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Gallai-Ramsey numbers for three graphs on at most five vertices

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Abstract. A Gallai k-coloring is a k-edge-coloring of a complete graph in which there are no rainbow triangles. For given graphs G_1, G_2, G_3 and nonnegative integers r, s, t with $k = r+s+t$, the k-colored Gallai-Ramsey number $gr_k(K_3 : r\cdot G_1,\ s\cdot G_2,\ t\cdot G_3)$ is the minimum integer n such that every Gallai k-colored K_n contains a monochromatic copy of G_1 colored by one of the first r colors or a monochromatic copy of G_2 colored by one of the middle s colors or a monochromatic copy of G_3 colored by one of the last t colors. In this paper, we determine the value of Gallai-Ramsey number in the case that $G_1 = B_3^+$, $G_2 = S_3^+$ and $G_3 = K_3$. Then the Gallai-Ramsey numbers $gr_k(K_3 : B_3^+)$, $gr_k(K_3 : S_3^+)$ and $gr_k(K_3 : K_3)$ are obtained, respectively. Furthermore, the Gallai-Ramsey numbers $gr_k(K_3 : r \cdot B_3^+,\ (k-r)\cdot S_3^+)$, $gr_k(K_3 : r\cdot B_3^+,\ (k-r)\cdot K_3)$ and $gr_k(K_3 : s\cdot S_3^+,\ (k-s)\cdot K_3)$ are obtained, respectively.

§1 Introduction

All graphs considered in this paper are finite, simple and undirected. For a graph G, we use $|G|$ to denote the number of vertices of G, say the *order* of G. The complete graph of order n is denoted by K_n. For a subset $S \subseteq V(G)$, let $G[S]$ be the subgraph of G induced by S. For two disjoint subsets A and B of $V(G)$, $E_G(A,B) = \{ab \in E(G) \mid a \in A, b \in B\}$. For any positive integer k, we write $[k]$ for the set $\{1, 2, \cdots, k\}$. An edge-colored graph is called *monochromatic* if all edges are colored by the same color, and *rainbow* if no two edges are colored by the same color.

Given graphs $H_1, H_2, \cdots, H_k$, the multicolor Ramsey number $R(H_1, H_2, \cdots, H_k)$ is the smallest positive integer n such that for every k-edge colored K_n with the color set $[k]$, there exists some $i \in [k]$ such that K_n contains a monochromatic copy of H_i colored by i. When $H = H_1 = \cdots = H_k$, we simply denote $R(H_1, \cdots, H_k)$ by $R_k(H)$. In this paper, we study Ramsey

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number in Gallai-coloring. A *Gallai-coloring* is an edge-coloring of a complete graph without rainbow triangle. Gallai-coloring naturally arises in several areas including: information theory [10]; the study of partially ordered sets, as in Gallai's original paper [6] (his result was restated in [8] in the terminology of graphs); and the study of perfect graphs [2]. A Gallai k-coloring is a Gallai-coloring that uses k colors. Given a positive integer k and graphs $H_1, H_2, \cdots, H_k$, the *Gallai-Ramsey number* $gr_k(K_3 : H_1, H_2, \cdots, H_k)$ is the smallest integer n such that every Gallai k-colored K_n contains a monochromatic copy of H_i in color i for some $i \in [k]$. Clearly, $gr_k(K_3 : H_1, H_2, \cdots, H_k) \leq R(H_1, H_2, \cdots, H_k)$ for any k and $gr_2(K_3 : H_1, H_2) = R(H_1, H_2)$. When $H = H_1 = \cdots = H_k$, we simply denote $gr_k(K_3 : H_1, H_2, \cdots, H_k)$ by $gr_k(K_3 : H)$. When $H = H_1 = \cdots = H_s (0 \leq s \leq k)$ and $G = H_{s+1} = \cdots = H_k$, we use the following shorthand notation

$$gr_k(K_3 : s \cdot H, (k-s) \cdot G) = gr_k(K_3 : \underbrace{H, \cdots, H}_{s \text{ times}}, \underbrace{G, \cdots, G}_{(k-s) \text{ times}}).$$

For nonnegative integers r, s, t, when $G_1 = H_1 = \cdots = H_r$, $G_2 = H_{r+1} = \cdots = H_{r+s}$, and $G_3 = H_{r+s+1} = \cdots = H_{r+s+t}$ with $k = r + s + t$, we use the following shorthand notation

$$gr_k(K_3 : r \cdot G_1, s \cdot G_2, t \cdot G_3) = gr_k(K_3 : \underbrace{G_1, \cdots, G_1}_{r \text{ times}}, \underbrace{G_2, \cdots, G_2}_{s \text{ times}}, \underbrace{G_3, \cdots, G_3}_{t \text{ times}}).$$

The Gallai-Ramsey numbers $gr_k(K_3 : H)$ for all the graphs H on five vertices and at most seven edges are obtained (see [11, 16, 17, 18]). There are two graphs on five vertices and eight edges, one of which is the wheel graph W_4, and the other is the graph B_3^+ obtained from the book graph B_3 by adding an edge between two vertices with degree two. Song et al. [15] and Mao et al. [12] obtained the Gallai-Ramsey number $gr_k(K_3 : W_4)$. In this paper, we determine the Gallai-Ramsey number $gr_k(K_3 : B_3^+)$. In order to get $gr_k(K_3 : B_3^+)$, we actually investigate the Gallai-Ramsey number $gr_k(K_3 : r \cdot B_3^+, s \cdot S_3^+, t \cdot K_3)$, where S_3^+ denotes the graph on 4 vertices obtained from K_3 by adding a pendant edge, as stated in Theorem 1.

Theorem 1. *For nonnegative integers r, s, t, let $k = r + s + t$. Then*

$$gr_k(K_3 : r \cdot B_3^+, s \cdot S_3^+, t \cdot K_3) = \begin{cases} 17^{\frac{r}{2}} \cdot 5^{\frac{t}{2}} + 1, & \text{if } r, t \text{ are even, } (c_1) \\ 2 \cdot 17^{\frac{r}{2}} \cdot 5^{\frac{t-1}{2}} + 1, & \text{if } r \text{ is even, } t \text{ is odd, } (c_2) \\ 8 \cdot 17^{\frac{r-1}{2}} \cdot 5^{\frac{t-1}{2}} + 1, & \text{if } r, t \text{ are odd, } (c_3) \\ 4 \cdot 17^{\frac{r-1}{2}} \cdot 5^{\frac{t}{2}} + 1, & \text{if } r \text{ is odd, } t \text{ is even, } (c_4) \\ 6 \cdot 17^{\frac{r}{2}} \cdot 5^{\frac{s+t-2}{2}} + 1, & \text{if } r \text{ and } s+t \text{ are even, } (c_5) \\ 3 \cdot 17^{\frac{r}{2}} \cdot 5^{\frac{s+t-1}{2}} + 1, & \text{if } r \text{ is even and } s+t \text{ is odd, } (c_6) \\ 9 \cdot 17^{\frac{r-1}{2}} + 1, & \text{if } r \text{ is odd and } s = 1 \text{ and } t = 0, (c_7) \\ 48 \cdot 17^{\frac{r-1}{2}} \cdot 5^{\frac{s+t-3}{2}} + 1, & \text{if } r \text{ and } s+t \text{ are odd and } t \neq 0 \text{ or } \\ & s \neq 1, (c_8) \\ 24 \cdot 17^{\frac{r-1}{2}} \cdot 5^{\frac{s+t-2}{2}} + 1, & \text{if } r \text{ is odd and } s+t \text{ is even, } (c_9) \end{cases}$$

where $s = 0$ for Condition c_1 to Condition c_4 and $s \geq 1$ for Condition c_5 to Condition c_9.

In Theorem 1, when we set $s = t = 0$, we can get the following Theorem 2.

Theorem 2. *For any integer $r \geq 1$,*

$$gr_r(K_3 : B_3^+) = \begin{cases} 17^{r/2} + 1, & \text{if } r \text{ is even,} \\ 4 \cdot 17^{(r-1)/2} + 1, & \text{if } r \text{ is odd.} \end{cases}$$

When we set $r = s = 0$ and $r = t = 0$, respectively, we can get the Gallai-Ramsey numbers $gr_t(K_3 : K_3)$ (see [3, 7]) and $gr_s(K_3 : S_3^+)$ (see [13]), respectively. When we set $r = 0$, $s = 0$ and $t = 0$, respectively, we can get the following Theorem 3 to Theorem 5, respectively.

Theorem 3. *Let k be a positive integer and s be an integer such that $0 \leq s \leq k$. Then*

$$gr_k(K_3 : s \cdot S_3^+, (k-s) \cdot K_3) = \begin{cases} 5^{\frac{k}{2}} + 1, & \text{if } s = 0 \text{ and } k \text{ is even,} \\ 2 \cdot 5^{\frac{k-1}{2}} + 1, & \text{if } s = 0 \text{ and } k \text{ is odd,} \\ 6 \cdot 5^{\frac{k-2}{2}} + 1, & \text{if } s \geq 1 \text{ and } k \text{ is even,} \\ 3 \cdot 5^{\frac{k-1}{2}} + 1, & \text{if } s \geq 1 \text{ and } k \text{ is odd.} \end{cases}$$

Theorem 4. *Let k be a positive integer and r be an integer such that $0 \leq r \leq k$. Then*

$$gr_k(K_3 : r \cdot B_3^+, (k-r) \cdot K_3) = \begin{cases} 17^{\frac{r}{2}} \cdot 5^{\frac{k-r}{2}} + 1, & \text{if both } r \text{ and } (k-r) \text{ are even,} \\ 2 \cdot 17^{\frac{r}{2}} \cdot 5^{\frac{k-r-1}{2}} + 1, & \text{if } r \text{ is even and } (k-r) \text{ is odd,} \\ 8 \cdot 17^{\frac{r-1}{2}} \cdot 5^{\frac{k-r-1}{2}} + 1, & \text{if both } r \text{ and } (k-r) \text{ are odd,} \\ 4 \cdot 17^{\frac{r-1}{2}} \cdot 5^{\frac{k-r}{2}} + 1, & \text{if } r \text{ is odd and } (k-r) \text{ is even.} \end{cases}$$

Theorem 5. *Let k be a positive integer and r be an integer such that $0 \leq r \leq k$. Then*

$$gr_k(K_3 : r \cdot B_3^+, (k-r) \cdot S_3^+) = \begin{cases} 6 \cdot 17^{\frac{r}{2}} \cdot 5^{\frac{k-r-2}{2}} + 1, & \text{if } r < k \text{ and both } r \text{ and } (k-r) \text{ are even,} \\ 17^{\frac{k}{2}} + 1, & \text{if } r = k \text{ and } k \text{ is even,} \\ 3 \cdot 17^{\frac{r}{2}} \cdot 5^{\frac{k-r-1}{2}} + 1, & \text{if } r \text{ is even and } (k-r) \text{ is odd,} \\ 48 \cdot 17^{\frac{r-1}{2}} \cdot 5^{\frac{k-r-3}{2}} + 1, & \text{if } r < k-1 \text{ and both } r \text{ and } (k-r) \text{ are odd,} \\ 9 \cdot 17^{\frac{k-2}{2}} + 1, & \text{if } r = k-1 \text{ and } k \text{ is even,} \\ 24 \cdot 17^{\frac{r-1}{2}} \cdot 5^{\frac{k-r-2}{2}} + 1, & \text{if } r < k \text{ and } r \text{ is odd and } (k-r) \text{ is even,} \\ 4 \cdot 17^{\frac{k-1}{2}} + 1, & \text{if } r = k \text{ and } k \text{ is odd.} \end{cases}$$

To prove Theorem 1, the following theorem is useful.

Theorem 6. [1, 6, 8] (Gallai-partition) *For any Gallai-coloring of a complete graph G, there exists a partition of $V(G)$ into at least two parts such that there are at most two colors on the edges between the parts and there is only one color on the edges between each pair of parts. The partition is called a Gallai-partition.*

§2 Proof of Theorem 1

First, recall some known classical Ramsey numbers which are useful.

Lemma 1. [4, 5, 9, 14]

$$R_2(K_3)=6, R_2(S_3^+)=7, R_2(B_3^+)=R_2(K_4)=18, R(K_3,S_3^+)=7,$$
$$R(K_3,B_3^+)=R(K_3,K_4)=9, R(S_3^+,B_3^+)=R(S_3^+,K_4)=10.$$

A *critical graph* of the Ramsey number $R(H_1,H_2)$, denoted by $C_{(H_1,H_2)}$, is a 2-edge colored $K_{R(H_1,H_2)-1}$ with red and blue such that there is neither a red copy of H_1 nor a blue copy of H_2. For example, $C_{(K_3,S_3^+)}$ is a 2-edge colored K_6 with red and blue such that there is neither a red copy of K_3 nor a blue copy of S_3^+.

For the sake of notation, let $f(r,s,t)=gr_k(K_3: r\cdot B_3^+, s\cdot S_3^+, t\cdot K_3)-1$, claimed in Theorem 1. It is easy to check the following inequalities:

$$\frac{f(r,s,t-1)}{f(r,s,t)}\le\frac{1}{2}\quad(1),\ \frac{f(r,s-1,t)}{f(r,s,t)}\le\frac{1}{2}\quad(2),\ \frac{f(r-1,s,t)}{f(r,s,t)}\le\begin{cases}\frac{1}{3}, & \text{if } s=1 \text{ and } t=0,\\ \frac{5}{16}, & \text{others,}\end{cases}\quad(3),$$

$$\frac{f(r-1,s+1,t)}{f(r,s,t)}\le\frac{3}{4}\quad(4),\ \frac{f(r-1,s,t+1)}{f(r,s,t)}\le\frac{2}{3}\quad(5),\ \frac{f(r,s,t-2)}{f(r,s,t)}\le\frac{1}{5}\quad(6),$$

$$\frac{f(r,s-1,t-1)}{f(r,s,t)}\le\frac{1}{5}\quad(7),\ \frac{f(r,s-2,t)}{f(r,s,t)}\le\frac{1}{5}\quad(8),\ \frac{f(r-1,s,t-1)}{f(r,s,t)}\le\frac{1}{8}\quad(9),$$

$$\frac{f(r-1,s-1,t)}{f(r,s,t)}\le\frac{1}{8}\quad(10),\ \frac{f(r-1,s+1,t-1)}{f(r,s,t)}\le\frac{3}{8}\quad(11),\ \frac{f(r-1,s-1,t+1)}{f(r,s,t)}\le\frac{5}{16}\quad(12),$$

$$\frac{f(r-2,s+1,t+1)}{f(r,s,t)}\le\frac{15}{34}\quad(13),\ \frac{f(r-2,s+1,t)}{f(r,s,t)}\le\frac{3}{17}\quad(14),\ \frac{f(r-2,s,t+2)}{f(r,s,t)}\le\frac{16}{51}\quad(15),$$

$$\frac{f(r-2,s,t+1)}{f(r,s,t)}\le\frac{8}{51}\quad(16),\ \frac{f(r-2,s,t)}{f(r,s,t)}=\frac{1}{17}\quad(17).$$

Now we prove Theorem 1.

Proof. We first prove that $gr_k(K_3: r\cdot B_3^+, s\cdot S_3^+, t\cdot K_3)\ge f(r,s,t)+1$ by constructing a Gallai k-colored complete graph G_k with order $f(r,s,t)$ which contains no monochromatic copy of B_3^+ colored by one of the first r colors and no monochromatic copy of S_3^+ colored by one of the middle s colors and no monochromatic copy of K_3 colored by one of the remaining t colors. For this construction, we use critical graphs of classical Ramsey results. Let $Q_1=C_{(K_3,K_3)}$, $Q_2=C_{(K_3,S_3^+)}$, $Q_3=C_{(K_3,K_4)}$, $Q_4=C_{(S_3^+,S_3^+)}$, $Q_5=C_{(S_3^+,K_4)}$ and $Q_6=C_{(K_4,K_4)}$. We construct our critical graph G_k by taking blow-ups of these critical graphs Q_j ($j\in[6]$). A *blow-up* of a 2-edge-colored graph Q_j with two new colors on an i-edge-colored graph G_i is a new graph G_{i+2} obtained from Q_j by replacing each vertex of Q_j with G_i and replacing each edge e of Q_j with a monochromatic complete bipartite graph $(V(G_i),V(G_i))$ in the same color with e. By induction on i, suppose that we have constructed graphs G_i, where G_i is i-edge-colored such that G_i contains no rainbow triangle and no appropriately colored monochromatic B_3^+ or S_3^+ or K_3. If $i=k$, then the construction is completed. Otherwise, we construct G_{i+2} by taking a

blow-up of 2-edge-colored Q_j with two new colors on G_i by distinguishing the following cases.
Case a. If the two new colors are in the first r colors, then we construct G_{i+2} by making a blow-up of Q_6 on G_i.
Case b. If the two new colors are in the middle s colors or in the last t colors, then we construct G_{i+2} by making a blow-up of Q_1 on G_i.

The base graphs (i.e., the first graphs in the induction) for this construction are constructed as follows.

For **Condition** c_1, the base graph G_0 is a single vertex. For **Condition** c_2, the base graph G_1 is a K_2 colored by one of the last t colors. For **Condition** c_3, the base graph G_2 is a Q_3 colored by two colors in which one is in the first r colors and the other is in the last t colors. For **Condition** c_4, the base graph G_1 is a monochromatic K_4 colored by one of the first r colors. For **Condition** c_5, the base graph G_2 is a Q_4 colored by two colors which are in the middle s colors if s, t are both even and the base graph G_2 is a Q_2 colored by two colors in which one is in the middle s colors and the other is in the last t colors if s, t are both odd. For **Condition** c_6, if s is odd and t is even, the base graph G_1 is a monochromatic copy of K_3 colored by one of the middle s colors. If s is even and t is odd, the base graph G_3 is a blow-up of Q_1 on a monochromatic K_3, where K_3 is colored by one of the middle s colors and Q_1 is colored by two new colors one of which is in the middle s colors and the other is in the last t colors. For **Condition** c_7, the base graph G_2 is a Q_5 with two colors in which one is in the first r colors and the other is in the middle s colors. For **Condition** c_8, if $t = 0$ and $s \geq 3$, the base graph G_4 is a blow-up of Q_3 on Q_4, where Q_4 is colored by two colors which are in the middle s colors and Q_3 is colored by two new colors one of which is in the first r colors and the other is in the middle s colors. Otherwise, if s is even and t is odd, the base graph G_4 is a blow-up of Q_3 on Q_4, where Q_4 is colored by two colors which are in the middle s colors and Q_3 is colored by two colors one of which is in the first r colors and the other is in the last t colors. If s is odd and t is even, then we first construct G_3. G_3 is a blow-up of Q_3 on a monochromatic K_3, where K_3 is colored by one of middle s colors and Q_3 is colored by two colors one of which is in the first r colors and the other is in the last t colors. The base graph G_4 is a blow-up of a monochromatic K_2 on G_3, where K_2 is colored by a new color in the last t colors. For **Condition** c_9, if s and t are both odd, the base graph G_3 is a blow-up of Q_3 on a monochromatic K_3, where K_3 is colored by one of the middle s colors and Q_3 is colored by two colors one of which is in the first r colors and the other is in the last t colors. If s and t are even, the base graph G_3 is a blow-up of Q_3 on a monochromatic K_3, where K_3 is colored by one of the middle s colors and Q_3 is colored by two new colors one of which is in the first r colors and the other is in the middle s colors.

It is easy to check that G_k is a Gallai k-colored complete graph which contains no appropriately colored monochromatic B_3^+ or S_3^+ or K_3 with order $f(r, s, t)$. Therefore, we see that $gr_k(K_3 : r \cdot B_3^+, s \cdot S_3^+, t \cdot K_3) \geq f(r, s, t) + 1$.

Now we prove that $gr_k(K_3 : r \cdot B_3^+, s \cdot S_3^+, t \cdot K_3) \leq f(r, s, t) + 1$ by induction on $3r + 2s + t$. The cases for $3r + 2s + t = 1$ or $k = 1$ are trivial. The statement holds in the case that

$3r+2s+t=2$ or $k=2$ by Lemma 1. So we can assume that $k \geq 3$, $3r+2s+t \geq 3$, and the statement holds for any r', s' and t' such that $3r'+2s'+t' < 3r+2s+t$. Let $n = f(r,s,t)+1$ and G be a Gallai k-colored complete graph of order n. Then $n \geq f(0,0,3)+1 = 11$. Suppose, to the contrary, that G contains neither a monochromatic copy of B_3^+ in any one of the first r colors nor a monochromatic copy of S_3^+ in any one of the middle s colors nor a monochromatic copy of K_3 in any one of the last t colors. By Theorem 6, there exists a Gallai-partition of $V(G)$. Choose a Gallai-partition with the smallest number of parts, say $(V_1, V_2, \cdots, V_q)$ and let $H_i = G[V_i]$ for each part V_i. Then $q \geq 2$.

We first consider the case that $q = 2$. W.L.O.G, suppose that the color on the edges between two parts is red. First suppose that red is in the last t colors or in the middle s colors. Then H_1 and H_2 both have no red edges, otherwise, there exists a red K_3 or a red S_3^+, a contradiction. Hence by the induction hypothesis, $|H_i| \leq f(r,s,t-1)$ if red is in the last t colors and $|H_i| \leq f(r,s-1,t)$ if red is in the middle s colors for each $i \in [2]$. By the inequalities (1) and (2), we have that

$$|G| = |H_1| + |H_2| \leq 2 \times \frac{1}{2} f(r,s,t) < n,$$

a contradiction. Next suppose that red is in the first r colors. If both H_1 and H_2 have a red edge, then G has a red B_3^+, a contradiction. First suppose that H_1 and H_2 both have no red edges. By the induction hypothesis and Inequality (3),

$$|G| = |H_1| + |H_2| \leq 2f(r-1,s,t) \leq \frac{2}{3} f(r,s,t) < n,$$

a contradiction. Then suppose that H_1 has a red edge, but H_2 has no red edges. Clearly, H_1 contains no red S_3^+, otherwise, G contains a red B_3^+, a contradiction. We first consider the case that H_1 contains a red K_3. To avoid a red B_3^+, we have that $|H_2| = 1$. Then H_1 can be seen that red is moved in the middle s colors. By the induction hypothesis and Inequality (4), we get that

$$|G| = |H_1| + |H_2| \leq f(r-1,s+1,t) + 1 \leq \frac{3}{4} f(r,s,t) + 1 < n,$$

a contradiction. So we can assume that H_1 contains no red K_3. Thus H_1 can be seen that red is moved in the last t colors. By the induction hypothesis and the inequalities (5) and (3), we get that

$$|G| = |H_1| + |H_2| \leq f(r-1,s,t+1) + f(r-1,s,t) \leq \left(\frac{2}{3} + \frac{1}{3}\right) f(r,s,t) < n,$$

a contradiction.

Now we can assume that $q \geq 3$ and the two colors appeared in the Gallai-partition (V_1, V_2, $\cdots$, V_q) are red and blue. If there exists one part (say V_1) such that all edges joining V_1 to the other parts are colored by the same color, then we can find a new Gallai-partition with two parts $(V_1, V_2 \bigcup \cdots \bigcup V_q)$, which contradicts with q is smallest. It follows that $q \neq 3$ and the following fact holds.

Fact 1. *For each part V_i, there exist both red and blue edges connecting V_i and the other parts.*

Now we can assume that $q \geq 4$. Then we have the following facts.

Fact 2. *If red is in the middle s colors, then G contains no red K_3 and the statement holds for blue symmetrically.*

Otherwise, suppose that G has a red K_3, say $v_1v_2v_3$. Let U be the union of parts containing a vertex in $\{v_1, v_2, v_3\}$. Then U contains at most 3 parts of Gallai-partition in G. If there exists a red edge in $E_G(U, V(G) \setminus U)$, then G contains a red S_3^+, a contradiction. It follows that all edges in $E_G(U, V(G) \setminus U)$ are blue. Then $(U, V(G) \setminus U)$ is a new Gallai-partition, which contradicts with $q \geq 4$ and q is smallest.

By Fact 1 and Fact 2, we have the following fact.

Fact 3. *If red is in the middle s colors, then every H_i has no red edges and the statement holds for blue symmetrically.*

Now we consider the following cases.

Case 1. *Neither red nor blue is in the first r colors.*

First we prove the following claim.

Claim 1. *G contains neither a red K_3 nor a blue K_3.*

Proof. If red is in the last t colors, then G contains no red K_3 clearly. If red is in the middle s colors, then G contains no red K_3 by Fact 2. The statement holds for blue symmetrically. □

Since $R_2(K_3) = 6$, we have that $q \leq 5$ by Claim 1. By Fact 1 and Claim 1, each H_i contains neither red nor blue edges. By the induction hypothesis, for each H_i, we have that

$$|H_i| \leq \begin{cases} f(r, s, t-2), \text{if red and blue are both in the last } t \text{ colors}, \\ f(r, s-2, t), \text{if red and blue are both in the middle } s \text{ colors}, \\ f(r, s-1, t-1), \text{if red is in the middle } s \text{ colors and blue in the last } t \text{ colors}. \end{cases}$$

By the inequalities (6)-(8), we have that

$$|G| = \sum_{i=1}^{q} |H_i| \leq 5 \times \frac{1}{5} f(r, s, t) < n,$$

a contradiction. The proof of Case 1 is completed.

Now we consider the case that red or blue are in the first r colors, we have the following fact and claim.

Fact 4. *If red is in the first r colors and both H_i and H_j contain red edges, then the edges in $E_G(V_i, V_j)$ must be blue and the statement holds for blue symmetrically.*

Otherwise, suppose that the edges in $E_G(V_i, V_j)$ are red. To avoid a red B_3^+, all edges in $E_G(\{V_i, V_j\}, V(G) \setminus \{V_i, V_j\})$ are blue. Then $(V_i \cup V_j, V(G) \setminus (V_i \cup V_j))$ is a new Gallai-partition with $q = 2$ which contradicts with $q \geq 4$ and q is smallest.

Claim 2. *If red is in the first r colors and X is the union of p parts of the Gallai-partition such that $G[X]$ contains no blue edges. Then*
(i) $|X| \leq f(r-1,s,t)+f(r-1,s,t-1)$ if blue is in the last t colors.
(ii) $|X| \leq f(r-1,s-1,t+1)+f(r-1,s-1,t)$ if blue is in the middle s colors.
(iii) $|X| \leq f(r-2,s,t+1)+f(r-2,s,t)$ if blue is in the first r colors.
The statement holds for blue symmetrically.

Proof. Since red is in the first r colors, $G[X]$ contains no red B_3^+. To avoid a red B_3^+, by Fact 1, for each part V_i in X, H_i contains no red S_3^+. Since $G[X]$ has no blue edges, every pair of parts in X are joined by red edges in $G[X]$. It follows that $p \leq 4$. By Fact 4, there is at most one part in $G[X]$ containing red edges. Then we distinguish two cases.
Case I. Each part in $G[X]$ has no red edges.
If $p \leq 3$, then

$$|X| \leq \begin{cases} 3f(r-1,s,t-1), \text{if blue is in the last } t \text{ colors,} \\ 3f(r-1,s-1,t), \text{if blue is in the middle } s \text{ colors,} \\ 3f(r-2,s,t), \text{if blue is in the first } r \text{ colors.} \end{cases}$$

By the definition of $f(r,s,t)$, we can check that $3f(r-1,s,t-1) \leq f(r-1,s,t)+f(r-1,s,t-1)$, $3f(r-1,s-1,t) \leq f(r-1,s-1,t+1)+f(r-1,s-1,t)$ and $3f(r-2,s,t) \leq f(r-2,s,t+1)+f(r-2,s,t)$. Then the statement of Claim 2 holds in this case. If $p=4$, to avoid a red B_3^+, then we have that $|X|=4$. Clearly, the statement still holds.
Case II. There is a unique part containing a red edge in $G[X]$, say H_1.
If H_1 contains a red K_3, then to avoid a red B_3^+, $|X \backslash V_1| \leq 1$. So

$$|X| \leq |H_1|+1 \leq \begin{cases} f(r-1,s+1,t-1)+1, \text{if blue is in the last } t \text{ colors,} \\ f(r-1,s,t)+1, \text{if blue is in the middle } s \text{ colors,} \\ f(r-2,s+1,t)+1, \text{if blue is in the first } r \text{ colors.} \end{cases}$$

In this case, the statement of Claim 2 holds. If H_1 contains no red K_3, then

$$|H_1| \leq \begin{cases} f(r-1,s,t), \text{if blue is in the last } t \text{ colors,} \\ f(r-1,s-1,t+1), \text{if blue is in the middle } s \text{ colors,} \\ f(r-2,s,t+1), \text{if blue is in the first } r \text{ colors.} \end{cases}$$

Since $G[X]$ has no red B_3^+ and H_1 has a red edge, we have that $p \leq 3$. Furthermore, $|X \setminus V_1|=2$ if $p=3$ and $|X|=|H_1|$ if $p=1$. It is easy to check that the statement still holds. If $p=2$, then the other part in $G[X]$, say H_2, contains no red edges. So

$$|X|=|H_1|+|H_2| \leq \begin{cases} f(r-1,s,t)+f(r-1,s,t-1), \text{if blue is in the last } t \text{ colors,} \\ f(r-1,s-1,t+1)+f(r-1,s-1,t), \text{if blue is in the middle } s \text{ colors,} \\ f(r-2,s,t+1)+f(r-2,s,t), \text{if blue is in the first } r \text{ colors.} \end{cases}$$

Complete the proof of Claim 2. □

For the part V_1 of Gallai-partition, let V_R (V_B) be the union of parts V_i such that the edges in $E_G(V_1, V_i)$ are red (blue) and $H_R = G[V_R]$, $H_B = G[V_B]$.

Case 2. *Exactly one of red and blue is in the first r colors.*

W.L.O.G., suppose that red appears in the first r colors. Then G has no red B_3^+. Since blue in the middle s colors or in the last t colors, G has no blue K_3 by Fact 2. Since $R(B_3^+, K_3) = R(K_4, K_3) = 9$, $q \le 8$. By Fact 1, each H_i contains neither a red S_3^+ nor a blue edge. If each H_i contains no red edges, then by the induction hypothesis and the inequalities (9) and (10), we get that

$$|G| = \sum_{i=1}^{q} |H_i| \le \begin{cases} 8f(r-1,s,t-1) \le f(r,s,t) < n, \text{if blue is in the last } t \text{ colors,} \\ 8f(r-1,s-1,t) \le f(r,s,t) < n, \text{if blue is in the middle } s \text{ colors.} \end{cases}$$

a contradiction. It follows that there is at least one part containing red edges. W.L.O.G., suppose that H_1 contains a red edge. So we consider the following two Subcases.

Subcase 2.1. *H_1 has a red K_3.*

To avoid a red B_3^+ or a blue K_3, we get that $|H_R| = 1$ and H_B contains no blue edges. By the induction hypothesis and the inequalities (11), (1), (3) and (2), we get that

$$\begin{aligned} |G| &= |H_1| + |H_R| + |H_B| \\ &\le \begin{cases} f(r-1,s+1,t-1) + 1 + f(r,s,t-1) \le \frac{7}{8} f(r,s,t) + 1 < n, \text{if blue is in the last } t \\ \text{colors,} \\ f(r-1,s,t) + 1 + f(r,s-1,t) \le \frac{5}{6} f(r,s,t) + 1 < n, \text{if blue is in the middle } s \\ \text{colors.} \end{cases} \end{aligned}$$

a contradiction.

Subcase 2.2. *H_1 has no red K_3.*

To avoid a blue K_3, H_B contains no blue edges. By Claim 2, we have that

$$|H_B| \le \begin{cases} f(r-1,s,t) + f(r-1,s,t-1), \text{if blue is in the last } t \text{ colors,} \\ f(r-1,s-1,t+1) + f(r-1,s-1,t), \text{if blue is in the middle } s \text{ colors.} \end{cases}$$

Suppose that $|H_R| \ge 3$. To avoid a red B_3^+, H_R contains no red edges. It follows that the edges between each pair of parts in H_R are blue. Since G has no blue K_3, H_R contains at most two parts of Gallai-partition. Furthermore, by Fact 1 and Fact 3, each part in H_R contains no blue edges. By the induction hypothesis,

$$|H_R| \le \begin{cases} 2f(r-1,s,t-1), \text{if blue is in the last } t \text{ colors,} \\ 2f(r-1,s-1,t), \text{if blue is in the middle } s \text{ colors.} \end{cases}$$

If $|H_R| \le 2$, then the above inequality still holds. By the induction hypothesis and the inequalities (3) (under the condition that $t \geq 1$), (9), (12) and (10), we get that

$$|G| = |H_1| + |H_B| + |H_R| \leq \begin{cases} 2f(r-1,s,t) + 3f(r-1,s,t-1) \leq f(r,s,t) < n, \text{if blue is in the last } t \text{ colors,} \\ 2f(r-1,s-1,t+1) + 3f(r-1,s-1,t) < n, \text{if blue is in the middle } s \text{ colors.} \end{cases}$$

a contradiction. The proof of Case 2 is completed.

Case 3. *Both red and blue are in the first r colors.*

In this case, the graph G contains neither red nor blue B_3^+. Then each H_i contains neither red nor blue S_3^+. Since $R_2(B_3^+) = R_2(K_4) = 18$, we have $q \leq 17$. First we prove the following claim.

Claim 3. *Each part H_i of G contains neither red nor blue K_3.*

Proof. Suppose, to the contrary, that there exists a part containing a red K_3 or a blue K_3, say H_1. If H_1 contains both red K_3 and blue K_3, then $|H_R| = |H_B| = 1$. (Otherwise, G has a red or blue B_3^+, a contradiction.) It follows that $q = 3$, which contradicts with $q \geq 4$. Now, W.L.O.G., suppose that H_1 contains a blue K_3, but contains no red K_3. Then $|H_B| = 1$. If H_1 contains no red edges, then to avoid a red B_3^+, H_R contains no red K_3. By the induction hypothesis and the inequalities (14) and (5), we have that

$$|G| = |H_1| + |H_R| + |H_B| \leq f(r-2,s+1,t) + f(r-1,s,t+1) + 1 \leq \frac{43}{51} f(r,s,t) + 1 < n,$$

a contradiction. Now we assume that H_1 contains a red edge. Suppose that $|H_R| \geq 3$. To avoid a red B_3^+, H_R contains no red edges. By the induction hypothesis, $|H_R| \leq f(r-1,s,t)$. If $|H_R| \leq 2$, then $|H_R| \leq f(r-1,s,t)$ clearly. By the inequalities (13) and (3), we have that

$$|G| = |H_1| + |H_R| + |H_B| \leq f(r-2,s+1,t+1) + f(r-1,s,t) + 1 \leq \frac{79}{102} f(r,s,t) + 1 < n,$$

a contradiction. Complete the proof of Claim 3. □

Now we consider the following subcases.

Subcase 3.1. *There exists a part H_1 containing both a red and a blue edge.*

Suppose that $|H_R| \geq 3$. To avoid a red B_3^+, H_R contains no red edges. By the induction hypothesis, we have that $|H_R| \leq f(r-1,s,t)$. If $|H_R| \leq 2$, then $|H_R| \leq f(r-1,s,t)$ clearly. Similarly, we can get that $|H_B| \leq f(r-1,s,t)$. By Claim 3 and the inequalities (15) and (3), we have that

$$|G| = |H_1| + |H_R| + |H_B| \leq f(r-2,s,t+2) + 2f(r-1,s,t) \leq \frac{50}{51} f(r,s,t) < n,$$

a contradiction.

Now we can assume that each part contains no red edges or no blue edges in the following. We call a part *free* if it contains neither red nor blue edges. We call a part *red (blue)* if it contains only red (blue) edges. By the induction hypothesis and Claim 3, we have that $|H_i| \leq f(r-2,s,t)$ if H_i is a free part and $|H_i| \leq f(r-2,s,t+1)$ if H_i is a red or blue part.

Subcase 3.2. *Each part H_i is a free part.*

By Inequality (17), we have that

$$|G| = \sum_{i=1}^{q} |H_i| \leq 17f(r-2,s,t) = f(r,s,t) < n,$$

a contradiction.

Subcase 3.3. *There exists one part H_1 which is red or blue.*

W.L.O.G., suppose that H_1 is red. If $|H_R| \geq 3$, then H_R has no red edges since G has no red B_3^+. Then by Claim 2, we have that $|H_R| \leq f(r-2,s,t) + f(r-2,s,t+1)$. If $|H_R| \leq 2$, then $|H_R| \leq 2 \leq f(r-2,s,t) + f(r-2,s,t+1)$ clearly. To avoid a blue B_3^+, H_B contains no blue K_3. Since $R(K_3, B_3^+) = R(K_3, K_4) = 9$, we have that H_B contains at most 8 parts of Gallai-partition. First suppose that all parts in H_B are free parts. Then $|H_B| \leq 8f(r-2,s,t)$. By the inequalities (16) and (17), we have that

$$|G| = |H_1| + |H_R| + |H_B| \leq 2f(r-2,s,t+1) + 9f(r-2,s,t) \leq \frac{43}{51}f(r,s,t) < n,$$

a contradiction. Next suppose that there is one part in H_B, say H_2, such that H_2 is blue or red part. Let V_{2R} (V_{2B}) be the union of parts V_i in V_B such that the edges in $E_G(V_2, V_i)$ are red (blue) and $H_{2R} = G[V_{2R}]$, $H_{2B} = G[V_{2B}]$. If H_2 is a blue part, then to avoid a red or blue B_3^+, H_{2R} contains neither red nor blue K_3 and $H_{2B} = \emptyset$. By the induction hypothesis and Claim 3, $|H_B| = |H_2| + |H_{2R}| \leq f(r-2,s,t+1) + f(r-2,s,t+2)$. By the inequalities (16), (17) and (15), we have that

$$\begin{aligned} |G| &= |H_1| + |H_R| + |H_B| \leq 3f(r-2,s,t+1) + f(r-2,s,t) + f(r-2,s,t+2) \\ &\leq \frac{43}{51}f(r,s,t) < n, \end{aligned}$$

a contradiction. If H_2 is a red part, then H_{2B} contains no blue edges since G contains no blue B_3^+. Then by Claim 2, $|H_{2B}| \leq f(r-2,s,t+1) + f(r-2,s,t)$. Now suppose that $|H_{2R}| \geq 3$. To avoid a red B_3^+, H_{2R} contains no red edges. It follows that the edges between each pair of parts in H_{2R} are blue. Since G has no blue B_3^+, H_{2R} contains no blue K_3. It follows that there are at most two parts in H_{2R}. If H_{2R} contains only one part, then H_{2R} is a free part or a blue part. We can get that $|H_{2R}| \leq f(r-2,s,t+1)$. If H_{2R} contains two parts, then both two parts are free parts. We have that $|H_{2R}| \leq 2f(r-2,s,t) \leq f(r-2,s,t+1)$. If $|H_{2R}| \leq 2$, then $|H_{2R}| \leq 2 \leq 2f(r-2,s,t) \leq f(r-2,s,t+1)$ clearly. So

$$|H_B| = |H_2| + |H_{2R}| + |H_{2B}| \leq 3f(r-2,s,t+1) + f(r-2,s,t).$$

By the inequalities (16) and (17), we have that

$$|G| = |H_1| + |H_R| + |H_B| \leq 5f(r-2,s,t+1) + 2f(r-2,s,t) \leq \frac{46}{51}f(r,s,t) < n,$$

a contradiction. Complete the proof of Case 3 and then the proof of Theorem 1. □

Declarations

Conflict of interest The authors declare no conflict of interest.

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