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New soliton solutions of (2+1)-dimensional Bogoyavlensky–Konopelchenko equation via two integration techniques

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Abstract. In this study, we investigate a variety of exact soliton solutions of general $(2+1)$-dimensional Bogoyavlensky–Konopelchenko equation via the $\exp(-\Phi(\eta))$-expansion method and modified Kudryashov method. The exact solutions are characterized in the form of hyperbolic, trigonometric and rational function solutions using $\exp(-\Phi(\eta))$-expansion method, whereas the solution in the form of hyperbolic function expression is obtained by the modified Kudryashov method. These exact solutions also include kink, bright, dark, singular and periodic soliton solutions. The graphical interpretation of the exact solutions is addressed for specific choices of the parameters appearing in the solutions.

§1 Introduction

Soliton is a wave packet that repeats itself, maintaining its shape when propagating at a constant speed. In 1834, Russell discovered the soliton phenomenon when he experienced a peculiar water wave on the Union Canal and later in 1844, he reported this discovery in a meeting [28]. In 1895, Korteweg and de Vries [16] presented a mathematical model for the propagation of waves, known by their name (KdV equation), and it could describe Russell's soliton phenomenon. The work of Zabusky and Kruskal [40] was a major milestone in the development of solitary theory, which correctly identified the nature of the solitary waves.

Soliton theory plays an effective role in fiber light distribution, switching modes of optical slab waveguides, nonlinear dielectric surface waves, laser physics, optical bistability, plasma physics and many other fluid dynamics phenomena. Therefore, the dynamics of soliton structure are an intriguing topic for the nonlinear evolution equations (NLEEs) that are derived from various physical phenomena of nonlinearity. The soliton theory remains still one of the most vibrant domains of mathematics and physics.

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During the last few decades, soliton theory gained a lot of interest to attain the exact solitary wave solutions of NLEEs through different approaches [1–9, 14]. Many researchers have spent considerable effort to study the exact explicit solutions of NLEEs [18–24] and various useful methods have been utilized, such as Osman [25] applied the unified method and its generalized form, siddique *et al.* [29] used the $\left(\frac{G'}{G}, \frac{1}{G}\right)$, $\frac{1}{G'}$ and modified $\frac{G'}{G^2}$-expansion methods, Srivastava *et al.* [30] applied the modified Kudryashov method, Yuan and Liu [39] obtained the analytical solutions by Hirota method and some more strategies are devised in [32–35].

The objective of this paper is to apply the $\exp(-\Phi(\eta))$-expansion method and modified Kudryashov method for obtaining the exact solutions of (2+1)-dimensional Bogoyavlensky–Konoplechenko (BK) equation, which retrieves the kink, bright, dark, singular soliton solutions and singular periodic solitary wave solutions.

1.1 Governing model

We consider the Bogoyavlensky–Konoplechenko (BK) equation [13, 31] in the form:

$$v_t + \gamma_1 v_{xxx} + \gamma_2 v_{xxy} + \gamma_3 v v_x + \gamma_4 (v v_y + v_x \partial_x^{-1} v_y) = 0, \tag{1}$$

where $\gamma_i, i = 1, \ldots, 4$ are constants, $\partial_x^{-1} = \int dx$ and $v = v(x, y, t)$ corresponds to the amplitude (or elevation) of the Riemann wave.
Using the transformation $v(x, y, t) = u_x(x, y, t)$ in Eq.(1) and integrating both sides with respect to x, it reduces to

$$u_t + \gamma_1 u_{xxx} + \gamma_2 u_{xxy} + \frac{\gamma_3}{2} {u_x}^2 + \gamma_4 u_x u_y = 0, \tag{2}$$

where the integration constants are chosen to be zero.

In [15], considering $\gamma_1 = \alpha,\ \gamma_2 = \beta,\ \gamma_3 = 6\alpha, \gamma_4 = 4\beta$, Eq.(1) describes the interaction of a Riemann wave and a long wave propagating along the y-axis and the x-axis, respectively. Exact solutions of BK equation are calculated by Prabhakar *et al.* in [26]. A special case of this equation developed in [10, 11], which is a two-dimensional form of the KdV equation. Mixed type of N-soliton solutions of BK equation with variable coefficients are obtained in [24] using the generalized unified method. Some special cases of BK equation to obtain the exact solutions have been studied by several authors in [5, 12, 17, 27, 36–38].

The rest of this paper is organized as follows: In section 2, the main steps of $\exp(-\Phi(\eta))$-expansion method and modified Kudryashov method are discussed in details. In sections 3 and 4, two different approaches for evaluating the exact solutions of the BK equations (1) and (2) are used. In section 5, specific values of the parameters are given to reveal the exact solutions graphically. Finally, in section 6, a conclusion is drawn for the results that have been obtained.

§2 Description of the methods

We consider a NLEE for $u = u(x, y, t)$ to be in the form:

$$F(u, u_x, u_y, u_t, u_{xx}, u_{yy}, u_{tt}, u_{xt}, \ldots) = 0, \tag{3}$$

where F is a polynomial of u(x,y,t) and its highest order partial derivatives and in which the nonlinear terms are involved. Using the wave transformation

$$u(x,y,t) = u(\eta), \qquad \eta = \kappa x + \omega y + \nu t, \tag{4}$$

where κ and ω are the soliton frequencies and ν is for wave speed and Eq.(3) can be transformed into the following nonlinear ordinary differential equation

$$Q(u, \kappa u^{'}, \omega u^{'}, \nu u^{'}, \kappa^2 u^{''}, \omega^2 u^{''}, \nu^2 u^{''}, \kappa\nu u^{''}, ...) = 0, \tag{5}$$

where prime denotes the derivatives with respect to η.

2.1 The $\exp(-\Phi(\eta))$-expansion method

The main steps of $\exp(-\Phi(\eta))$-expansion method are summarized as follows:

Step 1: Suppose that the traveling wave solution of Eq.(5) can be expressed as

$$u(\eta) = \sum_{i=0}^{N} a_i \exp(-\Phi(\eta))^i, \tag{6}$$

where a_i $(i = 1, 2, \ldots, N)$ are constants to be determined and $\Phi(\eta)$ satisfies the auxiliary ODE:

$$\Phi'(\eta) = \exp(-\Phi(\eta)) + \mu \exp(\Phi(\eta)) + \lambda. \tag{7}$$

The auxiliary Eq.(7) has the general solutions that are listed in the following few cases:

Case 1 (Hyperbolic function solutions): When $\lambda^2 - 4\mu > 0$ and $\mu \neq 0$, then

$$\Phi_1(\eta) = \ln\left(\frac{-\sqrt{\lambda^2 - 4\mu}\tanh\left(\frac{\sqrt{\lambda^2-4\mu}}{2}(\eta + C)\right) - \lambda}{2\mu}\right). \tag{8}$$

Case 2 (Trigonometric function solutions): If $\lambda^2 - 4\mu < 0$ and $\mu \neq 0$, then

$$\Phi_2(\eta) = \ln\left(\frac{\sqrt{4\mu - \lambda^2}\tan\left(\frac{\sqrt{4\mu-\lambda^2}}{2}(\eta + C)\right) - \lambda}{2\mu}\right). \tag{9}$$

Case 3 (Hyperbolic function solutions): If $\lambda^2 - 4\mu > 0, \mu = 0$ and $\lambda \neq 0$, then

$$\Phi_3(\eta) = -\ln\left(\frac{\lambda}{\cosh(\lambda(\eta + C)) + \sinh(\lambda(\eta + C)) - 1}\right). \tag{10}$$

Case 4 (Rational function solutions): If $\lambda^2 - 4\mu = 0, \mu \neq 0$ and $\lambda \neq 0$, then

$$\Phi_4(\eta) = \ln\left(-\frac{2(\lambda(\eta + C)) + 2}{\lambda^2(\eta + C)}\right). \tag{11}$$

Case 5: If $\lambda^2 - 4\mu = 0, \mu = 0$ and $\lambda = 0$, then

$$\Phi_5(\eta) = \ln(\eta + C), \tag{12}$$

where C is the constant of integration.

Step 2: The value of N can be determined after balancing the highest order derivative of u and the highest order nonlinear term in Eq.(5). Substituting Eq.(6) into Eq.(5) yields an algebraic equation involving $\exp(-\Phi(\eta))^i$ $(i = 0, 1, 2, ...)$. After equating the coefficients of each power of $\exp(-\Phi(\eta))$ to zero gives us a system of algebraic equations in a_i $(i = 0, 1, 2, ..., N)$, κ, ω, ν, λ

and μ.
Step 3: Substituting a_i in Eq.(6), a series of exact solutions of Eq.(3) can be carried out.

2.2 The modified Kudryashov method

The main steps of the modified Kudryashov method are summarized as follows:
Step 1: According to the modified Kudryashov method, the solution of Eq.(5) has the form:

$$u(\eta) = \sum_{i=0}^{N} b_i P^i(\eta), \tag{13}$$

where b_i $(i = 1, 2, \ldots, N)$ are the constants to be obtained. Moreover,

$$P(\eta) = \frac{1}{1 + kB^{\eta}}, \tag{14}$$

where k and B are nonzero constants with $B > 0$ and $B \neq 1$ and $P(\eta)$ satisfies the ODE:

$$P'(\eta) = P(\eta)(P(\eta) - 1)\ln B. \tag{15}$$

Step 2: Substituting Eq.(13) into Eq.(5) yields an algebraic equation involving $P^i(\eta)$ $(i = 0, 1, 2, ...)$. After equating the coefficients of each power of $P(\eta)$ to zero, gives us a system of algebraic equations in b_i $(i = 0, 1, 2, ..., N)$, $\kappa,\ \omega,\ \nu,\ \lambda$ and μ.
Step 3: Substituting b_i in Eq.(13), the exact solution of Eq.(3) can be carried out.

§3 Application of the $\exp(-\Phi(\eta))$-expansion method

The wave transformation $u(x, y, t) = u(\eta)$ and $\eta = \kappa x + \omega y + \nu t$, converts Eq.(2) to the nonlinear ODE as

$$\nu u' + (\gamma_1\kappa^3 + \gamma_2\kappa^2\omega)u''' + \left(\frac{\gamma_3}{2}\kappa^2 + \gamma_4\kappa\omega\right)(u')^2 = 0. \tag{16}$$

Balancing the highest order derivative term u''' with the nonlinear term $(u')^2$ in Eq.(16), leads to the balancing number as $N = 1$. The exact traveling wave solutions of the Eqs. (1) and (2) are achieved using the $\exp(-\Phi(\eta))$-expansion method as follows:
According to $\exp(-\Phi(\eta))$-expansion method, the solution of Eq.(16) is assumed as

$$u(\eta) = a_0 + a_1\exp(-\Phi(\eta)). \tag{17}$$

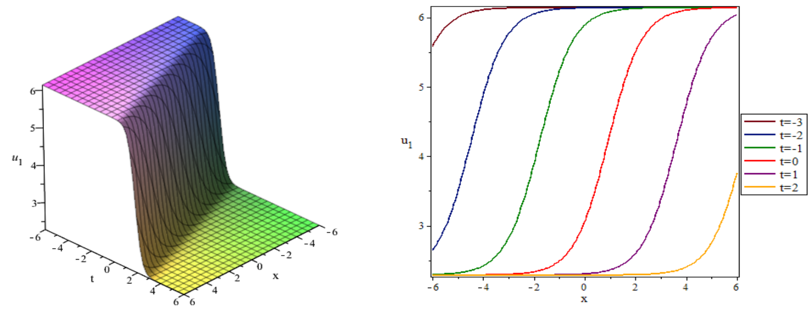
The direct substitution of Eq.(17) into Eq.(16) provides us a polynomial in $\exp(-\Phi(\eta))$. Equating the coefficient of each power of $\exp(-\Phi(\eta))$ to zero provides a system of algebraic equations:

$$\begin{aligned}
\exp(-\phi(\eta))^{-4}:&\quad -\frac{1}{2}a_1\left(12\gamma_2\kappa^2\omega + 12\gamma_1\kappa^3 - a_1\kappa^2\gamma_3 - 2a_1\gamma_4\kappa\omega\right) = 0,\\
\exp(-\phi(\eta))^{-3}:&\quad -\frac{1}{2}a_1\left(-4a_1\gamma_4\kappa\omega\lambda + 24\gamma_1\kappa^3\lambda + 24\gamma_2\kappa^2\omega\lambda - 2a_1\kappa^2\gamma_3\lambda\right) = 0,\\
\exp(-\phi(\eta))^{-2}:&\quad -\frac{1}{2}a_1\big(14\gamma_2\kappa^2\omega\lambda^2 - 4a_1\gamma_4\kappa\omega\mu + 16\gamma_2\kappa^2\omega\mu + 16\gamma_1\kappa^3\mu + 2\nu - a_1\kappa^2\gamma_3\lambda^2\\
&\quad - 2a_1\gamma_4\kappa\omega\lambda^2 - 2a_1\kappa^2\gamma_3\mu + 14\gamma_1\kappa^3\lambda^2\big) = 0,
\end{aligned}$$

$$\exp(-\phi(\eta))^{-1}: \quad -\frac{1}{2}\,a_1\big(2\,\gamma_1\kappa^3\lambda^3+16\,\gamma_1\kappa^3\lambda\,\mu+2\,\nu\,\lambda-4\,a_1\gamma_4\kappa\,\omega\,\mu\,\lambda+16\,\gamma_2\kappa^2\omega\,\lambda\,\mu$$
$$+\,2\,\gamma_2\kappa^2\omega\,\lambda^3-2\,a_1\kappa^2\gamma_3\mu\,\lambda\big)=0,$$
$$\exp(-\phi(\eta))^{0} \quad : \quad -\frac{1}{2}\,a_1\big(-2\,a_1\gamma_4\kappa\,\omega\,\mu^2+4\,\gamma_1\kappa^3\mu^2+2\,\gamma_1\kappa^3\mu\,\lambda^2+4\,\gamma_2\kappa^2\omega\,\mu^2-a_1\kappa^2\gamma_3\mu^2$$
$$+\,2\,\gamma_2\kappa^2\omega\,\mu\,\lambda^2+2\,\nu\,\mu\big)=0.$$

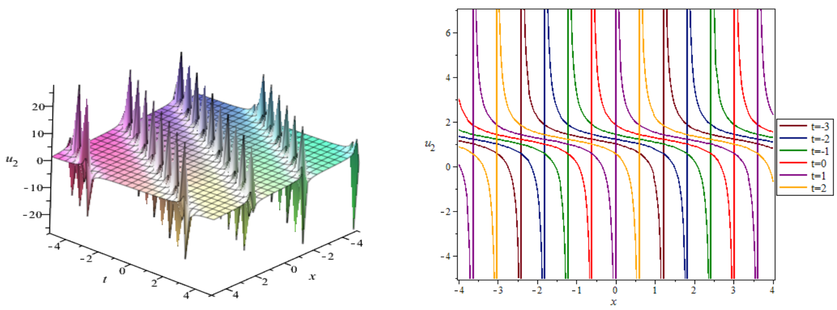
Solving the resulting system, we have

$$a_0=a_0,\qquad a_1=\frac{12\,\kappa\,(\gamma_1\kappa+\gamma_2\omega)}{\gamma_3\kappa+2\,\gamma_4\omega},\qquad \nu=\kappa^2\,(\gamma_1\kappa+\gamma_2\omega)\left(4\,\mu-\lambda^2\right). \tag{18}$$



(a) (b)

Figure 1. 3D and 2D-plots of u_1 with $\gamma_1=1,\ \gamma_2=1,\ \gamma_3=4,\ \gamma_4=4,\ \lambda=2.5,\ \mu=1,\ \kappa=-1,$ $\omega=-0.2,\ C=0,\ a_0=1.$



(a) (b)

Figure 2. 3D and 2D-plots of u_2 with $\gamma_1=1,\ \gamma_2=1,\ \gamma_3=4,\ \gamma_4=4,\ \lambda=1,\ \mu=1,\ \kappa=-1,$ $\omega=1.2,\ y=0,\ C=0,\ a_0=1.$

The exact solutions of BK equations (1) and (2) are obtained as the following different cases:

Case 1: (Hyperbolic function solutions): When $\lambda^2-4\mu>0$ and $\mu\neq 0$, then

$$u_1(x,y,t)=a_0+\frac{24\,\kappa\,(\gamma_1\kappa+\gamma_2\omega)\mu}{(\gamma_3\kappa+2\,\gamma_4\omega)\left(-\sqrt{\lambda^2-4\,\mu}\,\tanh\left(\frac{\sqrt{\lambda^2-4\,\mu}}{2}\left(\kappa\,x+\omega\,y+\kappa^2(\gamma_1\kappa+\gamma_2\omega)(4\,\mu-\lambda^2)t+C\right)\right)-\lambda\right)} \tag{19}$$

and

$$v_1(x,y,t)=\frac{2\left(\gamma_1\kappa+\gamma_2\omega\right)\kappa^2\mu\left(\lambda^2-4\mu\right)\left(\operatorname{sec h}\left(\frac{\sqrt{\lambda^2-4\mu}}{2}\left(\kappa x+\omega y+\kappa^2(\gamma_2\omega+\gamma_1\kappa)\left(4\mu-\lambda^2\right)t+C\right)\right)\right)^2}{(\gamma_3\kappa+2\gamma_4\omega)\left(-\sqrt{\lambda^2-4\mu}\tanh\left(\frac{\sqrt{\lambda^2-4\mu}}{2}\left(\kappa x+\omega y+\kappa^2(\gamma_2\omega+\gamma_1\kappa)(4\mu-\lambda^2)t+C\right)\right)-\lambda\right)^2}. \tag{20}$$

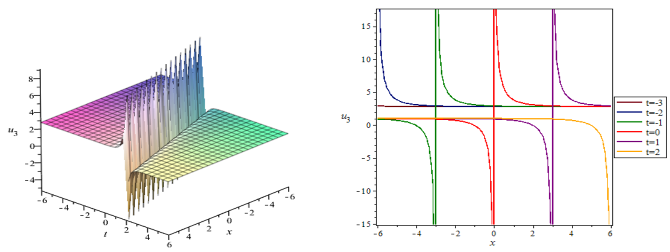
Case 2: (Trigonometric function solutions): When $\lambda^2-4\mu<0$ and $\mu\neq 0$,

$$u_2(x,y,t)=a_0+\frac{24\kappa\left(\gamma_1\kappa+\gamma_2\omega\right)\mu}{(\gamma_3\kappa+2\gamma_4\omega)\left(\sqrt{-\lambda^2+4\mu}\tan\left(\frac{\sqrt{4\mu-\lambda^2}}{2}\left(\kappa x+\omega y+\kappa^2(\gamma_1\kappa+\gamma_2\omega)(4\mu-\lambda^2)t+C\right)\right)-\lambda\right)} \tag{21}$$

and

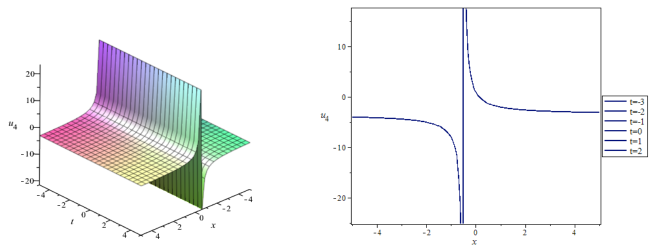
$$v_2(x,y,t)=\frac{-12\left(\gamma_1\kappa+\gamma_2\omega\right)\kappa^2\mu\left(4\mu-\lambda^2\right)\left(\sec\left(\frac{\sqrt{4\mu-\lambda^2}}{2}\left(\kappa x+\omega y+\kappa^2(\gamma_1\kappa+\gamma_2\omega)\left(4\mu-\lambda^2\right)t+C\right)\right)\right)^2}{(\gamma_3\kappa+2\gamma_4\omega)\left(\sqrt{-\lambda^2+4\mu}\tan\left(\frac{\sqrt{4\mu-\lambda^2}}{2}\left(\kappa x+\omega y+\kappa^2(\gamma_1\kappa+\gamma_2\omega)(4\mu-\lambda^2)t+C\right)\right)-\lambda\right)^2}. \tag{22}$$

Case 3: (Hyperbolic function solutions): When $\lambda^2-4\mu>0, \mu=0$ and $\lambda\neq 0$, then



(a) **(a)** (b)

Figure 3. 3D and 2D-plots of u_3 with $\gamma_1=1,\ \gamma_2=1,\ \gamma_3=4,\ \gamma_4=4,\ \lambda=1,\ \mu=0,\ \kappa=-1,$ $\omega=-2,\ C=0,\ a_0=1.$



(a) (b)

Figure 4. 3D and 2D-plots of u_4 with $\gamma_1=1,\ \gamma_2=1,\ \gamma_3=4,\ \gamma_4=4,\ \lambda=2,\ \mu=1,\ \kappa=2,$ $\omega=1,\ C=0,\ a_0=1.$

$$u_3(x,y,t)=a_0+\frac{12\left(\gamma_1\kappa+\gamma_2\omega\right)\kappa\lambda}{(\gamma_3\kappa+2\gamma_4\omega)(\cosh(\lambda(\kappa x+\omega y+\nu t+C))+\sinh(\lambda(\kappa x+\omega y+\nu t+C))-1)} \tag{23}$$

and

$$v_3(x,y,t)=-\frac{12\left(\gamma_1\kappa+\gamma_2\omega\right)\kappa^2\lambda^2\left(\sinh(\lambda(\kappa x+\omega y+\nu t+C))+\cosh(\lambda(\kappa x+\omega y+\nu t+C))\right)}{(\gamma_3\kappa+2\gamma_4\omega)\left(\cosh(\lambda(\kappa x+\omega y+\nu t+C))+\sinh(\lambda(\kappa x+\omega y+\nu t+C))-1\right)^2}, \tag{24}$$

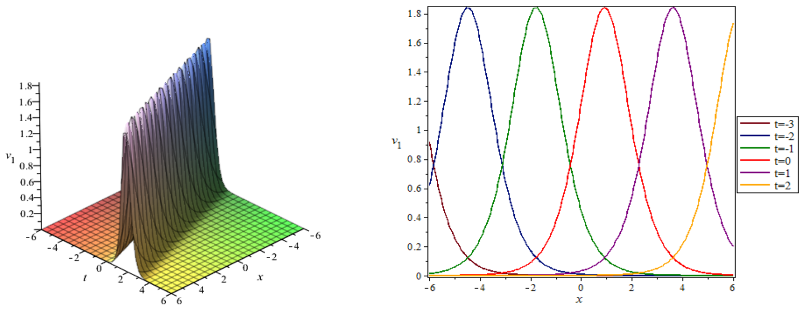
where $\nu = -\kappa^2 \lambda^2 (\gamma_1 \kappa + \gamma_2 \omega)$.

Case 4: (Rational function solutions): When $\lambda^2 - 4\mu = 0, \mu \neq 0$ and $\lambda \neq 0$, then

$$u_4(x,y,t) = \quad a_0 - \frac{6\,\kappa\,\lambda^2(\gamma_1\kappa+\gamma_2\omega)(\kappa\,x+\omega\,y+C)}{(\gamma_3\kappa+2\,\gamma_4\omega)(\lambda\,\kappa\,x+\lambda\,\omega\,y+\lambda\,C+2)} \tag{25}$$

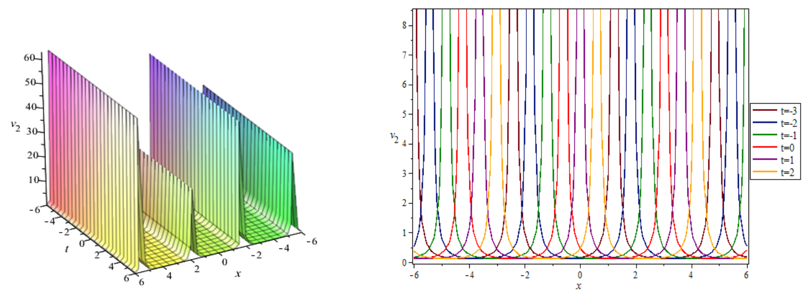
and

$$v_4(x,y,t) = \quad -\frac{12\,\lambda^2\kappa^2(\gamma_1\kappa+\gamma_2\omega)}{(\gamma_3\kappa+2\,\gamma_4\omega)(\lambda\,\kappa\,x+\lambda\,\omega\,y+\lambda\,C+2)^2}. \tag{26}$$



(a) (b)

Figure 5. 3D and 2D-plots of v_1 with $\gamma_1 = 1,\ \gamma_2 = 1,\ \gamma_3 = 6,\ \gamma_4 = -4, \lambda = 2.5,\ \mu = 1,\ \kappa = -1,$ $\omega = -0.2,\ C = 0,\ a_0 = 1.$



(a) (b)

Figure 6. 3D and 2D-plots of v_2 with $\gamma_1 = 1,\ \gamma_2 = 1,\ \gamma_3 = 6,\ \gamma_4 = -4, \lambda = 1,\ \mu = 1,\ \kappa = -1,$ $\omega = 1.2,\ y = 0,\ C = 0,\ a_0 = 1.$

Case 5: When $\lambda^2 - 4\mu = 0, \mu = 0$ and $\lambda = 0$, then

$$u_5(x,y,t) = \quad a_0 + \frac{12\,\kappa\,(\gamma_1\kappa+\gamma_2\omega)}{(\gamma_3\kappa+2\,\gamma_4\omega)(\kappa\,x+\omega\,y+C)} \tag{27}$$

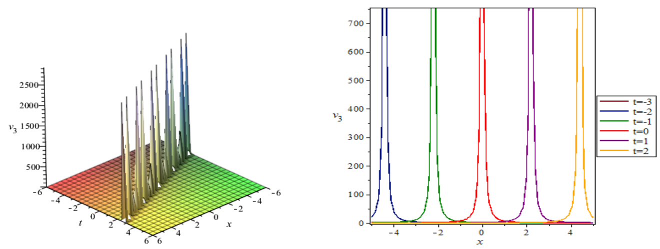
and

$$v_5(x,y,t) = \quad -\frac{12\,\kappa^2(\gamma_1\kappa+\gamma_2\omega)}{(\gamma_3\kappa+2\,\gamma_4\omega)(\kappa\,x+\omega\,y+C)^2}, \tag{28}$$

where C is the integration constant.

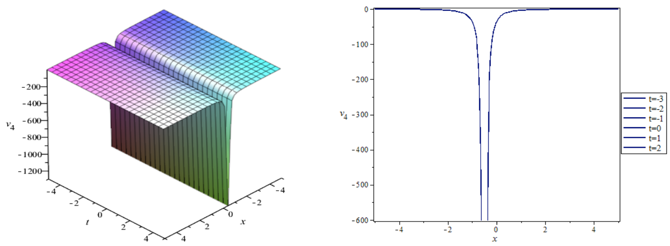
Note that the wave speed ν of traveling wave solutions for BK equations becomes zero in cases

4 and 5, whenever $\lambda^2 - 4\mu = 0$.



(a) (b)

Figure 7. 3D and 2D-plots of v_3 with $\gamma_1 = 1,\ \gamma_2 = 1,\ \gamma_3 = 6,\ \gamma_4 = -4,\ \lambda = 1,\ \mu = 0,\ \kappa = -1,$ $\omega = -1.2,\ C = 0,\ a_0 = 1$.



(a) (b)

Figure 8. 3D and 2D-plots of v_4 with $\gamma_1 = 1,\ \gamma_2 = 1,\ \gamma_3 = 6,\ \gamma_4 = -4,\ \lambda = 2,\ \mu = 1,\ \kappa = 2,$ $\omega = 1,\ C = 0,\ a_0 = 1$.

§4 Application of the modified Kudryashov method

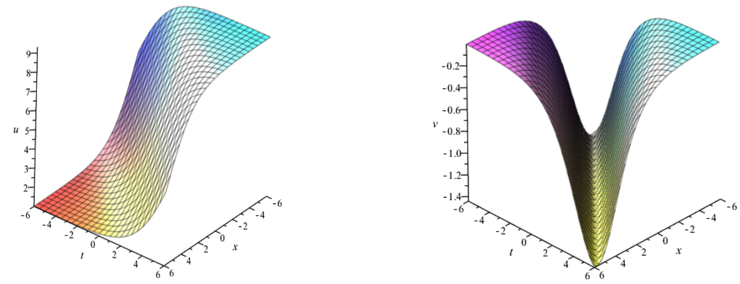
According to the modified Kudryashov method, the solution of Eq.(16) has the form:

$$u(\eta) = b_0 + b_1 P(\eta), \tag{29}$$

where b_0 and b_1 are the constants.

By the direct substitution of Eq.(29) along with Eq.(15) into Eq.(16), and equating the coefficients of each power of $P(\eta)$ to zero, we can obtain a system of algebraic equations:

$$\begin{aligned}
(P(\eta))^4 :&\ \frac{1}{2} b_1 \kappa\, (\ln B)^2 \left(12\, \gamma_2 \kappa\, \omega \ln B + b_1 \kappa\, \gamma_3 + 12\, \gamma_1 \kappa^2 \ln B + 2\, b_1 \gamma_4 \omega\right) = 0,\\
(P(\eta))^3 :&\ - b_1 \kappa\, (\ln B)^2 \left(12\, \gamma_2 \kappa\, \omega \ln B + b_1 \kappa\, \gamma_3 + 12\, \gamma_1 \kappa^2 \ln B + 2\, b_1 \gamma_4 \omega\right) = 0,\\
(P(\eta))^2 :&\ - \frac{1}{2} b_1 \ln B \big(14\, \gamma_2 \kappa^2 \omega\, (\ln B)^2 + b_1 \kappa^2 \gamma_3 \ln B + 14\, \gamma_1 \kappa^3\, (\ln B)^2\\
&\ + 2\, b_1 \gamma_4 \kappa\, \omega \ln B + 2\, \nu\big) = 0,\\
P(\eta)\ :&\ - b_1 \ln B \left(\nu + \gamma_2 \kappa^2 \omega\, (\ln B)^2 + \gamma_1 \kappa^3\, (\ln B)^2\right) = 0.
\end{aligned}$$



(a) (b)

Figure 9. 3D plots of u and v with $\gamma_1 = 1,\ \gamma_2 = 1,\ \gamma_3 = 6,\ \gamma_4 = -4,\ \kappa = 1,\ \omega = 1,\ B = 2,$ $k = 1, b_0 = 1.$

Solving the resulting system, we obtain

$$b_0 = b_0, \qquad b_1 = -\frac{12\,\kappa\ (\gamma_1\kappa + \gamma_2\omega)\ln B}{\kappa\,\gamma_3 + 2\,\gamma_4\omega}, \qquad \nu = -\gamma_2\kappa^2\omega\,(\ln B)^2 - \gamma_1\kappa^3\,(\ln B)^2\,. \tag{30}$$

In view of Eqs. (29), (14) and (30), we have the exact solution of Eq.(2),

$$u(x,y,t) = \quad b_0 - \frac{12\,\kappa\,(\gamma_1\kappa+\gamma_2\omega)\ln B}{(\kappa\,\gamma_3+2\,\gamma_4\omega)\Big(1+k(\sinh[(\kappa\,x+\omega\,y+\nu\,t)\ln B]+\cosh[(\kappa\,x+\omega\,y+\nu\,t)\ln B])\Big)} \tag{31}$$

and the exact solution of Eq.(1),

$$v(x,y,t) = \quad \frac{12\,k\kappa^2(\gamma_1\kappa+\gamma_2\omega)(\ln B)^2\Big(\sinh[(\kappa\,x+\omega\,y+\nu\,t)\ln B]+\cosh[(\kappa\,x+\omega\,y+\nu\,t)\ln B]\Big)}{(\kappa\,\gamma_3+2\,\gamma_4\omega)\Big(1+k(\sinh[(\kappa\,x+\omega\,y+\nu\,t)\ln B]+\cosh[(\kappa\,x+\omega\,y+\nu\,t)\ln B])\Big)^2}, \tag{32}$$

where $\nu = -\gamma_2\kappa^2\omega\,(\ln B)^2 - \gamma_1\kappa^3\,(\ln B)^2$.

§5 Results and discussion

In this section, we give the physical interpretation of different analytical or singular exact solutions of the considered BK equations (1) and (2). Here we represent the 3D and 2D plots of the solutions for $y = 0$.

Selecting $\gamma_1 = 1,\ \gamma_2 = 1,\ \gamma_3 = 4,\ \gamma_4 = 4$, the solutions u_1–u_4 are revealed in Figs. 1–4. We reveal the solutions u_1–u_4 given by Eqs. (19), (21), (23), (25) with the proper selections of other involving parameters $\lambda,\ \mu,\ \kappa$ and ω. Fig. 1 represents the solution u_1 and we observe the propagation of kink type waves along the x-axis. Fig. 2 represents the solution u_2 and in this case, we reveal the propagation of singular periodic solitary waves along the x-axis. We obtain the propagation of singular soliton solution u_3 in Fig. 3 along the x-axis. Fig. 4 represents the singular soliton solution u_4. We observe the wave profile remains the same at each instant of time, which also indicates the fact that the wave speed ν is zero for the solution u_4.

Selecting the parameters $\gamma_1 = 1,\ \gamma_2 = 1,\ \gamma_3 = 6,\ \gamma_4 = -4$, we reveal the solutions v_1–v_4 provided by Eqs. (20),(22),(24),(26) in Figs. 5–8. The propagation of solutions v_1–v_4 along the x-axis are shown with the proper selections of other involving parameters $\lambda,\ \mu,\ \kappa$ and ω.

They represent the bright soliton solution in Fig. 5, singular periodic solitary wave solution in Fig. 6 and singular soliton solutions in Figs. 7 and 8. We observe for the solution v_4 in Fig. 8 that the wave motion remains constant at each instant of time.

Choosing the parameters $\gamma_1 = 1,\ \gamma_2 = 1,\ \gamma_3 = 6,\ \gamma_4 = -4$, with the proper selection of other parameters, we reveal the solutions u and v given by Eqs. (31) and (32) in Figs. 9 (a) and (b), respectively. They represent kink and dark soliton solutions in Figs. 9 (a) and (b), respectively.

§6 Conclusion

In this paper, we retrieved the exact solutions for the two general forms of BK equations using two effective approaches. In addition to providing a standardized formulation for the exact solutions of NLEEs, the $\exp(-\Phi(\eta))$-expansion method and modified Kudryashov method provide a complete framework for classifying the forms of these solutions. The adopted techniques are quite impressive, efficient and offer several new exact solutions in the form of kink, bright, dark, singular soliton solutions and singular periodic solitary wave solutions for the considered BK equations. The findings of this study are novel and can be a valuable contribution to the fields of applied mathematics, mathematical physics and fluid dynamics. In addition to the solutions described, the process utilized in this paper can be applied to address comparable challenges. However, the study of the fractional form of BK equation employing various methods will be an interesting topic to be done in our future work.

Declarations

Conflict of interest The authors declare no conflict of interest.

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