_0\Big(\int_{s_0}^t\int_\Omega e^{(p+1)s}u^{p+1}dxds + \|v_0\|^{p+1}_{W^{2,p+1}}\Big) + \frac{C_4}{p+1} \\ \leq\ & e^{-(p+1)(t-s_0)}y_p(s_0) - \frac{C_3}{2}\int_{s_0}^t e^{-(p+1)(t-s)}y_{p+1}ds + \gamma_1\mu^{-p}\lambda_0\int_{s_0}^t e^{-(p+1)(t-s)}y_{p+1}(s)ds \\ & + \gamma_2\mu^{-p}\lambda_0\int_{s_0}^t e^{-(p+1)(t-s)}y_{p+1}(s)ds + C_5\lambda_0e^{-(p+1)t}\|v_0\|^{p+1}_{W^{2,p+1}} + \frac{C_4}{p+1} \\ =\ & e^{-(p+1)(t-s_0)}y_p(s_0) - \Big(\frac{C_3}{2} - \lambda_0C_5\Big)\int_{s_0}^t e^{-(p+1)(t-s)}y_{p+1}(s)ds \\ & + C_5\lambda_0e^{-(p+1)t}\|v_0\|^{p+1}_{W^{2,p+1}} + \frac{C_4}{p+1} \\ \leq\ & e^{-(p+1)(t-s_0)}y_p(s_0) + C_5\lambda_0e^{-(p+1)t}\|v_0\|^{p+1}_{W^{2,p+1}} + \frac{C_4}{p+1} \\ \leq\ & C_6, \end{aligned}$$

where $\lambda_0 \in (0, \frac{C_3}{2C_5})$, $C_5 = (\gamma_1+\gamma_2)\mu^{-p}$ and $C_6 > 0$, with $\mu > \mu_0$ for all $t \in (s_0, T_{max})$. Hence we get that the boundedness of u in $L^p(\Omega)$. □

Corollary 3.1 *Let* $n \geq 3$, $T_{max} \in (0,\infty)$, initial data (u_0, v_0, w_0) satisfy (1.2)-(1.4), $\chi > 0$ *and* ξ fulfill (1.8)-(1.9). Then there exists C (independent of T) such that

$$\|v(\cdot,t)\|_{W^{1,\infty}(\Omega)} \leq C \quad for \quad all \quad t \in (0, T_{max}). \tag{3.11}$$

Proof On the basis of the $L^p - L^q$ estimates for the Neumann heat semigroup on bounded domains in [23] and Lemma 3.3, we can get (3.11) immediately. □

We are now in the position to prove our main result.

Proof of Theorem 1.1 Under the condition that ξ satisfies (1.8)-(1.9), we obtain the boundedness of u in $L^\infty(\Omega)$ by the combination of Corollary 3.1 and the well-known Moser-Alikakos iteration in [16,17]. $\square$

Declarations

Conflict of interest The authors declare no conflict of interest.

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