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Stability analysis of approximately multidimensional additive mappings on fuzzy spaces

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Abstract. The intention of this paper is to study new additive kind multi-dimensional functional equations inspired by several applications of difference equations in biology, control theory, economics, and computer science, as well as notable implementation of fuzzy ideas in certain situations involving ambiguity or vagueness. In the context of different fuzzy spaces, we demonstrate their various fundamental stabilities related to Ulam stability theory. An appropriate example is given to show how stability result fails when the singular case occurs. The findings of this study suggest that stability results are valid in situations with uncertain or imprecise data. The stability results obtained under these fuzzy spaces are compared with previous stability results.

§1 Introduction

The question concerning the stability of several functional inequalities and equations evolved through a renowned question in [30] regarding homomorphism arising in group theory and it is responded via a partial answer under Banach spaces [10]. Later, this stability problem is dealt for additive mappings [2] and for linear functional inequality with bounded above by the sum of exponents of norms [25]. Further, stability problem is solved by substituting a common control function in place of the sum of exponents of norms [8]. Over the last few years, the solutions to the problems pertaining to the stability of many functional, functional-integral, difference equations, inequalities were solved by many mathematicians [3, 4, 9, 18, 20, 27, 28] in various modern spaces such as real and complex Banach spaces, in various fuzzy settings, random complete normed space, non-Archimedean complete normed space, etc.

An additive function is one of the basic functions. There are many applications of additive function in electronics, physics, economics, chemistry and in many other areas. Moreover, many

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ideas arising in theory of functions and their associated topics depend mainly on the notions of additive functions. The properties of the pioneering Cauchy additive equation

$$a(u+v) = a(u) + a(v) \tag{1}$$

are used to develop other functional equations, applied significantly in physical sciences and social sciences. The investigation of the approximate solution of equation (1) and its generalization in several variables are solved under fuzzy Banach spaces using direct approach and fixed point alternative approach in [5, 12, 13, 22, 32]. The stability analysis of uncertain heat equations and singular systems is studied in [15, 29]. The stabilities of nonlinear fractional order fuzzy systems and nonlinear systems with time-varying delays are investigated in [16, 24].

The fuzzy set was first introduced in [31]. The fuzzy notion is applied in various branches of Mathematics. For constructing a fuzzy topological form, a fuzzy norm is employed under a vector space. There are a few other views of the definition of fuzzy norms in [7, 11, 17, 21]. The idea of fuzzy is a powerful tool to deal with situations involving vague or imprecise data. Fuzzy concepts are applicable to deal with the situations by assigning a membership function to the data belonging to the domain. Moreover, there are many contexts where it is not easy to find the norm of a vector. In such contexts, we can apply the idea of intuitionistic fuzzy norms [19, 23, 26]. In other words, such situations can be dealt through modelling the uncertainty via intuitionistic fuzzy norm.

The investigation of this work is motivated via the remarkable applications of fuzzy ideas and the Fibonacci difference equation

$$a_s = a_{s-1} + a_{s-2} \tag{2}$$

with initial conditions $a(0) = 0$ and $a(1) = 1$. The equation (2) is a second-order homogenous linear difference equation and its solution is

$$a_s = \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^s - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^s.$$

Consider the difference equation

$$a_s = a_{s-1} - a_{s-2} \tag{3}$$

which has only one minus-sign different from (2) on its right hand side with the same initial conditions $a(0) = 0$ and $a(1) = 1$. Then, the solution of (3) is

$$a_s = \frac{1}{\sqrt{-3}}\left(\frac{1+\sqrt{-3}}{2}\right)^s - \frac{1}{\sqrt{-3}}\left(\frac{1-\sqrt{-3}}{2}\right)^s.$$

If the solution of equation (3) is computed numerically, then it will have little residual imaginary parts, which can be further simplified using `val.real` in Python programming to get real value.

Inspired through interesting difference equations (2) and (3), this study focuses on the ensuing advanced additive type multi-dimensional functional equations

$$a\left(\sum_{j=1}^{s} u_j\right) - a\left(\frac{1}{s}\sum_{j=1}^{s} u_j\right) = \frac{s-1}{s}\sum_{j=1}^{s} a(u_j) \tag{4}$$

and

$$a\left(\sum_{j=1}^{s} u_j\right) + a\left(\frac{1}{s}\sum_{j=1}^{s} u_j\right) = \frac{s+1}{s}\sum_{j=1}^{s} a(u_j). \tag{5}$$

It is interesting to check that an additive type mapping $a : \mathbb{R} \longrightarrow \mathbb{R}$ given by $a(u) = ku$, $u \in \mathbb{R}$ is a solution of both the equations (4) and (5) even though the difference between the equations (4) and (5) is minus-sign. The equation (4) involves the difference of functional values on its left-hand side and hence we call it an additive difference functional equation. The equation (5) has positive-sign different from (4) and hence we call it as an additive adjoint functional equation of (4). We determine the fuzzy solutions to stability problems of equations (4) and (5) under different fuzzy settings. We demonstrate an apt counter-example to disprove that the stability results of (4) and (5) for a critical case.

Remark 1.1. When $s = 2$, the equations (4) and (5) produce the following equations, respectively:

$$a(u_1 + u_2) - a\left(\frac{u_1 + u_2}{2}\right) = \frac{1}{2}\left[a(u_1) + a(u_2)\right]$$

and

$$a(u_1 + u_2) + a\left(\frac{u_1 + u_2}{2}\right) = \frac{3}{2}\left[a(u_1) + a(u_2)\right].$$

§2 Preliminaries

Here, we refer to a few basic terminologies and definitions connected with fuzzy space, which will be significant to obtain the major outcomes of this study.

Definition 2.1 [14] Suppose $\mathcal{F}$ is a vector space of reals. Then a fuzzy norm on $\mathcal{F}$ is a function H from $\mathcal{F} \times \mathbb{R}$ to $[0, 1]$, if the upcoming conditions hold, for all $u, v \in \mathcal{F}$, and all $\lambda, \mu \in \mathbb{R}$:

(N_1) $H(u, \lambda) = 0$ for any $\lambda \leq 0$,

(N_2) $u = 0$ if and only if $H(u, \lambda) = 1$ for all $\lambda > 0$,

(N_3) $H(pu, \lambda) = H\left(u, \frac{\lambda}{|p|}\right)$ if $p \neq 0$,

(N_4) $H(u + v, \lambda + \mu) \geq \min\{H(u, \lambda), H(v, \mu)\}$,

(N_5) $H(u, \cdot)$ is a not a decreasing function of $\mathbb{R}$, and $\lim_{\lambda\to\infty} H(u, \lambda) = 1$,

(N_6) for $u \neq 0$, $H(u, \cdot)$ is continuous over $\mathbb{R}$.

Moreover, $\mathcal{F}$ is said to be a vector space with the fuzzy norm H.

In the upcoming definitions, we presume that $\mathcal{F}$ be a vector space with a fuzzy norm H and let $\{s_m\}$ be a sequence in $\mathcal{F}$.

Definition 2.2 [14] The limit of the sequence $\{s_m\}$ is s, denoted as $H\text{-}\lim_{m\to\infty} s_m = s$ if

$\lim_{m\to\infty} H(s_m - s, \lambda) = 1$ for all $\lambda > 0$.

Definition 2.3 [14] The existence of a $m_0 \in \mathbb{N}$ such that for all $m > 0$ and all $k > 0$, we obtain $H(s_{m+k} - s_n, \lambda) > 1 - \epsilon$ implies that the sequence $\{s_m\}$ is Cauchy.

Clearly, in a fuzzy normed vector space, every convergent sequence is Cauchy. If every Cauchy sequence converges, then the fuzzy normed space becomes a fuzzy Banach space.

§3 Approximation of fuzzy almost additive mappings by additive mappings in fuzzy version

In the present section, we approximate a fuzzy almost additive mapping by an additive mapping as the solution for equation (4) under fuzzy normed spaces. Assume that $\mathcal{A}$ is a vector space and $(\mathcal{B}, H)$ is a fuzzy Banach space. For easy computation to derive the main results in a fuzzy version, let us consider a difference operator $\mathfrak{D}a : \underbrace{\mathcal{A} \times \cdots \times \mathcal{A}}_{s \text{ times}} \longrightarrow \mathcal{B}$ be defined as

$$\mathfrak{D}a(u_1, \ldots, u_s) = a\left(\sum_{j=1}^{s} u_j\right) - \left(\frac{1}{s}\sum_{j=1}^{s} u_j\right) - \frac{s-1}{s}\sum_{j=1}^{s} a(u_j)$$

for all $u_1, \ldots, u_s \in \mathcal{A}$.

Theorem 3.1 Let $\ell = \pm 1$ and let a function $\zeta : \underbrace{\mathcal{A} \times \cdots \times \mathcal{A}}_{s \text{ times}} \longrightarrow [0, \infty)$ with the condition that

$$\tilde{\zeta}(u_1, \ldots, u_s) = \sum_{m=0}^{\infty} s^{-\ell m} \zeta\left(s^{\ell m} u_1, \ldots, s^{\ell m} u_s\right) < \infty \tag{6}$$

for all $u_1, \ldots, u_s \in \mathcal{A}$. Let $a : \underbrace{\mathcal{A} \times \cdots \times \mathcal{A}}_{s \text{ times}} \longrightarrow \mathcal{B}$ be a mapping such that

$$\lim_{u\to\infty} H(\mathfrak{D}a(u_1, \ldots, u_s), u\zeta(u_1, \ldots, u_s)) = 1 \tag{7}$$

uniformly on $\underbrace{\mathcal{A} \times \cdots \times \mathcal{A}}_{s \text{ times}}$. Then for each $u \in \mathcal{A}$, there exists a unique additive mapping $\mathfrak{F} : \mathcal{A} \longrightarrow \mathcal{B}$ described by $\mathfrak{F}(u) = H\text{-}\lim_{m\to\infty} \frac{a(s^{\ell m} u)}{s^{\ell m}}$ with the condition that if for some $\alpha > 0$

$$H(\mathfrak{D}a(u_1, \ldots, u_s), \zeta(u_1, \ldots, u_s)) \geq \alpha \tag{8}$$

for all $u_1, \ldots, u_s \in \mathcal{A}$, then

$$H\left(a(u) - \mathfrak{F}(u), \frac{1}{s^{\ell}}\widetilde{\zeta}(\underbrace{u, \ldots, u}_{s \text{ times}})\right) \geq \alpha \tag{9}$$

for all $u \in \mathcal{A}$. Moreover, the mapping $\mathfrak{F} : \mathcal{A} \longrightarrow \mathcal{B}$ satisfies

$$\lim_{v\to\infty} H(a(u) - \mathfrak{F}(u), v\tilde{\zeta}(\underbrace{u, \ldots, u}_{s \text{ times}})) = 1 \tag{10}$$

uniformly on $\mathcal{A}$.

Proof. For a given $\eta > 0$, and by the condition (7), we can find some $u_0 > 0$ such that

$$H(\mathfrak{D}a(u_1,\dots,u_s), v\zeta(u_1,\dots,u_s)) \geq 1-\eta \tag{11}$$

for all $v \geq v_0$. Let $m \in \mathbb{N}$. Proceeding further and using a mathematical approach on m, we obtain that

$$H\left(s^{\ell m}a(u) - \mathfrak{F}\left(s^{\ell m}u\right), u\sum_{k=0}^{m-1} s^{\ell(m-k-1)}\zeta\left(\underbrace{s^{\ell k}u,\dots,s^{\ell k}u}_{s\text{ times}}\right)\right) \geq 1-\eta \tag{12}$$

for all $v \geq v_0$, all $u \in \mathcal{A}$, and all $s \in \mathbb{N}$. Letting $u_1 = \cdots = u_s = u$ in (11), we obtain

$$H(sa(u) - a(su), v\zeta(\underbrace{u,\dots,u}_{s\text{ times}})) \geq 1-\eta \tag{13}$$

for all $u \in \mathcal{A}$ and all $v \geq v_0$. Hence (12) is valid when $m = 1$. Next, let us presume that (12) is valid when $m \in \mathbb{N}$. Then

$$\begin{aligned}
&H\left(s^{\ell(m+1)}a(u) - a\left(s^{\ell(m+1)}u\right), u\sum_{k=0}^{m} s^{\ell(m-k)}\zeta\left(\underbrace{s^{\ell k}u,\dots,s^{\ell k}u}_{s\text{ times}}\right)\right)\\
&\quad\geq \min\Bigg\{H\left(s^{\ell(m+1)}a(u) - s^{\ell m}a\left(s^{\ell m}u\right), v_0\sum_{k=0}^{m-1} s^{\ell(m-k)}\zeta\left(\underbrace{s^{\ell k}u,\dots,s^{\ell k}u}_{s\text{ times}}\right)\right),\\
&\qquad\qquad H\left(s^{\ell m}a\left(s^{\ell m}u\right) - a\left(s^{\ell(m+1)}u\right), v_0\zeta\left(\underbrace{s^{\ell m}u,\dots,s^{\ell m}u}_{s\text{ times}}\right)\right)\Bigg\}\\
&\quad\geq \min\{1-\eta, 1-\eta\} = 1-\eta.
\end{aligned}\tag{14}$$

Hence (12) is true for any $m \in \mathbb{N}$. Letting $u = u_0$, and reinstating m by r and u by $s^m u$ in (12), we obtain

$$\begin{aligned}
H\left(\frac{a\left(s^{\ell r}u\right)}{s^{\ell r}} - \frac{a\left(s^{\ell(m+r)}u\right)}{s^{\ell(m+r)}}, \frac{u_0}{s^{\ell(m+r)}}\sum_{m=0}^{r-1} s^{\ell(-m-r-1)}\zeta\left(\underbrace{s^{\ell(m+r)}u,\dots,s^{\ell(m+r)}u}_{s\text{ times}}\right)\right)&\\
\geq 1-\eta&
\end{aligned}\tag{15}$$

for all integers $m \geq 0$, $r > 0$. In view of (6), and the following equality

$$\begin{aligned}
&\sum_{m=0}^{r-1} s^{\ell(-m-r-1)}\zeta\left(\underbrace{s^{\ell(m+r)}u,\dots,s^{\ell(m+r)}u}_{s\text{ times}}\right)\\
&\qquad = \frac{1}{r}\sum_{m=n}^{n+r-1} s^{-\ell m}\zeta\left(\underbrace{s^{\ell m}u,\dots,s^{\ell m}u}_{s\text{ times}}\right)
\end{aligned}\tag{16}$$

that for some $\delta > 0$, we can find a positive integer n_0 so that

$$\frac{u_0}{k}\sum_{m=n}^{n+r-1} s^{-\ell m}\zeta\left(s^{\ell m}u,\dots,s^{\ell m}u\right) < \delta \tag{17}$$

for all $n \geq n_0$, and $r > 0$. Also, one can derive from (15) that

$$H\left(\frac{a\left(s^{\ell m}u\right)}{s^{\ell m}} - \frac{a\left(s^{\ell(m+n)}u\right)}{s^{\ell(m+n)}}, \delta\right)$$

$$\geq H\left(\frac{a\left(s^{\ell m}u\right)}{s^{\ell m}} - \frac{a\left(s^{\ell(m+n}u\right)}{s^{\ell(m+n)}}, \frac{u_0}{s^{\ell(m+n)}}\sum_{k=0}^{n-1} s^{(-m-k-1)}\zeta\left(\underbrace{s^{\ell(m+k)}u,\ldots,s^{\ell(m+k)}u}_{s \quad \text{times}}\right)\right)$$

$$\geq 1-\eta \tag{18}$$

for all $m \geq m_0$, and all $n > 0$. Hence the sequence $\left\{\frac{a\left(s^{\ell m}u\right)}{s^{\ell m}}\right\}$ turns out to be Cauchy in $\mathcal{B}$ and hence it converges to some $\mathfrak{F}(u) \in \mathcal{B}$, as $\mathcal{B}$ is complete. This induces us to describe a mapping $\mathcal{F}: \mathcal{A} \longrightarrow \mathcal{B}$ via $\mathcal{F}(u) = H\text{-}\lim_{m\to\infty} \frac{a\left(s^{\ell m}u\right)}{s^{\ell m}}$ such that for $v > 0$ and for all $u \in \mathcal{A}$, $\lim_{m\to\infty} H\left(\frac{a\left(s^{\ell m}u\right)}{s^{\ell m}} - \mathcal{F}(u), v\right) = 1$. Let $u_1,\ldots,u_s \in \mathcal{A}$. Fix $v > 0$ and $0 < \eta < 1$. Since $\lim_{m\to\infty} s^{-\ell m}\zeta\left(s^{\ell m}u_1,\ldots,s^{\ell m}u_s\right) = 0$, there is an $m_1 > m_0$ such that $v_0\zeta\left(s^{\ell m}u_1,\ldots,s^{\ell m}u_s\right) < \frac{s^{\ell m}u}{s^{\ell}}$ for all $m \geq m_1$. Hence for all $m \geq m_1$, we have

$$\begin{aligned}
&H(\mathfrak{D}a(u_1,\ldots,u_s),v)\\
&\geq \min\Big\{H\left(\mathcal{F}\left(\sum_{j=1}^{s}u_j\right) - s^{-\ell m}a\left(s^{\ell m}\left(\sum_{j=1}^{s}u_j\right)\right), \frac{v}{4}\right),\\
&\quad H\left(\mathcal{F}\left(\frac{1}{s}\sum_{j=1}^{s}u_j\right) - s^{-\ell m}a\left(s^{\ell m}\left(\frac{1}{s}\sum_{j=1}^{s}u_j\right)\right), \frac{v}{4}\right),\\
&\quad H\left(\frac{s+1}{s}\sum_{j=1}^{s}\mathcal{F}(u_j) - s^{-\ell m}\left(s^{\ell m}\left(\frac{s+1}{s}\sum_{j=1}^{s}a(u_j)\right)\right), \frac{v}{4}\right),\\
&\quad H\Big(s^{-\ell m}a\left(s^{\ell m}\sum_{j=1}^{s}u_j\right) + s^{-\ell m}a\left(s^{\ell m}\left(\frac{1}{s}\sum_{j=1}^{s}u_j\right)\right)\\
&\qquad - \left(s^{-\ell m}\left(s^{\ell m}\left(\frac{s+1}{s}\sum_{j=1}^{s}a(u_j)\right)\right), \frac{v}{4}\right)\Big\}.
\end{aligned} \tag{19}$$

From the preceding inequality, by letting $m \to \infty$, we find that the first three terms on the right-hand side approach 1 and the fourth term on the right-hand side approaches not less than to

$$H\Big(a\left(s^{\ell m}\sum_{j=1}^{s}u_j\right) + a\left(s^{\ell m}\left(\frac{1}{s}\sum_{j=1}^{s}u_j\right)\right) - s^{\ell m}\left(\frac{s+1}{s}\sum_{j=1}^{s}a(u_j)\right), \frac{s^{\ell m}v}{4}\Big) \tag{20}$$

which is further not less than to $1-\eta$. Thus

$$H(\mathcal{D}\mathcal{F}(u_1,\ldots,u_s),v) \geq 1-\eta \tag{21}$$

for all $u > 0$. Since $H(\mathcal{D}\mathcal{F}(u_1,\ldots,u_s),v) = 1$ for all $v > 0$, by $H(\mathcal{D}\mathcal{F}(u_1,\ldots,u_s),v) =$

0 for all $u_1, \ldots, u_s \in \mathcal{A}$. This implies that $\mathcal{F}$ is additive mapping, which indicates that, $\mathcal{D}\mathcal{F}(u_1, \ldots, u_s) = 0$, for all $u_1, \ldots, u_s \in \mathcal{A}$. Now, suppose for some $\delta > 0$, (18) hold. Let

$$\zeta_m(\underbrace{u, \ldots, u}_{s \text{ times}}) = \sum_{n=0}^{m-1} s^{-\ell(n+1)} \xi \left(\underbrace{s^{\ell n} u, \ldots, s^{\ell n} u}_{s \text{ times}} \right) \tag{22}$$

for all $u \in \mathcal{A}$. Let $u \in \mathcal{A}$. By similar arguments at the starting point of the proof, one can derive from (8) that

$$H\left(s^{\ell m} a(u) - a\left(s^{\ell m} u\right), \delta \sum_{n=0}^{m-1} s^{\ell(m-n-1)} \zeta \left(\underbrace{s^{\ell n} u, \ldots, s^{\ell n} u}_{s \text{ times}} \right) \right) \geq \alpha \tag{23}$$

for all integers $m > 0$. Let $v > 0$. Then, we have

$$\begin{aligned} &H(a(u) - \mathcal{F}(u), \zeta_m(\underbrace{u, \ldots, u}_{s \text{ times}}) + v) \\ &\geq \min \left\{ H\left(a(u) - \frac{a\left(s^{\ell m} u\right)}{s^{\ell m}}, \zeta_m(\underbrace{u, \ldots, u}_{s \text{ times}}) \right), H\left(\frac{a\left(s^{\ell m} u\right)}{s^{\ell m}} - \mathcal{F}(u), v \right) \right\}. \end{aligned} \tag{24}$$

Consolidating (23), and (24) and $\lim_{m \to \infty} H\left(\frac{a(s^{\ell m} u)}{s^{\ell m}} - \mathcal{F}(u), v \right) = 1$, we find that

$$H(a(u) - \mathcal{F}(u), \zeta_m(\underbrace{u, \ldots, u}_{s \text{ times}}) + v) \geq \alpha \tag{25}$$

for sufficiently large $m \in \mathbb{N}$. By the continuity of the function $H(a(u) - \mathcal{F}(u), \cdot)$, we observe that $H\left(a(u) - \mathcal{F}(u), \frac{1}{s}\tilde{\zeta}(\underbrace{u, \ldots, u}_{s \text{ times}}) + v \right) \geq \alpha$. Letting $u \to 0$, we conclude that

$$H\left(a(u) - \mathcal{F}(u), \frac{1}{s}\tilde{\zeta}(\underbrace{u, \ldots, u}_{s \text{ times}}) \right) \geq \alpha. \tag{26}$$

The remaining part of the proof is to show $\mathcal{F}$ is unique. Suppose let us assume that $\mathcal{G}$ be some other additive mapping satisfying (10). Choose $c > 0$. Given that $\eta > 0$, utilizing (10) for $\mathcal{F}$ and $\mathcal{G}$, we can obtain some $u_0 > 0$ so that

$$\begin{aligned} &H\left(a(u) - \mathcal{F}(u), v\tilde{\zeta}(\underbrace{u, \ldots, u}_{s \text{ times}}) \right) \geq 1 - \eta, \\ &H\left(a(u) - \mathcal{G}(u), v\tilde{\zeta}(\underbrace{u, \ldots, u}_{s \text{ times}}) \right) \geq 1 - \eta \end{aligned} \tag{27}$$

for all $u \in \mathcal{A}$ and all $v \geq 2v_0$. For some specified $u \in \mathcal{A}$, we can obtain some integer m_0 such that

$$u_0 \sum_{n=m}^{\infty} s^{-\ell n} \zeta \left(\underbrace{s^{\ell n} u, \ldots, s^{\ell n} u}_{s \text{ times}} \right) < \frac{c}{2} \tag{28}$$

for all $m \geq m_0$. As

$$\sum_{n=m}^{\infty} s^{-\ell n}\zeta\left(\underbrace{s^{\ell n}u,\ldots,s^{\ell n}u}_{s\text{ times}}\right) = \frac{1}{s^{\ell m}}\sum_{n=m}^{\infty} s^{-\ell(n-m)}\zeta\left(\underbrace{s^{\ell(n-m)}\left(s^{\ell m}u\right),\ldots,s^{\ell(n-m)}\left(s^{\ell m}u\right)}_{s\text{ times}}\right)$$

$$= \frac{1}{s^{\ell m}}\sum_{n=0}^{\infty} s^{-\ell n}\zeta\left(\underbrace{s^{\ell n}\left(s^{\ell m}u\right),\ldots,s^{\ell n}\left(s^{\ell m}u\right)}_{s\text{ times}}\right)$$

$$= \frac{1}{s^{\ell m}}\tilde{\zeta}\left(\underbrace{s^{\ell m}u,\ldots,s^{\ell m}u}_{s\text{ times}}\right), \tag{29}$$

we have

$$\begin{aligned}
&H(\mathcal{F}(u)-\mathcal{G}(u),v)\\
&\geq \min\left\{H\left(\frac{a\left(s^{\ell m}u\right)}{s^{\ell m}}-\mathcal{F}(u),\frac{v}{2}\right),H\left(\mathcal{G}(u)-\frac{a\left(s^{\ell m}u\right)}{s^{\ell m}},\frac{v}{2}\right)\right\}\\
&= \min\left\{H\left(a\left(s^{\ell m}u\right)-\mathcal{F}\left(s^{\ell m}u\right),s^{\ell(m-1)}2v\right),H\left(\mathcal{G}\left(s^{\ell m}u\right)-a\left(s^{\ell m}u\right),s^{\ell(m-1)}2v\right)\right\}\\
&\geq \min\Bigg\{H\left(a\left(s^{\ell m}u\right)-\mathcal{F}\left(s^{\ell m}u\right),s^{\ell m}v_0\sum_{n=m}^{\infty}s^{-\ell n}\zeta\left(\underbrace{s^{\ell n}u,\ldots,s^{\ell n}u}_{s\text{ times}}\right)\right),\\
&\quad H\left(\mathcal{G}\left(s^{\ell m}u\right)-a\left(s^{\ell m}u\right),s^{\ell m}t_0\sum_{n=m}^{\infty}s^{-\ell n}\zeta\left(\underbrace{s^{\ell n}u,\ldots,s^{\ell n}u}_{s\text{ times}}\right)\right)\Bigg\}\geq 1-\eta.
\end{aligned} \tag{30}$$

Hence, it implies for all $v > 0$ that $H(\mathcal{F}(u)-\mathcal{G}(u),v) = 1$. Consequently, this leads to $\mathcal{F}(u) = \mathcal{G}(u)$ for all $u \in \mathcal{A}$. Hence the theorem follows. □

The succeeding outcomes follow directly from Theorem 3.1 and are associated with Hyers-Ulam and T. Rassias stabilities. In the following corollaries, let $a : \underbrace{\mathcal{A}\times\cdots\times\mathcal{A}}_{s\text{ times}} \longrightarrow \mathcal{B}$ be a mapping.

Corollary 3.2 Let $\epsilon > 0$ and the mapping a satisfies

$$\lim_{v\to\infty} H(\mathcal{D}a(u_1,\ldots,u_s),v\epsilon) = 1$$

uniformly on $\underbrace{\mathcal{A}\times\cdots\times\mathcal{A}}_{s\text{ times}}$. Then for each $u \in \mathcal{A}$, there exists a unique additive mapping $\mathcal{F} : \mathcal{A} \longrightarrow \mathcal{B}$ described by $\mathcal{F}(u) = H\text{-}\lim_{m\to\infty}\frac{a\left(s^{\ell m}u\right)}{s^{\ell m}}$ with the condition that if for some $\alpha > 0$

$$H(\mathcal{D}a(u_1\ldots,u_s),\epsilon) \geq \alpha$$

for all $u_1,\ldots,u_s \in \mathcal{A}$, then

$$H\left(a(u)-\mathcal{F}(u),\frac{s\epsilon}{s-1}\right) \geq \alpha$$

for all $u \in \mathcal{A}$. Moreover, the mapping $\mathcal{F}$ satisfies

$$\lim_{v\to\infty} H\left(a(u) - \mathcal{F}(u), \frac{vs\epsilon}{s-1}\right) = 1$$

uniformly on $\mathcal{A}$.

Proof. The proof goes along with Theorem 3.1 by assuming $\zeta(u_1, \ldots, u_s) = \epsilon$ when $\ell = -1$. $\square$

Corollary 3.3 Let $\mu \geq 0$ and $0 < r < 1$. Let the mapping a satisfy

$$\lim_{v\to\infty} H\Big(\mathcal{D}a(u_1, \ldots, u_s), v\epsilon\left(\sum_{j=1}^{s} \|u_j\|^r\right)\Big) = 1 \tag{31}$$

uniformly on $\underbrace{\mathcal{A} \times \cdots \times \mathcal{A}}_{s \text{ times}}$. Then for each $u \in \mathcal{A}$, there exists a unique additive mapping $\mathcal{F} : \mathcal{A} \longrightarrow \mathcal{B}$ described by $\mathcal{F}(u) = H\text{-}\lim_{m\to\infty} \frac{a\left(s^{\ell m}u\right)}{s^{\ell m}}$ with the condition that if for some $\alpha > 0$

$$H\Big(\mathcal{D}a(u_1, \ldots, u_s), \epsilon\left(\sum_{j=1}^{s} \|u_j\|^r\right)\Big) \geq \alpha$$

for all $u_1, \ldots, u_s \in A$, then

$$H\left(a(u) - \mathcal{F}(u), \frac{s^r\epsilon}{|1 - s^{r-1}|}\|u\|^r\right) \geq \alpha$$

for all $u \in \mathcal{A}$. Further, the mapping $\mathcal{F}$ satisfies

$$\lim_{v\to\infty} H\left(a(u) - \mathcal{F}(u), \frac{vs^r\epsilon}{|1 - s^{r-1}|}\|u\|^r\right) = 1$$

uniformly on $\mathcal{A}$.

Proof. The required outcome is achieved by assuming $\zeta(u_1, \ldots, u_s) = \epsilon\left(\sum_{j=1}^{s} \|u_j\|^r\right)$, and following with similar arguments as in Theorem 3.1. $\square$

Similarly, the stability results pertaining to equation (5) can be obtained.

§4 Approximation of intuitionistic fuzzy almost additive mappings in intuitionistic fuzzy version

Let us evoke the fundamental concepts and basic definitions concerning intuitionistic fuzzy normed spaces [1], which are useful to obtain the solution to stability problems.

Definition 4.1. A binary operation T_N from $[0,1] \times [0,1]$ to $[0,1]$ is known as a continuous triangular norm (abbreviated as continuous t-norm) satisfying the ensuing conditions:

(T_N1) $T_N(u_1, u_2) = T_N(u_2, u_1)$ (commutativity),

(T_N2) $T_N(u_1, T_N(u_2, u_3)) = T_N(T_N(u_1, u_2), u_3)$ (associativity),

(T_N3) $u_2 \leq u_3 \Longrightarrow T_N(u_1, u_2) \leq T_N(u_1, u_3)$ (monotonicity),

(T_N4) $T_N(u,1) = u$ (neutral element 1) and

(T_N5) T_N is continuous.

Example 4.2. $T_{\min}(u_1, u_2) = \min(u_1, u_2)$ (minimum or Gödel t-norm).

Example 4.3. $T_{\text{prod}}(u_1, u_2) = u_1 \cdot u_2$ (product t-norm).

Example 4.4. $T_{\max}(u_1, u_2) = \max(u_1 + u_2 - 1, 0)$ (Lukasiewicz t-norm).

Definition 4.5. The dual concept of a t-norm T_N is a continuous triangular conorm (abbreviated as continuous t-conorm, also known as s-norm), denoted as TC_N. The neutral element in this case is 0 instead of 1, and all other conditions are unaltered.

(TC_N1) $TC_N(u_1, u_2) = TC_N(u_2, u_1)$ (commutativity)

(TC_N2) $TC_N(u_1, TC_N(u_2, u_3)) = TC_N(TC_N(u_1, u_2), u_3)$ (associativity),

(TC_N3) $u_2 \leq u_3 \Longrightarrow TC_N(u_1, u_2) \leq TC_N(u_1, u_3)$ (monotonicity),

(TC_N4) $TC_N(u, 0) = u$ (neutral element 0) and

(TC_N5) TC_N is continuous.

Example 4.6. $TC_{\max}(u_1, u_2) = \max(u_1, u_2)$ (maximum or Gödel t-conorm).

Example 4.7. $TC_{\text{prod}}(u_1, u_2) = u_1 + u_2 - u_1 \cdot u_2$ (product t-conorm or probabilistic sum).

Example 4.8. $TC_{\min}(u_1, u_2) = \min(u_1 + u_2, 1)$ (Lukasiewicz t-conorm or bounded sum).

Definition 4.9. The five-tuple $(\mathcal{L}, \mu, \nu, \star, \diamond)$ is called an intuitionistic fuzzy normed space (abbreviated as IFNS) if $\mathcal{L}$ is a vector space with $\star$ and $\diamond$ as a continuous t-norm and a continuous t-conorm, respectively, and μ, ν are fuzzy sets defined on $\mathcal{L} \times (0, \infty)$ so that for all $u_1, u_2 \in \mathcal{L}$ and $s, t > 0$ satisfying the subsequent conditions.

(1) $\mu(u_1, t) + nu(u_1, t) \leq 1$,

(2) $\mu(u_1, t) > 0$,

(3) $\mu(u_1, t) = 1$ if and only if $u_1 = 0$,

(4) $\mu(\beta u_1, t) = \mu\left(u_1, \frac{t}{|\beta|}\right)$ for each $\beta \neq 0$,

(5) $\mu(u_1, t) \star \mu(u_2, s) \geq \mu(u_1 + u_2, t + s)$,

(6) $\mu(u_1, \cdot) : (0, \infty) \to [0, 1]$ is continuous,

(7) $\lim\limits_{t\to\infty} \mu(u_1, t) = 1$ and $\lim\limits_{t\to 0} \mu(u_1, t) = 0$,

(8) $\nu(u_1,t)<1$,

(9) $\nu(u_1,t)=0$ if and only if $u_1=0$,

(10) $\nu(\beta u_1,t)=\nu\left(u_1,\frac{t}{|\beta|}\right)$ for each $\beta\neq 0$,

(11) $\nu(u_1,t)\diamond\nu(u_2,s)\leq\nu(u_1+u_2,t+s)$,

(12) $\nu(u_1,\cdot):(0,\infty)\to[0,1]$ is continuous,

(13) $\lim_{t\to\infty}\nu(u_1,t)=0$ and $\lim_{t\to 0}\nu(u_1,t)=1$.

In the following definitions, let $(\mathcal{L},\mu,\nu,\star,\diamond)$ be an IFNS and $\{u_n\}$ be a sequence in $\mathcal{L}$.

Definition 4.10. The sequence $\{u_n\}$ converges to $\ell\in\mathcal{L}$, if $\lim_{m\to\infty}\mu(u_m-\ell,t))=1$ and $\lim_{m\to\infty}\nu(u_m-\ell,t)=0$ for all $t>0$.

Definition 4.11. The sequence $\{u_n\}$ is Cauchy, if $\lim_{m\to\infty}\mu(u_{m+r}-u_m,t)=1$ and $\lim_{m\to\infty}\nu(u_{m+r}-u_m,t)=0$ for all $t>0$ and $r=1,2\ldots$.

Definition 4.12 The IFNS $\mathcal{L}$ is complete (or Banach space), if every Cauchy sequence in $\mathcal{L}$ converges in $\mathcal{L}$.

In the following theorem, we find the solutions to the stability problems of equation (4) under intuitionistic fuzzy complete normed spaces. Assume that $\mathcal{L}$ is a linear space and $(\mathcal{G},\mu,\nu)$ is an intuitionistic fuzzy complete space and $a:\underbrace{\mathcal{L}\times\cdots\times\mathcal{L}}_{s\text{ times}}\longrightarrow\mathcal{G}$ is a mapping in this section.

Theorem 4.1 Let a function $\zeta:\underbrace{\mathcal{L}\times\cdots\times\mathcal{L}}_{s\text{ times}}\longrightarrow[0,\infty)$ with the condition that

$$\tilde{\zeta}(u_1,\ldots,u_s)=\sum_{n=0}^{\infty}s^{-m}\zeta\left(s^m u_1,\ldots,s^m u_s\right)<\infty \tag{32}$$

for all $u_1,\ldots,u_s\in\mathcal{L}$. Let a mapping $a:\underbrace{\mathcal{L}\times\cdots\times\mathcal{L}}_{s\text{ times}}\longrightarrow\mathcal{G}$ satisfy the following with reference to ζ

$$\lim_{t\to\infty}\mu(\mathcal{D}a(u_1,\ldots,u_s),t\zeta(u_1,\ldots,u_s))=1,$$

and

$$\lim_{t\to\infty}\nu(\mathcal{D}a(u_1,\ldots,u_s),t\zeta(u_1,\ldots,u_s))=0 \tag{33}$$

for all $u_1,\ldots,u_2\in\mathcal{L}$. Then for every $u\in\mathcal{L}$, there exists an additive mapping $\mathcal{F}:\underbrace{\mathcal{L}\times\cdots\times\mathcal{L}}_{s\text{ times}}$ $\longrightarrow\mathcal{G}$ described by $\mathcal{F}(x)=(\mu,\nu)\text{-}\lim_{m\to\infty}\frac{a\left(s^{\ell m}x\right)}{s^{\ell m}}$ with the condition that if for some $\alpha>0$ and all $u_1,\ldots,u_s\in\mathcal{L}$,

$$\mu(\mathcal{D}a(u_1,\ldots,u_s),\zeta(u_1,\ldots,u_s))>\alpha,$$

and

$$\nu(\mathcal{D}a(u_1,\ldots,u_s),\zeta(u_1,\ldots,u_s))<1-\alpha, \tag{34}$$

then $\mu\left(a(u)-\mathcal{F}(u),\tilde{\zeta}(u,\ldots,u)\right)>\alpha,\quad$ and $\quad\nu\left(a(u)-\mathcal{F}(u),\tilde{\zeta}(u,\ldots,u)\right)<1-\alpha.$

Proof. Let $\epsilon > 0$ be given by (33), one can notice some $t_0 > 0$ such that

$$\mu(\mathcal{D}a(u_1,\ldots,u_s), t\zeta(u_1,\ldots,u_s)) \geq 1-\epsilon,$$

and

$$\nu(\mathcal{D}a(u_1,\ldots,u_s), t\zeta(u_1,\ldots,u_s)) \leq \epsilon \tag{35}$$

for all $u_1,\ldots,u_s \in \mathcal{L}$, and all $t \geq t_0$. Employing induction process on m, we find that

$$\mu\left(a\left(s^m u\right) - s^m a(u), t\sum_{n=0}^{m-1} s^{m-n-1}\zeta\left(s^n u,\ldots,s^n u\right)\right) \geq 1-\epsilon,$$

and (36)

$$\nu\left(a\left(s^m u\right) - s^m a(u), t\sum_{n=0}^{m-1} s^{m-n-1}\zeta\left(s^n u,\ldots,s^n u\right)\right) \leq \epsilon,$$

for all $u \in \mathcal{L}$, $t \geq t_0$, and all positive integers m. Letting $u_1 = \cdots = u_s = u$ in (35), we obtain (36) for $m = 1$. Let (36) hold for integer $m > 0$. Then

$$\begin{aligned}
&\mu\left(a\left(s^{m+1}u\right) - s^{m+1}a(u), t\sum_{n=0}^{m} s^{(m-n)}\zeta\left(s^n u,\ldots,s^n u\right)\right)\\
&\qquad \geq \mu\left(a\left(s^{m+1}u\right) - s^m a\left(s^m u\right), t_0\zeta\left(s^m u,\ldots,s^m u\right)\right)\\
&\qquad\qquad \star\, \mu\left(s^m a\left(s^m u\right) - s^{m+1}a(u), t_0\sum_{n=0}^{m-1} s^{(m-n)}\zeta\left(s^n u,\ldots,s^n u\right)\right)\\
&\qquad \geq (1-\epsilon)\star(1-\epsilon) = 1-\epsilon,
\end{aligned}$$

and

$$\begin{aligned}
&\nu\left(a\left(s^{m+1}u\right) - s^{m+1}a(u), t\sum_{n=0}^{m} s^{(m-n)}\zeta\left(s^n u,\ldots,s^n u\right)\right)\\
&\qquad \leq \nu\left(a\left(s^{m+1}u\right) - s^m a\left(s^m u\right), t_0\zeta\left(s^m u,\ldots,s^m u\right)\right)\\
&\qquad\qquad \Diamond\nu\left(s^m a\left(s^m u\right) - s^{m+1}a(u), t_0\sum_{n=0}^{m-1} s^{(m-n)}\zeta\left(s^n u,\ldots,s^n u\right)\right) \leq \epsilon\Diamond\epsilon = \epsilon.
\end{aligned}$$

Hence, by induction (36) is true. Take $t = t_0$ and substitute $m = p$. Then by replacing u by $s^m u$ in (36), we get

$$\mu\left(\frac{a\left(s^{m+p}u\right)}{s^{m+p}} - \frac{a\left(s^m u\right)}{s^m}, \frac{t_0}{s^{m+p}}\sum_{n=0}^{p-1} s^{p-n-1}\zeta\left(s^{m+p}u,\ldots,s^{m+p}u\right)\right) \geq 1-\epsilon,$$

and

$$\nu\left(\frac{a\left(s^{m+p}u\right)}{s^{m+p}} - \frac{a\left(s^m u\right)}{s^m}, \frac{t_0}{s^{m+p}}\sum_{n=0}^{p-1} s^{p-n-1}\zeta\left(s^{m+n}u,\ldots,s^{m+n}u\right)\right) \leq \epsilon, \tag{37}$$

for all integers $m \geq 0$ and $u > 0$. Since (32) is convergent and

$$\sum_{n=0}^{u-1} s^{-(m+n+1)}\zeta\left(s^{m+n}u,\ldots,s^{m+n}u\right) = \frac{1}{2}\sum_{n=m}^{m+p-1} s^{-n}\zeta\left(s^n u,\ldots,s^n u\right)$$

imply that for given $\delta > 0$ there is an integer $m_0 > 0$ with

$$\frac{t_0}{2} \sum_{n=m}^{m+p-1} s^{-n} \zeta\left(s^n u, \ldots, s^n u\right) < \delta,$$

for all $m \geq m_0$ and all $p > 0$. Now, we deduce that from (37) that

$$\begin{aligned}
&\mu\left(\frac{a\left(s^{m+p} u\right)}{s^{m+p}} - \frac{a\left(s^m u\right)}{s^m}, \delta\right) \\
&\quad \geq \mu\left(\frac{a\left(s^{m+p} u\right)}{s^{m+p}} - \frac{a\left(s^m u\right)}{s^m}, \frac{t_0}{s^{m+p}} \sum_{n=0}^{p-1} s^{p-n-1} \zeta\left(s^{m+n} u, \ldots, s^{m+n} u\right)\right) \\
&\quad \geq 1 - \epsilon,
\end{aligned}$$

and

$$\begin{aligned}
&\nu\left(\frac{a\left(s^{m+p} u\right)}{s^{m+p}} - \frac{a\left(s^m u\right)}{s^m}, \delta\right) \\
&\quad \leq \nu\left(\frac{a\left(s^{m+p} u\right)}{s^{m+p}} - \frac{a\left(s^m u\right)}{s^m}, \frac{t_0}{s^{m+p}} \sum_{n=0}^{p-1} s^{p-n-1} \zeta\left(s^{m+n} u, \ldots, s^{m+n} u, \right)\right) \\
&\quad \leq \epsilon,
\end{aligned}$$

for all $n \geq n_0$ and all $p > 0$. Hence, $\left\{\frac{a(s^m u)}{s^u}\right\}$ is Cauchy and it converges to $\mathcal{F}(u) \in \mathcal{G}$ as $\mathcal{G}$ is complete. Hence, we can describe a mapping $\mathcal{F} : \underbrace{\mathcal{L} \times \cdots \times \mathcal{L}}_{s \text{ times}} \longrightarrow \mathcal{G}$ such that $\mathcal{F}(u) = (\mu, \nu)$-$\lim_{m \to \infty} \frac{a(s^m u)}{s^m}$, namely, for each $t > 0$, and $u \in \mathcal{L}$,

$$\mu\left(\mathcal{F}(u) - \frac{a\left(s^m u\right)}{s^m}, t\right) = 1, \qquad \text{and} \qquad \nu\left(\mathcal{F}(u) - \frac{a\left(s^m u\right)}{s^m}, t\right) = 0.$$

Now, let $u_1, \ldots, u_s \in \mathcal{L}$. Select any fixed value of $t > 0$, and $0 < \epsilon < 1$.
Since $\lim_{m \to \infty} s^{-m} \zeta\left(s^m u, \ldots, s^m u\right) = 0$, there exists $m_1 > m_0$ such that $t_0 \zeta\left(s^m u, \ldots, s^m u\right) < \frac{s^m t}{s}$ for all $m \geq m_1$. Hence for each $m \geq m_1$, we have

$$\begin{aligned}
&\mu(\mathcal{D}a(u_1, \ldots, u_s), t) \\
&\quad \geq \mu\left(\mathcal{F}\left(\sum_{j=1}^s u_j\right) - \frac{a\left(s^m\left(\sum_{j=1}^s u_j\right)\right)}{s^m}, \frac{t}{4}\right) \\
&\quad\quad \star\, \mu\left(\mathcal{F}\left(\frac{1}{s} \sum_{j=1}^s u_j\right) - \frac{a\left(s^m\left(\frac{1}{s} \sum_{j=1}^s u_j\right)\right)}{s^m}, \frac{t}{4}\right) \\
&\quad\quad \star\, \mu\left(\frac{s+1}{s} \mathcal{F}\left(\sum_{j=1}^s u_j\right) - \left(\frac{s+1}{s}\right) \frac{a\left(s^m\left(\sum_{j=1}^s u_j\right)\right)}{s^m}, \frac{t}{4}\right) \\
&\quad\quad \star\, \mu\left(\mathcal{D}a(s^m u_1, \ldots, s^m u_s), \frac{s^m t}{4}\right),
\end{aligned} \tag{38}$$

and also

$$\mu\left(\mathcal{D}a(s^m u_1, \ldots, s^m u_s), \frac{s^m t}{4}\right)$$

$$\geq \mu\left(\mathcal{D}a(s^m u_1, \ldots, s^m u_s), t\zeta\left(s^m u_1, \ldots, s^m u_s\right)\right). \tag{39}$$

Allowing $m \to \infty$ in (38) and using (35), (39), we obtain

$$\mu(\mathcal{D}a(u_1, \ldots, u_s), t) \geq 1 - \epsilon$$

for all $t > 0$ and $0 < \epsilon < 1$. With the similar arguments, we obtain

$$\nu(\mathcal{DF}(u_1, \ldots, u_s), t) \leq \epsilon$$

for all $t > 0$ and $0 < \epsilon < 1$. It implies that $\mu(\mathcal{DF}(u_1, \ldots, u_s), t) = 1$, and $\nu(\mathcal{DF}(u_1, \ldots, u_s), t) = 0$, for all $t > 0$ and hence $\mathcal{F}$ satisfies (4).

Finally, assume that for some α, (34) is true, and

$$\zeta_m(u_1, \ldots, u_s) = \sum_{n=0}^{m-1} s^{-(n+1)}\zeta\left(s^n u_1, \ldots, s^n u_s\right),$$

for all $u_1, \ldots, u_s \in \mathcal{L}$. Also, using parallel arguments as in the beginning, one can derive from (34) that

$$\mu\left(a\left(s^m u\right) - s^m a(u), \sum_{n=0}^{m-1} s^{m-n-1}\zeta\left(s^n u, \ldots, s^n u\right)\right) \geq \alpha,$$

and (40)

$$\nu\left(a\left(s^m u\right) - s^m a(u), \sum_{n=0}^{m-1} s^{m-n-1}\zeta\left(s^n u, \ldots, s^n u\right)\right) \leq 1 - \alpha,$$

for all integers $m > 0$. For $y > 0$, we find that

$$\mu(a(u) - \mathcal{F}(u), \zeta_m(u, \ldots, u) + y)$$
$$\geq \mu\left(a(u) - \frac{a\left(s^m u\right)}{s^m}, \zeta_m(u, \ldots, u)\right) \star \mu\left(\frac{a\left(s^m u\right)}{s^m} - \mathcal{F}(u), y\right),$$

and

$$\nu(a(u) - \mathcal{F}(u), \zeta_m(u, \ldots, u) + y)$$
$$\leq \nu\left(a(u) - \frac{a\left(s^m u\right)}{s^m}, \zeta_m(u, \ldots, u)\right) \Diamond \nu\left(\frac{a\left(s^m u\right)}{s^m} - \mathcal{F}(u), y\right). \tag{41}$$

Combining (40) with (41), and using the fact that

$$\lim_{m\to\infty} \mu\left(\frac{a\left(s^m u\right)}{s^m} - \mathcal{F}(u), y\right) = 1,$$

and

$$\lim_{m\to\infty} \nu\left(\frac{a\left(s^m u\right)}{s^m} - \mathcal{F}(u), y\right) = 0,$$

we obtain

$$\mu(a(u) - \mathcal{F}(u), \zeta_m(u, \ldots, u) + y) \geq \alpha,$$

and

$$\nu(a(u) - \mathcal{F}(u), \zeta_m(u, \ldots, u) + y) \leq 1 - \alpha,$$

for sufficiently large m. In view of the continuity of functions

$$\mu(a(u) - \mathcal{F}(u), \cdot), \qquad \text{and} \qquad \nu(a(u) - \mathcal{F}(u), \cdot),$$

we find that

$$\mu\left(a(u)-\mathcal{F}(u), \tilde{\zeta}(u,\ldots,u)+y\right) \geq \alpha,$$

and

$$v\left(a(u)-\mathcal{F}(u), \tilde{\zeta}(u,\ldots,u)+y\right) \leq 1-\alpha.$$

By allowing the limit $y \to \infty$, one can obtain that

$$\mu\left(a(u)-\mathcal{F}(u), \tilde{\zeta}(u,\ldots,u)\right) \geq \alpha,$$

and

$$v\left(a(u)-\mathcal{F}(u), \tilde{\zeta}(u,\ldots,u)\right) \leq 1-\alpha.$$

This concludes the theorem. □

We solve the stability problem with an upper bound as $\epsilon > 0$ in the following corollary.

Corollary 4.2 Suppose the mapping a satisfies the following

$$\lim_{t\to\infty} \mu(\mathcal{D}a(x_1,\ldots,x_s), t\epsilon) = 1, \quad \text{and} \quad \lim_{t\to\infty} \nu(\mathcal{D}a(u_1,\ldots,u_s), t\epsilon) = 0 \tag{42}$$

for all $u_1,\ldots,u_s \in \mathcal{L}$. Then for each $u \in \mathcal{L}$, there exists an additive mapping $\mathcal{F} : \underbrace{\mathcal{L}\times\cdots\times\mathcal{L}}_{s\text{ times}} \longrightarrow \mathcal{G}$ which is unique and described by $\mathcal{F}(u) = (\mu,\nu)\text{-}\lim_{m\to\infty} \frac{a(s^m u)}{s^m}$ with the condition that if for some $\alpha > 0$ and all $u_1,\ldots,u_s \in \mathcal{L}$,

$$\mu(\mathcal{D}a(u_1,\ldots,u_s), \epsilon) > \alpha, \quad \text{and} \quad \nu(\mathcal{D}a(u_1,\ldots,u_s), \epsilon) < 1-\alpha, \tag{43}$$

then $\quad \mu\left(\mathcal{F}(u)-a(u), \frac{s\epsilon}{s-1}\right) > \alpha, \quad \text{and} \quad \nu\left(\mathcal{F}(u)-a(u), \frac{s\epsilon}{s-1}\right) < 1-\alpha.$

The following outcome is the dual of Theorem 4.1.

Theorem 4.3 Let $\zeta : \underbrace{\mathcal{L}\times\cdots\times\mathcal{L}}_{s\text{ times}} \longrightarrow [0,\infty)$ be a mapping satisfying (32). Let the mapping a satisfying (33) with reference to ζ. Then there is a unique additive mapping $\mathcal{F} : \underbrace{\mathcal{L}\times\cdots\times\mathcal{L}}_{s\text{ times}} \longrightarrow \mathcal{G}$ such that

$$\lim_{t\to\infty} \mu\left(a(u)-\mathcal{F}(u), t\tilde{\zeta}\left(\frac{u}{s},\ldots,\frac{u}{s}\right)\right) = 1,$$

and (44)

$$\lim_{t\to\infty} \nu\left(a(u)-\mathcal{F}(u), t\tilde{\zeta}\left(\frac{u}{s},\ldots,\frac{u}{s}\right)\right) = 0$$

uniformly in $\mathcal{L}$.

Proof. The limit (44) exists from Theorem 4.1. Now, it remains to show that $\mathcal{F}$ is unique. For this let us presume that $\mathcal{H}$ is some other additive mapping satisfying (44). Select $t > 0$. Then, for a given $\epsilon > 0$, there exists some $t_0 > 0$ so that from (44), we find that

$$\mu\left(a(u)-\mathcal{F}(u), t\tilde{\zeta}\left(\frac{u}{s},\ldots,\frac{u}{s}\right)\right) \geq 1-\epsilon, \quad \text{and} \quad \nu\left(a(u)-\mathcal{F}(u), t\tilde{\zeta}\left(\frac{u}{s},\ldots,\frac{u}{s}\right)\right) \leq \epsilon$$

for all $u \in \mathcal{L}$, and all $t \geq t_0$. For some $u \in \mathcal{L}$, we can obtain some integer m_0 so that

$$t_0 \sum_{n=m}^{\infty} s^n \zeta\left(\frac{u}{s^n},\ldots,\frac{u}{s^n}\right) < \frac{t}{2},$$

for all $m \geq m_0$. Since

$$\sum_{n=m}^{\infty} s^n \zeta\left(\frac{u}{s^n},\ldots,\frac{u}{s^n}\right) = s^m \sum_{n=m}^{\infty} s^{n-m} \zeta\left(\frac{u}{s^m},\ldots,\frac{u}{s^m}\right) = s^m \tilde{\zeta}\left(\frac{u}{s^m},\ldots,\frac{u}{s^m}\right),$$

we have

$$\begin{aligned}
\mu(a(u)-\mathcal{F}(u),t) &\geq \mu\left(s^m a\left(\frac{u}{s^m}\right)-\mathcal{F}(u),\frac{t}{2}\right)\star\mu\left(\mathcal{F}(u)-s^m a\left(\frac{u}{s^m}\right),\frac{t}{2}\right)\\
&=\mu\left(a\left(\frac{u}{s^m}\right)-\mathcal{F}\left(\frac{u}{s^m}\right),s^{m-1}t\right)\star\mu\left(\mathcal{F}\left(\frac{u}{s^m}\right)-a\left(\frac{u}{s^m}\right),s^{m-1}t\right)\\
&\geq \mu\left(a\left(\frac{u}{s^m}\right)-\mathcal{F}\left(\frac{u}{s^m}\right),s^m t_0\sum_{n=m}^{\infty}s^n\zeta\left(\frac{u}{s^n},\dots,\frac{u}{s^n}\right)\right)\\
&\qquad\star\mu\left(\mathcal{F}\left(\frac{u}{s^m}\right)-a\left(\frac{u}{s^m}\right),s^m t_0\sum_{n=m}^{\infty}s^n\zeta\left(\frac{u}{s^n},\dots,\frac{u}{s^n}\right)\right)\\
&=\mu\left(a\left(\frac{u}{s^m}\right)-\mathcal{F}\left(\frac{u}{s^m}\right),t_0\tilde{\zeta}\left(\frac{u}{s^m},\dots,\frac{u}{s^m}\right)\right)\\
&\qquad\star\mu\left(\mathcal{F}\left(\frac{u}{s^m}\right)-a\left(\frac{u}{s^m}\right),t_0\tilde{\zeta}\left(\frac{u}{s^m},\dots,\frac{u}{s^m}\right)\right)\geq 1-\epsilon,
\end{aligned}$$

and similarly

$$\begin{aligned}
\nu(\mathcal{F}(u)-\mathcal{H}(u),t) &\leq \nu\left(\frac{a\left(s^m u\right)}{s^m}-\mathcal{F}(u),\frac{t}{2}\right)\Diamond\nu\left(\mathcal{H}(u)-\frac{a\left(s^m u\right)}{s^m},\frac{t}{2}\right)\\
&\leq \nu\left(a\left(\frac{u}{s^m}\right)-\mathcal{F}\left(\frac{u}{s^m}\right),\frac{1}{s^m}t_0\sum_{n=m}^{\infty}s^n\zeta\left(\frac{u}{s^n},\dots,\frac{u}{s^n}\right)\right)\\
&\qquad\Diamond\nu\left(\mathcal{H}\left(\frac{u}{s^m}\right)-a\left(\frac{u}{s^m}\right),\frac{1}{s^m}t_0\sum_{n=m}^{\infty}s^n\zeta\left(\frac{u}{s^n},\dots,\frac{u}{s^n}\right)\right)\\
&=\nu\left(a\left(\frac{u}{s^m}\right)-\mathcal{F}\left(\frac{u}{s^m}\right),t_0\tilde{\zeta}\left(\frac{u}{s^m},\dots,\frac{u}{s^m}\right)\right)\\
&\qquad\Diamond\nu\left(\mathcal{H}\left(\frac{u}{s^m}\right)-a\left(\frac{u}{s^m}\right),t_0\tilde{\zeta}\left(\frac{u}{s^m},\dots,\frac{u}{s^m}\right)\right)\leq\epsilon.
\end{aligned}$$

It implies that $\mu(\mathcal{F}(u)-\mathcal{H}(u),t)=1$ and $\nu(\mathcal{F}(u)-\mathcal{H}(u),t)=0$ for all $t>0$. Consequently, we have $\mathcal{F}(u)=\mathcal{H}(u)$ for all $u\in\mathcal{L}$, which concludes the proof. □

The following corollary concerns with the T. Rassias stability taking the upper bound as the sum of exponents of several variables.

Corollary 4.4 Let the mapping a be such that

$$\lim_{t\to\infty}\mu\left(\mathcal{D}a(u_1,\dots,u_s),t\epsilon\left(\sum_{j=1}^{s}\|u_j\|^r\right)\right)=1,$$

uniformly in $\underbrace{\mathcal{L}\times\cdots\times\mathcal{L}}_{s\text{ times}}$, for all $\epsilon\geq 0$, $0\leq r\leq 1$. Then a mapping $\mathcal{F}:\underbrace{\mathcal{L}\times\cdots\times\mathcal{L}}_{s\text{ times}}\longrightarrow\mathcal{G}$ exists which is additive and unique with

$$\lim_{t\to\infty}\mu\left(\mathcal{F}(u)-a(u),\frac{t\epsilon\,\|u\|^r}{|1-s^{r-1}|}\right)=1,\quad\text{and}\quad\lim_{t\to\infty}\nu\left(\mathcal{F}(u)-a(u),\frac{t\epsilon\,\|u\|\,|^r}{|1-s^{r-1}|}\right)=0,$$

uniformly in $\mathcal{L}$.

Proof. The required outcome is achieved through Theorems 4.1, and 4.3 by plugging $\zeta(u_1,\dots,u_s)$ $=\epsilon\left(\sum_{j=1}^{s}\|u_j\|^r\right)$ for some $\theta>0$. □

§5 Counter-examples

Here, we will prove that equation (4) does not hold stability when $r = 1$ in Corollary 3.3. Let a function $\chi : \mathbb{R} \longrightarrow \mathbb{R}$ be defined via

$$\chi(x) = \begin{cases} cu, & \text{for} \quad -1 < x < 1 \\ c, & \text{for} \quad \text{otherwise.} \end{cases} \tag{45}$$

Then the function χ sets out as an appropriate example to show that (4) is not stable for the singularity $r = 1$ in the succeeding result.

Theorem 5.1 Assume a mapping $a : \mathbb{R} \longrightarrow \mathbb{R}$ is defined as

$$a(u) = \sum_{k=0}^{\infty} s^{-k} \chi(s^k u) \tag{46}$$

for all $u \in \mathbb{R}$. Let the mapping a defined in (46) satisfy

$$|\mathcal{D}a(u_1, \ldots, u_s)| \leq \frac{2(3s-1)}{s-1} \left(\sum_{j=1}^{s} |u_j| \right) \tag{47}$$

for all $u_1, \ldots, u_s \in \mathbb{R}$. Then an additive mapping $\mathcal{F} : \mathbb{R} \longrightarrow \mathbb{R}$ does not exist with a constant $\mu > 0$ with the condition that

$$|a(u) - \mathcal{F}(u)| \leq \mu\, |u| \tag{48}$$

for all $u \in \mathbb{R}$.

Proof. First of all, we shall prove that (47) is satisfied by a. By the definition of (45), we have

$$|a(u)| = \left| \sum_{k=0}^{\infty} s^{-k} \chi(s^k u) \right| \leq \sum_{k=0}^{\infty} \frac{c}{s^k} = \frac{s}{s-1} c.$$

The preceding inequality signifies that a is bounded by $\dfrac{sc}{s-1}$ on $\mathbb{R}$. Suppose $\sum_{j=0}^{s} |u_j| \geq 1$. Then, it is obvious that the expression on the left hand side of (47) is less than $\dfrac{2(3s-1)}{s-1}$. Now, suppose that $0 < \sum_{j=0}^{s} |u_j| < 1$. Therefore, there exists a $k \in \mathbb{Z}^+$ so that

$$\frac{1}{s^{k+1}} \leq \sum_{j=0}^{s} |u_j| < \frac{1}{s^k}. \tag{49}$$

The inequality (49) yields $p^k \left(\sum_{j=0}^{s} |u_j| \right) < 1$, or equivalently; $s^k u_j < 1$, for $j = 1, 2, \ldots, s$. Hence, from the last inequalities, we deduce that $s^{k-1}(u_j) < 1,\ j = 1, 2, \ldots, s$; $s^{k-1} \left(\sum_{j=1}^{s} u_j \right)$ < 1, $s^{k-1} \left(\frac{1}{s} \sum_{j=1}^{s} u_j \right) < 1$. Therefore, for every value of $k = 0, 1, 2, \ldots, n-1$, we obtain $s^n(u_j) < 1$, for $j = 1, 2, \ldots, s$; $s^n(\sum_{j=1}^{s} u_j) < 1$, $s^n \left(\frac{1}{s} \sum_{j=1}^{s} u_j \right) < 1$, and $\mathcal{D}a(s^n u_1, \ldots, s^n u_j) = 0$ for $k = 0, 1, 2, \ldots, n-1$. By virtue of (49) and using the definition of the mapping a, we find that

$$\frac{|\mathcal{D}a(u_1, \ldots, u_s)|}{\left(\sum_{j=1}^{s} u_j \right)} \leq \frac{2(3s-1)c}{s-1}$$

for all $u_1, \ldots, u_s \in \mathbb{R}$. From the above inequality, we infer that (47) is satisfied. Now, let us claim that (4) is not stable when $r = 1$ in Corollary 3.3. Suppose that there exists an additive mapping $\mathcal{F} : \mathbb{R} \longrightarrow \mathbb{R}$ satisfying (47). So, we obtain

$$|a(u)| \leq (|c| + 1)|u|. \tag{50}$$

On the other side, we can observe a positive integer k exists with $k+1 > |c|+1$. If $u \in \left(1, s^{-k}\right)$, then $s^n u \in (0, 1)$ for all $k = 0, 1, 2, \ldots, n-1$ and hence, we have

$$|a(u)| = \sum_{k=0}^{\infty} s^{-k}\chi(s^k u) \geq \sum_{k=0}^{n-1} s^{-k}\left(s^k u\right) = (k+1)u > (|c| + 1)u$$

which contradicts (50). Hence, equation (4) fails to be stable for $r = 1$ in Corollary 3.3. □

Remark 5.2. The function defined in (45) can be used to prove the stability result of equation (5) fails for a critical case.

§6 Discussion on the comparison of stability results

In the present section, a comparison study of the results pertaining to equation (4) obtained in fuzzy Banach spaces and intuitionistic fuzzy Banach spaces is discussed. The following table shows that the stabilities bounded above by the sum of exponents of norms pertaining to equation (4) is presented.

Corollary No.	**Fuzzy space**	**Result obtained**
Corollary 3.3	Fuzzy Banach space	$H\left(a(u) - \mathcal{F}(u), \frac{s^r \epsilon}{\lvert 1 - s^{r-1}\rvert}\right) \geq \alpha$
Corollary 4.4	Intuitionistic fuzzy Banach space	$\mu\left(\mathcal{F}(u) - a(u), \frac{\epsilon}{\lvert 1 - s^{r-1}\rvert}\right) > \alpha,$ $\nu\left(\mathcal{F}(u) - a(u), \frac{\epsilon}{\lvert 1 - s^{r-1}\rvert}\right) < 1 - \alpha$

The above table indicates that the approximate solution $\mathcal{F}(u)$ is very close to the exact solution $a(u)$ under intuitionistic fuzzy Banach space. Also, the stabilities pertaining to equation (4) with $\epsilon > 0$ as an upper bound are proved in Corollaries 3.2 and 4.2.

§7 Discussion and comparison of the results obtained with previous results

Here, we compare the results of the equation (4) obtained in this study with the previous results obtained by various other researchers. The results obtained in these results are better than the stability results proved in [1, 6]. We have obtained various stabilities of equations (4) and (5) in fuzzy setting and intuitionistic fuzzy setting. Employing various advanced fuzzy concepts, the stabilities are investigated in different fuzzy spaces.

§8 Conclusion

The inspiration initiated to study equations (4) and (5) is due to the fascinating roles of difference equations in various fields. Using the definitions of different fuzzy settings, the fuzzy approximations of equations (4) and (5) are achieved. The outcomes obtained in Section 3 involve only a membership function H, whereas in the outcomes of Section 4, both membership function μ and non-membership function ν are involved. An appropriate example is illustrated that the stability result fails for a critical case. The intuitionistic fuzzy normed space is a strong tool to model imprecision in real-life situations. The significant applications of the advanced fuzzy settings in real-life decision-making problems involving uncertainities could be dealt with the stability results of (4) and (5). These results can be validated in situations where vague or uncertain data are involved.

Declarations

Conflict of interest The authors declare no conflict of interest.

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