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Stochastic interpretation for a single server retrial queue with Bernoulli feedback and negative customers

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Abstract. In this paper, we introduce a qualitative analysis in order to study the monotonicity and comparability properties of a single-server retrial queueing model with Bernoulli feedback and negative customers, relative to stochastic orderings. Performance measures of such a system are available explicitly, while their forms are cumbersome (these formulas include integrals of Laplace transform, solutions of functional equations, etc.). Therefore, they are not exploitable from the application point of view. To overcome these difficulties, we present stochastic comparison methods in order to get qualitative estimates of these measures. In particular, we prove the monotonicity of the transition operator of the embedded Markov chain. In addition, we establish conditions for which transition operators as well as stationary probabilities, associated with two embedded Markov chains, having the same structure but with different parameters, are comparable relative to the given stochastic orderings. Further, numerical examples are carried out to illustrate the theoretical results.

§1 Introduction

Every day we experience many queueing instances in which service is not prompt and customers need to wait to get service [41,42]. The topic of retrial queues in queueing theory has been an interesting research topic in the last two decades. The main feature of a retrial queue is that incoming customers who find all servers occupied are obliged to quit the service area and join a retrial group, called orbit, in order to try their luck again after some random time. Retrial queueing models appear in stochastic modeling of several real-life situations including data transmission wherein a packet transmitted from the source to the destination may be returned and the process should be repeated until the packet is finally transmitted. For a detailed review of the main results and literature on this topic, the reader can be referred to [3,4,28,48,49].

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In recent years, a particular attention has been addressed to classical queues and queueing networks characterized by the presence of two types of arrivals, positive (regular) and negative. In its simplest version, a negative customer removes a positive customer in the queue (or being served) according to a specified killing discipline. The most common killing strategies considered in the literature are (i) the arrival of a negative customer who removes all customers present in the system, (ii) the arrival of a negative customer who deletes the customer from the head, including the one in service, (iii) the arrival of a negative customer who deletes the customer at the end of the queue (for more details see [26] and the references therein). All these kinds of negative customers have no effect on an empty system. The interest in this family of queueing networks, introduced by Gelenbe [30], was initially motivated by the modeling of neuronal networks where positive and negative arrivals represent the excitatory signals which increase the potential of the neuronal to produce impulsion and inhibitors, and decrease the potential of the neuronal in producing the impulsion, respectively. Further, their fields of application are extended to other complex systems as computer networks with virus infection [5], bulk arrival retrial queues [7], telecommunications systems [24], elimination of transactions in databases [29], production systems [31], inventory models [32], and feedback retrial queues [36].

In various queueing situations, customers may be unsatisfied with their service for some reasons (incomplete or unsatisfactory service). In such cases, they can come back to the system and retry again and again until a successful service is completed. Queueing systems with Bernoulli customer feedback appear in many real-world situations, such as communication networks where data transmissions must be guaranteed error-free within a specified probability threshold. Feedback schemes are used to request retransmission of packets that are lost or received in a corrupted form. This kind of queueing model is known as a feedback queue. The first pioneering work on this concept was given by Takàcs [47]. Since then, considerable research works have been devoted to the study of such queueing systems. The readers can be referred to [6,15,17,21–23,25,27,33,35,37–40] for comprehensive reviews.

Retrial queues have been successfully used to stochastically model many complex real-world systems problems. However, the analytical theory of these models is limited because of the complexity of the known results. Indeed, in most cases, we are faced with systems of equations whose resolution is complex, or having solutions not easily interpretable. For instance, Pollaczek-Khintchine formula requires a numerical inversion of the Laplace transforms to compute the distribution of the waiting time [32]. In many cases, even the Laplace transform or probability generating functions are not available in explicit forms. To overcome these difficulties, approximation methods are often used to obtain quantitative and/or qualitative estimates for certain performance measures.

Qualitative properties of stochastic models constitute an important theoretical basis for approximation methods. One of the important qualitative properties is monotonicity which can be studied using the general theory of stochastic orderings [43–45]. The monotonicity approach can be thought of as a descriptive approach to studying complex systems. Instead of studying performance measures quantitatively, this approach attempts to show the relationship between performance measures and system parameters. The main concern of this technique comes from the fact that we can arrive at a compromise between the role of these qualitative bounds and the complexity of solving certain complicated systems, in which certain parameters are not perfectly known. One of the monotonicity properties was provided in [34], where authors studied monotonicity properties of an $M/G/1$ retrial queue relative to given stochastic orderings. Then, Boualem et al. [19] dealt with some comparability and monotonicity problems for the analysis of the $M/G/1$ queue with constant retrials and server vacation using the general theory of stochastic orderings. Recently, Boualem and Touche [20] used the stochastic monotonicity approach to derive insensitive bounds for the stationary joint distribution of the embedded Markov chain of a non-Markovian priority retrial queue, which serves two types of customers. Numerous results related to the subject can be found in [1,2,8–14,16,17].

In this paper, we study the monotonicity properties of a single server retrial queue with Bernoulli feedback and negative customers, relative to the stochastic and convex orderings, in order to obtain simple bounds for the stationary distribution of the embedded Markov chain associated with the considered model. The proposed approach is quite different from that given by Kumar et al. [36], where the performance characteristics of the $M/G/1$ Bernoulli feedback retrial queue with negative customers have been expressed in terms of generating functions and Laplace transforms.

This work is motivated by several considerations. First, to the best of our knowledge, no previous studies have applied the monotonicity approach to this type of model. Furthermore, investigating the effect of negative customers on the system's stationary behavior is theoretically significant, as most queueing analysis focuses on models without negative customers. These systems are particularly relevant for describing real-world scenarios, such as communication systems affected by computer viruses or intentional external interventions, and transaction deletions in databases. It should be pointed out that the presence of negative arrivals has a great impact on the system. Therefore, it is important to understand how negative arrivals affect the system's performance. This point will be discussed through a numerical study, later in this paper.

The paper is organized as follows: Section 2 is devoted to the mathematical description of the considered model. In Section 3, we present some preliminary results and definitions which permit to make the comparison of the probabilities of the number of customers arrived during a service period. Monotonicity and comparability conditions of the transition operator associated with the embedded Markov chain are given in Section 4. Comparability conditions of stationary distributions are presented in Section 5. Section 6 is dedicated to some numerical examples, using the discrete event simulation approach, illustrating the interest of our study. Finally, we conclude the paper in Section 7.

§2 Model description

We consider a single-server retrial queueing system with Bernoulli feedback and negative customers, where positive and negative customers arrive at the system according to two independent Poisson processes with rates $\lambda > 0$ and $\delta > 0$, respectively. The model under consideration is schematically represented in Figure 1, where

$$I = \begin{cases} 0, \text{the positive customer is completely served,} \\ 1, \text{the service of the positive customer is interrupted.} \end{cases}$$

On arrival of a positive customer, if the server is idle, the positive customer is served immediately. Otherwise, the customer joins an orbit of infinite capacity and repeats his request after a random period of time. This process is repeated until the customer finds the server idle and gets the requested service. The time between successive repeated attempts is assumed to be exponentially distributed with parameter $n\mu$, when the number of customers in the orbit is $n \in \mathbb{N}$ [18].

The service times are independent and identically distributed with general distribution function $G(x)$ and probability density function $g(x)$.

Once the positive customer is completely served, he decides whether to join the orbit with probability θ_1 $(0 \le \theta_1 < 1)$ or leave definitively the system with probability $\overline{\theta}_1 = 1 - \theta_1$.

The negative customer, arriving during the service time of a positive customer, will remove the positive customer in service and the interrupted positive customer either enters into the orbit with probability θ_2 $(0 \le \theta_2 < 1)$ or leaves the system forever with probability $\overline{\theta}_2 = 1 - \theta_2$.

The inter-arrival times of positive and negative customers, retrial times, and service times are mutually independent random variables.

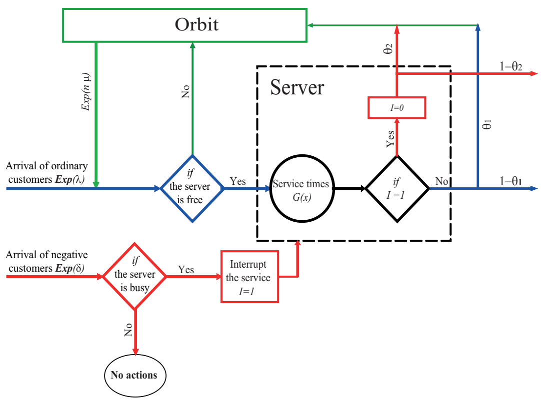

Figure 1. Schematic representation of the queueing model.

At an arbitrary instant t, the state of the system can be described by the continuous time stochastic process $\{X(t), t \ge 0\} = \{(C(t), N(t), \xi(t), t \ge 0\}$, where $C(t)$ designates the state of the server, with

$$C(t) = \begin{cases} 0, \text{if the server is free,} \\ 1, \text{if the server is busy.} \end{cases} \tag{1}$$

$N(t)$ is the number of customers in the orbit at time t. In addition, if $C(t) = 1$, then $\xi(t) > 0$ represents the elapsed service time of the positive customer currently being served, and $\xi(t) = 0$ if $C(t) = 0$.

2.1 Embedded Markov chain

Let t_n, $n \in \mathbb{Z}_+$ denote the output epochs, i.e., either the epoch of service completion of the ith positive customer or that at which the ith positive customer in service is removed by a negative customer, $t_0 = 0$. Let $N_n = N(t_n^+)$ be the number of customers in orbit just after the time epoch t_n. Then the sequence $\{N_n, n \in \mathbb{Z}_+\}$ of random variables forms a Markov chain with $\mathbb{Z}_+$ as state space, which is the embedded chain for our continuous time retrial queueing system.

From [36], we get the necessary and sufficient condition of ergodicity of the embedded Markov chain:

$$\theta_1 \beta^*(\delta) + \theta_2(1 - \beta^*(\delta)) + \frac{\lambda[1 - \beta^*(\delta)]}{\delta} < 1, \tag{2}$$

where, $\beta^*(s) = \int_0^{+\infty} e^{-sx} g(x)dx$ is the Laplace transform of the function $g(x)$.

The one-step transition probabilities of the Markov chain for the $M/G/1$ queueing system with retrials, Bernoulli feedback and negative customers are given as

$$P_{n,m} = \begin{cases} (1 - \delta_{0m})\,\theta_1 k_{m-1} + \overline{\theta_1} k_m + (1 - \delta_{0m})\,\theta_2 h_{m-1} + \overline{\theta_2} h_m, \text{if } n = 0,\ m \geq 0, \\ \dfrac{n\mu\overline{\theta_1}}{\lambda + n\mu} k_0 + \dfrac{n\mu\overline{\theta_2}}{\lambda + n\mu} h_0, \text{ if } n \geq 1,\ m = n - 1, \\ \dfrac{\lambda\theta_1}{\lambda + n\mu} k_{m-n-1} + \dfrac{\lambda\overline{\theta_1}}{\lambda + n\mu} k_{m-n} + \dfrac{n\mu\theta_1}{\lambda + n\mu} k_{m-n} + \dfrac{n\mu\overline{\theta_1}}{\lambda + n\mu} k_{m-n+1} \\ + \dfrac{\lambda\theta_2}{\lambda + n\mu} h_{m-n-1} + \dfrac{\lambda\overline{\theta_2}}{\lambda + n\mu} h_{m-n} + \dfrac{n\mu\theta_2}{\lambda + n\mu} h_{m-n} \\ + \dfrac{n\mu\overline{\theta_2}}{\lambda + n\mu} h_{m-n+1}, \text{if } n \geq 1,\ m > n - 1, \\ 0, \quad \text{otherwise,} \end{cases} \tag{3}$$

where,

$$\begin{aligned} k_m &= \int_0^{+\infty} \frac{(\lambda u)^m \exp\{-\lambda u\}}{m!} \exp\{-\delta u\}\, dG(u),\ m = 0, 1, 2, ..., \\ h_m &= \int_0^{+\infty} \frac{(\lambda u)^m \exp\{-\lambda u\}}{m!} \exp\{-\delta u\}\, \delta[1 - G(u)]du,\ m = 0, 1, 2, ..., \end{aligned}$$

and δ_{km} is the Kronecker delta function.

2.2 Notations

Consider two $M/G/1$ retrial queueing systems with a Bernoulli feedback and negative customers, having the same structure but with different parameters, noted by Σ_1 and Σ_2, respectively. For $i = 1,\ 2$, let

$\lambda^{(i)}$: be the arrival rate of positive customers in Σ_i.

$\delta^{(i)}$: be the arrival rate of negative customers in Σ_i.

$\mu^{(i)}$: be the retrial rate in Σ_i.

$\theta_1^{(i)}$: be the probability that a customer joins the orbit in Σ_i.

$\theta_2^{(i)}$: be the probability that the interrupted customer joins the orbit in Σ_i.

$G^{(i)}(u)$: be the distribution of service time in Σ_i.

$k_m^{(i)}$: be the number of new arrivals during the service of the n^{th} customer in Σ_i.

$h_m^{(i)}$: be the number of negative customers arriving during the service of the n^{th} customer in Σ_i.

$\pi_n^{(i)}$: be the stationary distribution of the number of customers in Σ_i.

§3 Preliminary results and definitions

3.1 Generalities

The objective of stochastic orders is the approximation of a complex model by a simpler model or by a model whose distributions are simpler. Important fields of application are, amongst others, operations research, queueing theory, reliability theory, decision theory, and insurance mathematics. For more discussion on these stochastic orderings and their applications, one can refer to [43–46].

In what follows, we recall definitions of some stochastic orders and aging concepts that are relevant to the main results developed in this paper.

Definition 3.1. Let X and Y be two non-negative random variables with distribution functions F and G, respectively. X is said to be smaller than Y with respect to

(a) Stochastic ordering ($X \leq_{st} Y$) iff $F(x) \geq G(x)$, $\forall x \geq 0$.

(b) Convex ordering ($X \leq_v Y$) iff $\int\limits_x^{+\infty} \overline{F}(u)d(u) \leq \int\limits_x^{+\infty} \overline{G}(u)d(u)$, $\forall x \geq 0$.

In the case where X and Y are discrete random variables taking values in the set of relative integers $\mathbb{Z}$, with distribution functions $\omega = (\omega_n)_{n\geq 0}$ and $\nu = (\nu_n)_{n\geq 0}$, for $i \in \mathbb{Z}$, respectively. Then,

(a) $\omega \leq_{st} \nu$ iff $\overline{\omega}_m = \sum_{n\geq m} \omega_n \leq \overline{\nu}_m = \sum_{n\geq m} \nu_n$, $\forall m$,

(b) $\omega \leq_v \nu$ iff $\overline{\overline{\omega}}_m = \sum_{n\geq m} \sum_{k\geq n} \omega_k \leq \overline{\overline{\nu}}_m = \sum_{n\geq m} \sum_{k\geq n} \nu_k$, $\forall m$.

Definition 3.2. Let X and X_τ be random variables representing respectively the lifetime and the residual lifetime of an element. Let F and F_τ be their respective distributions. We say that F is

(a) $NBUE$ (New Better than Used in Expectation) if $E(X_\tau) \leq E(X)$, $(0 < \tau < \infty)$,

(b) $NWUE$ (New Worse than Used in Expectation) if $E(X) \leq E(X_\tau)$, $(0 < \tau < \infty)$.

3.2 Preliminary inequalities

In this section, we use the general theory of stochastic orderings in order to study the monotonicity properties of the $M/G/1$ retrial queueing system with Bernoulli feedback and negative customers, relative to stochastic and convex orderings.

The following lemma gives the conditions on the parameters of $\Sigma_i, i = 1, 2$ systems under which the probabilities of the number of customers arriving during the service of a customer in the two waiting systems $\left\{k_m^{(i)}; m \in \mathbb{N}\right\}$ and $\left\{h_m^{(i)}; m \in \mathbb{N}\right\}$, are comparable relative to given stochastic and convex orderings.

Lemma 3.3. *Let Σ_1 and Σ_2 be two queueing systems.*

If $\lambda^{(1)} \leq \lambda^{(2)}, \delta^{(1)} \geq \delta^{(2)}$ *and* $G^{(1)} \leq_s G^{(2)}$, *then* $\left\{k_m^{(1)}\right\} \leq_s \left\{k_m^{(2)}\right\}$,

If $\lambda^{(1)} \leq \lambda^{(2)}, \delta^{(1)} \geq \delta^{(2)}$ *and* $\overline{G}^{(1)} \leq_s \overline{G}^{(2)}$, *then* $\left\{h_m^{(1)}\right\} \leq_s \left\{h_m^{(2)}\right\}$,

where $\leq_s$ *is one of the symbols* $\leq_{st}$ *or* $\leq_v$.

Proof. By definition, we have:

$$\overline{k}_n^{(i)} = \sum_{m\geq n} k_m^{(i)} = \int_0^{+\infty} \sum_{m\geq n} \frac{(\lambda^{(i)}u)^m}{m!} e^{-\lambda^{(i)}u} e^{-\delta^{(i)}u} dG^{(i)}(u),\ i = 1, 2,$$

$$\overline{\overline{k}}_n^{(i)} = \sum_{m\geq n} \overline{k}_m^{(i)} = \int_0^{+\infty} \sum_{m\geq n} \sum_{l\geq m} \frac{(\lambda^{(i)}u)^l}{l!} e^{-\lambda^{(i)}u} e^{-\delta^{(i)}u} dG^{(i)}(u),\ i = 1, 2,$$

$$\overline{h}_n^{(i)} = \sum_{m\geq n} h_m^{(i)} = \int_0^{+\infty} \sum_{m\geq n} \frac{(\lambda^{(i)}u)^m}{m!} e^{-\lambda^{(i)}u} e^{-\delta^{(i)}u} \delta^{(i)} d\overline{G}^{(i)}(u),\ i = 1, 2,$$

$$\overline{\overline{h}}_n^{(i)} = \sum_{m\geq n} \overline{h}_m^{(i)} = \int_0^{+\infty} \sum_{m\geq n} \sum_{l\geq m} \frac{(\lambda^{(i)}u)^l}{l!} e^{-\lambda^{(i)}u} e^{-\delta^{(i)}u} \delta^{(i)} d\overline{G}^{(i)}(u),\ i = 1, 2.$$

To prove that $\left\{k_m^{(1)}\right\} \leq_s \left\{k_m^{(2)}\right\}$ and $\left\{h_m^{(1)}\right\} \leq_s \left\{h_m^{(2)}\right\}$, we have to establish the usual numerical inequalities

$$\overline{k}_m^{(1)} = \sum_{n\geq m} k_n^{(1)} \leq \overline{k}_m^{(2)}, \text{ (for } \leq_s = \leq_{st}), \tag{4}$$

$$\overline{\overline{k}}_m^{(1)} = \sum_{n\geq m} \overline{k}_n^{(1)} \leq \overline{\overline{k}}_m^{(2)}, \text{ (for } \leq_s = \leq_{v}), \tag{5}$$

$$\overline{h}_m^{(1)} = \sum_{n\geq m} h_n^{(1)} \leq \overline{h}_m^{(2)}, \text{ (for } \leq_s = \leq_{st}), \tag{6}$$

$$\overline{\overline{h}}_m^{(1)} = \sum_{n\geq m} \overline{h}_n^{(1)} \leq \overline{\overline{h}}_m^{(2)}, \text{ (for } \leq_s = \leq_{v}). \tag{7}$$

To verify inequality (4), we use the fact that the function

$$f(\lambda,\ \delta,\ u) = \sum_{m\geq n} \frac{(\lambda u)^m}{m!} e^{-\lambda u} e^{-\delta u},$$

is increasing in λ, δ and u.

By hypothesis, we have $G^{(1)} \leq_s G^{(2)}$, for $\leq_s = \leq_{st}$. Then, with the help of Theorem 1.2.2 given in Stoyan [45] and by monotonicity of $f(\lambda,\ \delta,\ u)$ with respect to λ and δ, one can find that

$$\int_0^\infty f(\lambda^{(1)}, \delta^{(1)}, u) dG^{(1)}(u) \leq \int_0^\infty f(\lambda^{(2)}, \delta^{(2)}, u) dG^{(1)}(u) \leq \int_0^\infty f(\lambda^{(2)}, \delta^{(2)}, u) dG^{(2)}(u).$$

The verification of equations (5), (6) and (7) is similar to that of equation (4). It is sufficient to substitute $\overline{k}_m^{(1)} \leq \overline{k}_m^{(2)}$ by $\overline{\overline{k}}_m^{(1)} \leq \overline{\overline{k}}_m^{(2)}$, $\overline{h}_m^{(1)} \leq \overline{h}_m^{(2)}$, and $\overline{\overline{h}}_m^{(1)} \leq \overline{\overline{h}}_m^{(2)}$. □

§4 Monotonicity properties of the embedded Markov chain

Let τ be the transition operator of the embedded Markov chain. To every distribution $p = (p_n)_{n\geq 0}$, we associate a distribution $\tau_p = q = (q_m)_{m\geq 0}$, such that

$$q_m = \sum_{n\geq 0} p_n p_{nm},$$

where transition probabilities P_{nm} are defined in formula (3).

The operator τ is monotone with respect to the stochastic ordering iff

$$\overline{P}_{n-1m} \leq \overline{P}_{nm},\ \forall n, m. \tag{8}$$

The operator τ is monotone with respect to the convex ordering iff

$$2\overline{\overline{P}}_{nm} \leq \overline{\overline{P}}_{n-1m} + \overline{\overline{P}}_{n+1m},\ \forall n, m. \tag{9}$$

4.1 Monotonicity properties of the transition operator

The following theorems present the monotonicity condition of the transition operator τ relative to stochastic and convex orderings, respectively.

Theorem 4.1. *The transition operator τ is monotone with respect to the stochastic ordering, that is, for any two distributions $p^{(1)}$ and $p^{(2)}$, the inequality $p^{(1)} \leq_{st} p^{(2)}$ implies that $\tau p^{(1)} \leq_{st} \tau p^{(2)}$.*

Proof. The transition operator τ is monotone with respect to the stochastic ordering iff (8) is verified. Thus, we consider the following cases:

First case: $n=0$ and $m \geq 0$. We have:

$$\begin{aligned}
P_{nm} &= (1-\delta_{0m})\theta_1 k_{m-1} + (1-\theta_1)k_m + (1-\delta_{0m})\theta_2 h_{m-1} + (1-\theta_2)h_m. \\
\overline{P}_{nm} &= \sum_{\ell \geq m} P_{n\ell} = \sum_{\ell \geq m} [(1-\delta_{0\ell})\theta_1 k_{\ell-1} + (1-\theta_1) + (1-\delta_{0\ell})\theta_2 h_{\ell-1} + (1-\theta_2)h_\ell] \\
&= \theta_1 k_{m-1} + \bar{k}_m + \theta_2 h_{m-1} + \bar{h}_m. \\
\overline{P}_{n-1m} &= \sum_{\ell \geq m} P_{n-1\ell} = \theta_1 k_{m-1} + \overline{k}_m + \theta_2 h_{m-1} + \overline{h}_m.
\end{aligned}$$

Therefore,

$$\overline{P}_{nm} - \overline{P}_{n-1m} \geq 0.$$

Second case: $n \geq 1$ and $m = n-1$. We have:

$$\begin{aligned}
P_{nm} &= \frac{n\mu(1-\theta_1)}{\lambda + n\mu} k_0 + \frac{n\mu(1-\theta_2)}{\lambda + n\mu} h_0. \\
\overline{P}_{nm} &= \frac{n\mu(1-\theta_1)}{\lambda + n\mu} k_0 + \frac{n\mu(1-\theta_2)}{\lambda + n\mu} h_0. \\
\overline{P}_{n-1m} &= \frac{(n-1)\mu(1-\theta_1)}{\lambda + (n-1)\mu} k_0 + \frac{(n-1)\,\mu(1-\theta_2)}{\lambda + (n-1)\,\mu} h_0.
\end{aligned}$$

Thus,

$$\begin{aligned}
\overline{P}_{nm} - \overline{P}_{n-1m} &= \left[\frac{n\mu}{\lambda + n\mu} - \frac{(n-1)\mu}{\lambda + (n-1)\mu}\right](1-\theta_1)k_0 \\
&\quad + \left[\frac{n\mu}{\lambda + n\mu} - \frac{(n-1)\mu}{\lambda + (n-1)\mu}\right](1-\theta_2)h_0 \\
&= \frac{\lambda\mu(1-\theta_1)}{(\lambda + n\mu)(\lambda + (n-1)\mu)} k_0 + \frac{\lambda\mu(1-\theta_2)}{(\lambda + n\mu)(\lambda + (n-1)\mu)} h_0 \geq 0.
\end{aligned}$$

Third case: $n \geq 1$ and $m > n-1$, similarly as in the two above cases we have :

$$\begin{aligned}
\overline{P}_{nm} - \overline{P}_{n-1m} &= \frac{\lambda\theta_1}{\lambda + n\mu} k_{m-n-1} + \left[\frac{(n-1)\mu(1-\theta_1)}{\lambda + (n-1)\,\mu}\right] k_{m-n+1} \\
&+ \left[\frac{\lambda^2(1-\theta_1) + \lambda\mu(n-1) + n(n-1)\mu^2\theta_1}{(\lambda + n\mu)\,(\lambda + (n-1)\,\mu)}\right] k_{m-n} \\
&+ \frac{\lambda\theta_2}{\lambda + n\mu} h_{m-n-1} + \left[\frac{(n-1)\mu(1-\theta_2)}{\lambda + (n-1)\,\mu}\right] h_{m-n+1} \\
&+ \left[\frac{\lambda^2(1-\theta_2) + \lambda\mu(n-1) + n(n-1)\mu^2\theta_2}{(\lambda + n\mu)\,(\lambda + (n-1)\,\mu)}\right] h_{m-n} \geq 0.
\end{aligned}$$

Therefore, inequality (8) holds for any n and m, and the operator τ is monotone with respect to $\leq_{st}$. □

Theorem 4.2. *The transition operator τ is monotone with respect to the convex ordering, that is, for any two distributions $p^{(1)}$ and $p^{(2)}$, the inequality $p^{(1)} \leq_v p^{(2)}$ implies that $\tau p^{(1)} \leq_v \tau p^{(2)}$.*

Proof. The proof of this theorem is analogous to that of Theorem 4.1. By making use of the formula (9) (with respect to the convex ordering $(\leq_v)$), we obtain the desired result. □

4.2 Comparability of transition operators

Consider Σ_1 and Σ_2 two $M/G/1$ retrial queueing models with Bernoulli feedback and negative customers. Let $\tau^{(1)}$ and $\tau^{(2)}$ be the transition operators associated with the embedded Markov chains of each system.

The following theorems present comparability conditions of these operators with respect to partial orderings: Stochastic and convex.

Theorem 4.3. *If $\lambda^{(1)} \leq \lambda^{(2)}$, $\mu^{(1)} \geq \mu^{(2)}$, $\theta_1^{(1)} \leq \theta_1^{(2)}$, $\theta_2^{(1)} \leq \theta_2^{(2)}$, $\delta^{(1)} \geq \delta^{(2)}$, and $G^{(1)}(u) \leq_{st} G^{(2)}(u)$, then $\tau^{(1)} \leq_{st} \tau^{(2)}$, that is, for any distribution p, we have $\tau^{(1)}p \leq_{st} \tau^{(2)}p$.*

Proof. According to Theorem 2.4.2, given by Stoyan [45], we have to establish the following inequalities for the stochastic ordering,

$$\overline{P}_{nm}^{(1)} \leq \overline{P}_{nm}^{(2)}, \ \forall\ n,\ m, \tag{10}$$

which amounts to showing that

$$\begin{aligned} &\frac{\lambda^{(1)}\theta_1^{(1)}}{\lambda^{(1)}+n\mu^{(1)}}k_{m-n-1}^{(1)} - \frac{n\mu^{(1)}(1-\theta_1^{(1)})}{\lambda^{(1)}+n\mu^{(1)}}k_{m-n}^{(1)} + \overline{k}_{m-n}^{(1)} \\ &+\frac{\lambda^{(1)}\theta_2^{(1)}}{\lambda^{(1)}+n\mu^{(1)}}h_{m-n-1}^{(1)} - \frac{n\mu^{(1)}(1-\theta_2^{(1)})}{\lambda^{(1)}+n\mu^{(1)}}h_{m-n}^{(1)} + \overline{h}_{m-n}^{(1)} \\ &\leq \frac{\lambda^{(2)}\theta_1^{(2)}}{\lambda^{(2)}+n\mu^{(2)}}k_{m-n-1}^{(2)} - \frac{n\mu^{(2)}(1-\theta_1^{(2)})}{\lambda^{(2)}+n\mu^{(2)}}k_{m-n}^{(2)} + \overline{k}_{m-n}^{(2)} \\ &+\frac{\lambda^{(2)}\theta_2^{(2)}}{\lambda^{(2)}+n\mu^{(2)}}h_{m-n-1}^{(2)} - \frac{n\mu^{(2)}(1-\theta_2^{(2)})}{\lambda^{(2)}+n\mu^{(2)}}h_{m-n}^{(2)} + \overline{h}_{m-n}^{(2)}. \end{aligned} \tag{11}$$

According to Lemma 3.3 (relative to the stochastic ordering ($\leq_{st}$)), we have:

$$\left\{k_m^{(1)}\right\} \leq_{st} \left\{k_m^{(2)}\right\}, \quad \forall m \geq 0. \tag{12}$$

On the other hand,

$$\text{if } \lambda^{(1)} \leq \lambda^{(2)} \ \text{ and } \ \mu^{(1)} \geq \mu^{(2)}, \ \text{ then } \frac{\lambda^{(1)}}{\mu^{(1)}} \leq \frac{\lambda^{(2)}}{\mu^{(2)}} \ \text{ or } \ \frac{\mu^{(1)}}{\lambda^{(1)}} \geq \frac{\mu^{(2)}}{\lambda^{(2)}}. \tag{13}$$

In addition, as the function $x \to \dfrac{x}{x+n}$ is increasing with respect to x, thus we have

$$\frac{\lambda^{(1)}}{\lambda^{(1)}+n\mu^{(1)}} \leq \frac{\lambda^{(2)}}{\lambda^{(2)}+n\mu^{(2)}}. \tag{14}$$

Further, we have $\theta_1^{(1)} \leq \theta_1^{(2)}$ and $\theta_2^{(1)} \leq \theta_2^{(2)}$. Therefore,

$$\frac{\lambda^{(1)}\theta_1^{(1)}}{\lambda^{(1)}+n\mu^{(1)}} \leq \frac{\lambda^{(2)}\theta_1^{(2)}}{\lambda^{(2)}+n\mu^{(2)}} \ \text{ and } \ \frac{\lambda^{(1)}\theta_2^{(1)}}{\lambda^{(1)}+n\mu^{(1)}} \leq \frac{\lambda^{(2)}\theta_2^{(2)}}{\lambda^{(2)}+n\mu^{(2)}}. \tag{15}$$

Similarly, the function $\dfrac{x}{1+x}$ is increasing. Thus,

$$\frac{n\mu^{(1)}}{\lambda^{(1)}+n\mu^{(1)}} \geq \frac{n\mu^{(2)}}{\lambda^{(2)}+n\mu^{(2)}}. \tag{16}$$

Moreover, as

$$\theta_1^{(1)} \leq \theta_1^{(2)} \ \text{ and } \ \theta_2^{(1)} \leq \theta_2^{(2)}, \ \text{ then } \left(1-\theta_1^{(1)}\right) \geq \left(1-\theta_1^{(2)}\right) \ \text{ and } \ \left(1-\theta_2^{(1)}\right) \geq \left(1-\theta_2^{(2)}\right).$$

So,

$$-\frac{n\mu^{(1)}\left(1-\theta_1^{(1)}\right)}{\lambda^{(1)}+n\mu^{(1)}} \leq -\frac{n\mu^{(2)}\left(1-\theta_1^{(2)}\right)}{\lambda^{(2)}+n\mu^{(2)}} \text{ and } -\frac{n\mu^{(1)}\left(1-\theta_2^{(1)}\right)}{\lambda^{(1)}+n\mu^{(1)}} \leq -\frac{n\mu^{(2)}\left(1-\theta_2^{(2)}\right)}{\lambda^{(2)}+n\mu^{(2)}}. \quad (17)$$

Finally, making use of inequalities (12), (15), and (17), we obtain

$$\begin{aligned}
\overline{P}_{nm}^{(1)} &= \frac{\lambda^{(1)}\theta_1^{(1)}}{\lambda^{(1)}+n\mu^{(1)}}k_{m-n-1}^{(1)} - \frac{n\mu^{(1)}(1-\theta_1^{(1)})}{\lambda^{(1)}+n\mu^{(1)}}k_{m-n}^{(1)} + \overline{k}_{m-n}^{(1)} \\
&\quad + \frac{\lambda^{(1)}\theta_2^{(1)}}{\lambda^{(1)}+n\mu^{(1)}}h_{m-n-1}^{(1)} - \frac{n\mu^{(1)}(1-\theta_2^{(1)})}{\lambda^{(1)}+n\mu^{(1)}}h_{m-n}^{(1)} + \overline{h}_{m-n}^{(1)} \\
&\leq \frac{\lambda^{(2)}\theta_1^{(2)}}{\lambda^{(2)}+n\mu^{(2)}}k_{m-n-1}^{(1)} - \frac{n\mu^{(2)}(1-\theta_1^{(2)})}{\lambda^{(2)}+n\mu^{(2)}}k_{m-n}^{(1)} + \overline{k}_{m-n}^{(1)} \\
&\quad + \frac{\lambda^{(2)}\theta_2^{(2)}}{\lambda^{(2)}+n\mu^{(2)}}h_{m-n-1}^{(1)} - \frac{n\mu^{(2)}(1-\theta_2^{(2)})}{\lambda^{(2)}+n\mu^{(2)}}h_{m-n}^{(1)} + \overline{h}_{m-n}^{(1)} \\
&= \frac{\lambda^{(2)}\theta_1^{(2)}}{\lambda^{(2)}+n\mu^{(2)}}\overline{k}_{m-n-1}^{(1)} + \frac{\lambda^{(2)}(1-\theta_1^{(2)})}{\lambda^{(2)}+n\mu^{(2)}}\overline{k}_{m-n}^{(1)} \\
&\quad + \frac{\lambda^{(2)}\theta_2^{(2)}}{\lambda^{(2)}+n\mu^{(2)}}\overline{h}_{m-n-1}^{(1)} + \frac{\lambda^{(2)}(1-\theta_2^{(2)})}{\lambda^{(2)}+n\mu^{(2)}}\overline{h}_{m-n}^{(1)} \\
&\quad + \frac{n\mu^{(2)}\theta_1^{(2)}}{\lambda^{(2)}+n\mu^{(2)}}\overline{k}_{m-n}^{(1)} + \frac{n\mu^{(2)}(1-\theta_1^{(2)})}{\lambda^{(2)}+n\mu^{(2)}}\overline{k}_{m-n+1}^{(1)} \\
&\quad + \frac{n\mu^{(2)}\theta_2^{(2)}}{\lambda^{(2)}+n\mu^{(2)}}\overline{h}_{m-n}^{(1)} + \frac{n\mu^{(2)}(1-\theta_2^{(2)})}{\lambda^{(2)}+n\mu^{(2)}}\overline{h}_{m-n+1}^{(1)} \leq \overline{P}_{nm}^{(2)}.
\end{aligned}$$

□

Theorem 4.4. *If* $\lambda^{(1)} \leq \lambda^{(2)}$, $\mu^{(1)} \geq \mu^{(2)}$, $\theta_1^{(1)} \leq \theta_1^{(2)}$, $\theta_2^{(1)} \leq \theta_2^{(2)}$, $\delta^{(1)} \geq \delta^{(2)}$, *and* $G^{(1)}(u) \leq_v G^{(2)}(u)$, *then* $\tau^{(1)} \leq_v \tau^{(2)}$, *that is, for any distribution* p, *we have* $\tau^{(1)}p \leq_v \tau^{(2)}p$.

Proof. According to Theorem 2.4.2 from Stoyan [45], for the convex ordering, we have to show the following inequality

$$\overline{\overline{P}}_{nm}^{(1)} \leq \overline{\overline{P}}_{nm}^{(2)}, \quad \forall \ \ 0 \leq n \leq m,$$

that is,

$$\begin{aligned}
& \frac{\lambda^{(1)}\theta_1^{(1)}}{\lambda^{(1)}+n\mu^{(1)}}\overline{k}_{m-n-1}^{(1)} - \frac{n\mu^{(1)}(1-\theta_1^{(1)})}{\lambda^{(1)}+n\mu^{(1)}}\overline{k}_{m-n}^{(1)} + \overline{\overline{k}}_{m-n}^{(1)} \\
& + \frac{\lambda^{(1)}\theta_2^{(1)}}{\lambda^{(1)}+n\mu^{(1)}}\overline{h}_{m-n-1}^{(1)} - \frac{n\mu^{(1)}(1-\theta_1^{(1)})}{\lambda^{(1)}+n\mu^{(1)}}\overline{h}_{m-n}^{(1)} + \overline{\overline{h}}_{m-n}^{(1)} \\
\leq \ & \frac{\lambda^{(2)}\theta_1^{(2)}}{\lambda^{(2)}+n\mu^{(2)}}\overline{k}_{m-n-1}^{(2)} - \frac{n\mu^{(2)}(1-\theta_1^{(2)})}{\lambda^{(2)}+n\mu^{(2)}}\overline{k}_{m-n}^{(2)} + \overline{\overline{k}}_{m-n}^{(2)} \\
& + \frac{\lambda^{(2)}\theta_2^{(2)}}{\lambda^{(2)}+n\mu^{(2)}}\overline{h}_{m-n-1}^{(2)} - \frac{n\mu^{(2)}(1-\theta_2^{(2)})}{\lambda^{(2)}+n\mu^{(2)}}\overline{h}_{m-n}^{(2)} + \overline{\overline{h}}_{m-n}^{(2)}.
\end{aligned}$$

Following the same steps of the proof of Theorem 4.3 and making use of Lemma 3.3 (with respect to the convex ordering ($\leq_v$)), we obtain the desired result. □

§5 Stochastic inequalities for stationary distributions

In the following theorems, we establish the comparability conditions of stationary distributions of the number of customers for two $M/G/1$ retrial queueing systems with Bernoulli feedback and negative customers (having the same structure but with different parameters), with respect to partial orderings: Stochastic and convex.

Theorem 5.1. *Let $\pi_n^{(1)}$ and $\pi_n^{(2)}$ be the stationary distributions of the number of customers in Σ_1 and Σ_2, respectively. If*

$$\lambda^{(1)} \leq \lambda^{(2)},\ \mu^{(1)} \geq \mu^{(2)},\ \theta_1^{(1)} \leq \theta_1^{(2)},\ \theta_2^{(1)} \leq \theta_2^{(2)},\ \delta^{(1)} \geq \delta^{(2)},\ G^{(1)} \leq_{st,v} G^{(2)},$$

then, we have

$$\left\{\pi_n^{(1)}\right\} \leq_{st,v} \left\{\pi_n^{(2)}\right\}. \tag{18}$$

Proof. Via hypotheses given in Theorems 4.3–4.4, we have $\lambda^{(1)} \leq \lambda^{(2)}$, $\mu^{(1)} \geq \mu^{(2)}$, $\theta_1^{(1)} \leq \theta_1^{(2)}$, $\theta_2^{(1)} \leq \theta_2^{(2)}$ and $G^{(1)} \leq_{st,v} G^{(2)}$. This implies that $\tau^{(1)} \leq_{st,v} \tau^{(2)}$, that is, for any distribution p, we get

$$\tau^{(1)}p \leq_{st,v} \tau^{(2)}p. \tag{19}$$

In addition, we have $G^{(1)} \leq_{st,v} G^{(2)}$, then from Theorems 4.1–4.2, the operator τ associated with the embedded Markov chain of the second system is monotone, that is, for any distributions $p_1^{(2)}$ and $p_2^{(2)}$, such that $p_1^{(2)} \leq_{st,v} p_2^{(2)}$, we have

$$\tau^{(2)}p_1^{(2)} \leq_{st,v} \tau^{(2)}p_2^{(2)}. \tag{20}$$

However, from inequality (19), we obtain

$$\tau^{(1)}p^{(1)} \leq_{st,v} \tau^{(2)}p^{(1)}. \tag{21}$$

Then, there exists a probability $p_1^{(2)}$, such that the inequality

$$\tau^{(2)}p^{(1)} \leq_{st,v} \tau^{(2)}p_1^{(2)}, \tag{22}$$

takes place. From (20)–(22), for any two distributions $p^{(1)}$ and $p^{(2)}$ one can obtain the following result:

$$\tau^{(1)}p^{(1)} \leq_{st,v} \tau^{(2)}p^{(2)}, \tag{23}$$

Next, inequality (23) can be rewritten as follows

$$\begin{aligned} \tau^{(1)}p_n^{(1)} &= P(N_l^{(1)} = n) = P(N_l^{(1)} = n) \\ &\leq_{st,v} P(N_l^{(2)} = n) = P(N_l^{(2)} = n) = \tau^{(2)}p_n^{(2)}. \end{aligned}$$

Finally, when $l \to \infty$, we get the desired result, that is,

$$\left\{\pi_n^{(1)}\right\} \leq_{st,v} \left\{\pi_n^{(2)}\right\}.$$

□

Theorem 5.2. *If in the $M/G/1$ retrial queueing model with Bernoulli feedback and negative customers, the service time distribution is NBUE (resp. NWUE), then the stationary distribution of the number of customers in this system is lower (resp. higher), with respect to the convex ordering, than the stationary distribution of the number of customers in the $M/M/1$*

retrial queue with Bernoulli feedback and negative customers having the same parameters as in the former system.

Proof. Consider an $M/M/1$ retrial queue with Bernoulli feedback and negative customers, having the same parameters as in the $M/G/1$ retrial queueing system with Bernoulli feedback and negative customers: arrival rate λ, retrial rate μ, the probability of joining the orbit θ_1, the mean service time β_1, but with exponentially distributed service time, with parameter $\frac{1}{\beta_1}$, given by:

$$G^*(x) = \begin{cases} 1 - e^{-\frac{x}{\beta_1}}, & \text{if } x \geq 0, \\ 0, & \text{otherwise.} \end{cases} \tag{24}$$

According to Stoyan [45], if $G(x)$ is NBUE (resp. NWUE), then

$$G(x) \leq_v G^*(x) \ \text{ (respectively } G^*(x) \leq_v G(x)). \tag{25}$$

Further, as $G^{(1)} \leq_v G^{(2)}$, then via Theorem 5.1, we deduce that the stationary distribution of the number of customers in an $M/G/1$ retrial queueing system with Bernoulli feedback and negative customers is lower (resp. greater) than the stationary distribution of the number of customers in $M/M/1$ retrial queue with Bernoulli feedback and negative customers having the same parameters as in the former system. □

§6 Numerical results

In this section, we present some numerical examples in order to illustrate and confirm the results obtained in Theorem 5.2. We develop a simulator, under a Matlab environment, based on discrete event simulation, which describes the behavior of the $M/G/1$ retrial queueing model with Bernoulli feedback and negative customers. For service times, we choose two types of probability distribution:

NBUE (Erlang distribution $E_k(\lambda) = E_2(0.4)$),

NWUE (Gamma distribution $\Gamma(a,b) = \Gamma(0.5, 10)$, with $0 < a < 1$).

The results (mean number of customers in the system ($\overline{N}$) and in the orbit ($\overline{N}_o$)) are compared with those of the exponential distribution exp(0.20) with respect to convex ordering. In addition, we carry out this study to analyze the influence of the arrival rate of negative customers δ (with $\delta \in \{0.1, 0.2, 0.4, 0.5, 0.7, 0.9\}$) on the performance measures (the stationary distribution of the number of positive customers in the system (π_n)) of the considered model.

For the simulation study, we take $\lambda = 0.5$, $\theta_1 = 0.4$, and $\theta_2 = 0.6$. The simulation time $T_{max} = 2000$ time units, and $m = 30$ (number of replications).

A sample of the results obtained for these different previous parameters is listed in Table 1 and presented in Figures 2 and 3.

According to Table 1 and Figures 2–3, for different values of δ, we have:

- The stationary distribution of the number of customers in the $M/G/1$ retrial queue with Bernoulli feedback and negative customers with exponential service time is greater (respectively lower) than the stationary distribution of the number of customers in the $M/G/1$

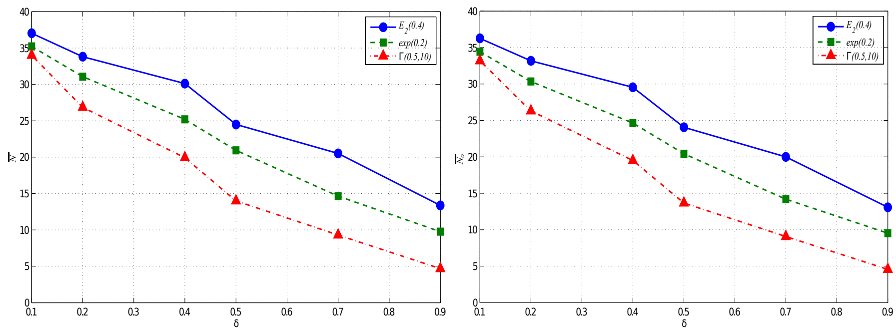

Figure 2. Mean number of customers in the system ($\overline{N}$) and in the orbit ($\overline{N}_o$) vs. δ.

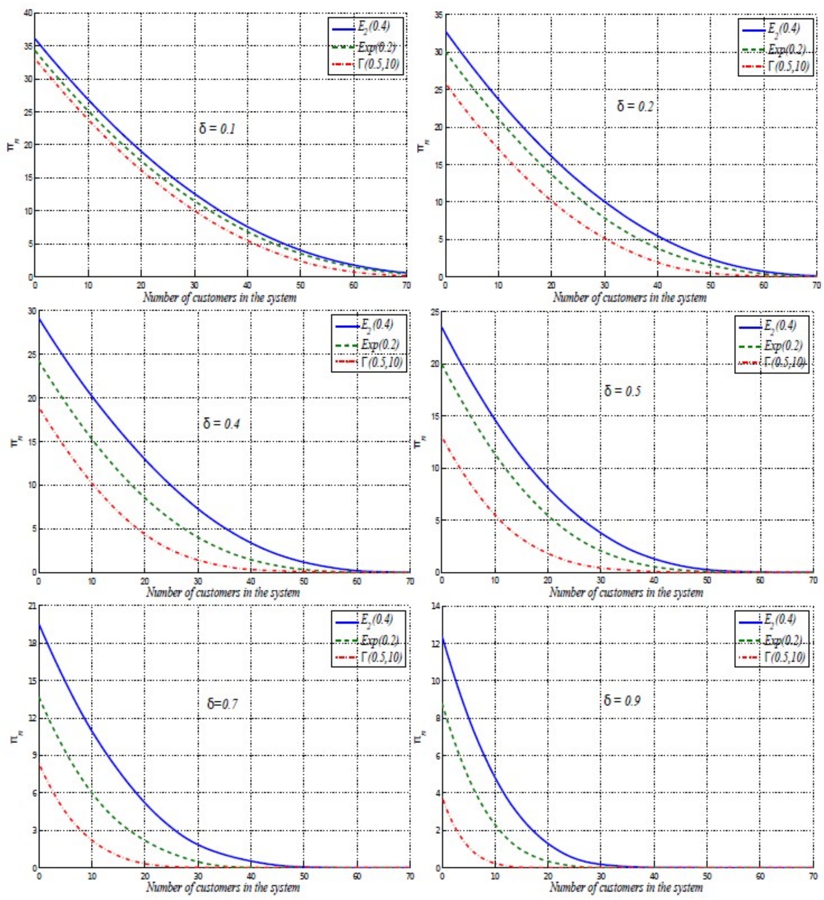

Figure 3. Variation of the stationary distributions of the number of customers in the system according to δ, with respect to the convex ordering.

Table 1. Different situations taken into consideration during the simulation study.

	NBUE $E_2(0.4)$		Exp $\exp(0.2)$		NWUE $\Gamma(0.5, 10)$	
δ	N	N_o	N	N_o	N	N_o
0.1	37.0304	36.2587	35.2227	34.4277	33.9921	33.1786
0.2	33.8065	33.1728	31.0738	30.3479	26.8490	26.3051
0.4	30.1202	29.5466	25.1960	24.6409	19.9299	19.5051
0.5	24.4961	24.0525	20.9393	20.4236	13.9915	13.6671
0.7	20.5071	20.0039	14.6426	14.1910	9.3054	9.0699
0.9	13.3664	13.0813	9.7615	9.5251	4.6790	4.5691

retrial queue with Bernoulli feedback and negative customers, where the service time distribution is NBUE (respectively NWUE). Shortly, the following inequality holds:

$$\left\{\pi_n^{(NBUE)}\right\} \leq_v \left\{\pi_n^{(exp)}\right\} \leq_v \left\{\pi_n^{(NWUE)}\right\}.$$

The obtained results match perfectly with those given in Theorem 5.2. In other words, these results give insensitive bounds for the stationary distribution of the considered embedded Markov chain.

- If the arrival rate of negative customers δ is small enough ($\delta = 0.1$), then the stochastic bounds are good approximations for the stationary probabilities of the model under consideration, whatever the distribution of the service times (NBUE or NWUE). Therefore, the performance measures of a such system can be estimated by those of the $M/M/1$ retrial queueing model with Bernoulli feedback and negative customers.

- With the increase in arrival rate of negative customers δ, the average number of positive customers in the system ($\overline{N}$) and in the orbit ($\overline{N}_o$) decrease (see Table 1 and Figure 2). As intuitively expected, a negative arrival has the effect of removing a regular customer from the queue.

- The results show that the increase of the arrival rate of negative customers has a significant impact on the stationary distribution of the number of positive customers in the system; it can be seen that when δ is large enough ($\delta \geq 0.2$), the characteristics of the system under study are different from those of the $M/M/1$ retrial queueing model with Bernoulli feedback and negative customers (see Figure 3). Otherwise, the considered $M/G/1$ retrial queueing model with Bernoulli feedback and negative customers behaves in the same way as the $M/M/1$ retrial queueing model with Bernoulli feedback and negative customers (see Figures 2–3).

- The stochastic bounds given in Theorem 5.2 depend on the arrival rate of negative customers δ (see Figure 3 for each δ). Indeed, the presence of negative customers affects the size of the queue since the average number of positive customers in the system diminishes.

This makes the system analysis complicated. Hence the interest of our monotonicity approach which makes it possible to provide sensitive bounds for its performance measures.

§7 Conclusion

In this work, we established conditions on the parameters of two systems, under which the probabilities of the number of customers arriving during a busy period in two systems, having the same structure but with different parameters, are comparable with respect to the given stochastic orderings. Then, we investigated the monotonicity of the transition operator of the embedded Markov chain with respect to stochastic and convex orderings. In addition, we established comparability conditions of the transition operators of the considered systems, associated with two Markov chains. Further, we obtained comparability conditions for which the stationary distribution of the number of customers in the $M/G/1$ retrial queue with Bernoulli feedback and negative customers is bounded above (resp. bounded below) by the stationary distribution of the number of customers in the $M/M/1$ retrial queue with Bernoulli feedback and negative customers, if the service time distribution is NBUE (resp. NWUE). Finally, based on numerical examples, we showed that analytical results match well with those of simulation.

Moreover, from the realized simulation, we observed that the presence of negative customers had a considerable effect on the stationary distribution of the number of positive customers in the system. These results may be helpful for practitioners who are often faced with such situations. In this case, if the rate of negative customers is small enough, we recommend to replacing the characteristics of an $M/G/1$ retrial queue with Bernoulli feedback and negative customers with those of the simplest $M/M/1$ retrial queue with Bernoulli feedback and negative customers. Otherwise, if the rate of negative customers is relatively important, theoretical bounds provided through this work may be used to contribute a qualitative analysis of the considered system. Our queueing model has broad applications in real-world scenarios, and our findings have significant implications for both quantitative and qualitative analysis. These results can be adapted to evaluate the performance and reliability of complex telecommunication systems, where negative customers represent viruses, providing robust bounds for system characteristics through the monotonicity approach. This work opens several directions for future research. One potential extension would be to consider different types of negative customers. Rather than focusing on negative customers in service, we could examine arrivals that remove customers from the orbit. Since this allows customers to exit the system without receiving service, it breaks the system's structural preservation, making analysis more complex. In such cases, the monotonicity approach could provide sensitive bounds for performance measures.

Declarations

Conflict of interest The authors declare no conflict of interest.

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