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Investigation on the new exact solutions of generalized Rosenau-Kawahara-RLW equation with p-th order nonlinearity occurring in ocean engineering models

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Abstract. The main objective of this study is to find novel wave solutions for the time-fractional generalized Rosenau-Kawahara-RLW equation, which occurs in unidirectional water wave propagation. The generalized Rosenau-Kawahara-RLW equation comprises three equations Rosenau equation, Kawahara equation, RLW equation and also p-th order nonlinear term. All these equations describe the wave phenomena especially the wave-wave and wave-wall interactions in shallow and narrow channel waters. The auxiliary equation method is employed to get the analytical results.

§1 Introduction

Explaining the nonlinear events in the nature has a great importance [29-34]. To understand the behavior and physical features of the considered natural event, scientists set up the mathematical models for this event [35-39]. It is understood that the best way to model the considered event is to establish a differential equation to express the development of the event. The differential equation arises after a great number of experimental data. At the beginning, scientists used integer order derivation to establish a model for regarded engineering or physical event. But it is seen that the integer order differential equations are inadequate and do not correspond to the experimental data completely. Hence fractional calculus arouse as an alternative way to state the chaotic, nonlinear and complex real world problems. The importance of fractional calculus has been proved to be very useful in different areas such as continuum mechanics, signal analysis, quantum mechanics and etc. Of course, it is not enough to set up models by using fractional derivatives and it is also important to produce solutions to these models. To deal with these sorts of models, analytical, numerical, and fractional integral transformations can be employed. For instance, Esen and Tasbozan [1] used quadratic B-Spline Galerkin method to express the approximate solutions of time fractional Burgers' equation in Caputo sense. Senol et al. [2] compared the RPSM and PIA for numerical solutions of fractional Rosenau-Hyman equation in Caputo sense. Owolabi considered [3] the stability analysis

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and numerical treatment of chaotic time-fractional differential equations in Riemann-Liouville sense. Celik and Duman [4] applied the Crank-Nicolson numerical scheme to get the approximate results for fractional diffusion equation in the sense of Riesz derivative. It is clearly seen that only approximate solutions of fractional differential equations (FDES) can be obtained for above linked references. But it has a great importance to acquire the exact solutions of nonlinear fractional partial differential equations (NFPDEs) to examine the efficiency and accuracy of the regarded numerical methods. In many scientific papers authors compared the approximate solutions with integer order analytical results. But it is not a true method from the scientific perspective. This is because both Riemann-Liouville, Caputo and Riesz fractional derivatives are not suitable to employ the known analytical methods. Also these derivatives do not hold some basic useful properties that are frequently used. For instance,

- If l is not a natural number, then the Riemann-Liouville formulation fails to satisfy $D^l 1 = 0$.
- The function is assumed to be differentiable in Caputo definition.
- Both formulations fail to satisfy the derivative of a product of two functions.
- The derivative of the quotient of two functions is not described in either definition.
- The chain rule is not satisfied by either definition.
- The index rule is not satisfied by either definition.

Conformable fractional derivative and integral which overcome the above mentioned deficiencies were expressed by Khalil et al. [5].

Definition 1. *Let $f : [0, \infty) \to R$ is a function $l-th$ order "conformable fractional derivative" of a defined by*

$$T_l(f)(t) = \lim_{\varepsilon \to 0} \frac{f(t + \varepsilon t^{1-l}) - (f)(t)}{\varepsilon} \tag{1}$$

for all $t > 0$, $l \in (0, 1)$.

Definition 2. *The conformable integral starting from $a \geqslant 0$ is expressed in [5] as*

$$I_l^a(f)(s) = \int_a^s \frac{f(t)}{t^{1-l}} dt. \tag{2}$$

Conformable fractional partial derivative [18] is expressed as follows:

Definition 3. *Let f be a function with n variables such as $x_1, \cdots, x_n$ and the conformable partial derivatives of f of order $l \in (0, 1]$ in x_i is defined as follows*

$$\frac{d^l}{dx_i^l} f(x_1, \cdots, x_n) = \lim_{\varepsilon \to 0} \frac{f(x_1, \cdots, x_{i-1}, x_i + \varepsilon x_i^{1-l}, \cdots, x_n) - f(x_1, \cdots, x_n)}{\varepsilon}. \tag{3}$$

The following are some fundamental properties of conformable derivative definition [5].

Theorem 1. *Let $l \in (0, 1]$ and f, g functions are l-differentiable at point $t > 0$, then*

1. $T_l(mf + ng) = mT_l(f) + nT_l(g)$ for all $m, n \in \mathbb{R}$.

2. $T_l(t^p) = pt^{p-l}$ for all p.
3. $T_l(f.g) = fT_l(g) + gT_l(f)$.
4. $T_l(\frac{f}{g}) = \frac{gT_l(f)-fT_l(g)}{g^2}$.
5. $T_l(c) = 0$, where c is a constant.
6. Furthermore, if f is differentiable, then $T_l(f)(t) = t^{1-l}\frac{df(t)}{dt}$

Scientists paid great attention to conformable fractional derivative and integral due to its applicability, lucidness and effectiveness. For example, Wang and Liu [24] obtained the analytical solutions of Jimbo-Miwa equation and fractional Zakharov-Kuznetsov equation with the aid of a complete discrimination system of polynomial method. Khater et al. [7] handled a modified auxiliary equation method to get the exact results for some fractional biological models. Senol et al. [8] used residual power series method for obtaining approximate solutions of Burgers' type equations. Tasbozan et al. [9] obtained both numerical and analytical results for conformable fractional Drinfeld-Sokolov-Wilson equation. Huang and Yang [10] obtained the exact solutions to local conformable time-fractional viscous Burgers system in their studies. Akinyemi et al. in [11] conformable derivative to studied integrable generalized $(2+1)$-dimensional nonlinear conformable Schrödinger system of equations. Alharbi et al. [12] used the conformable derivative to investigate the projectile motion in a resisting medium. The advantage of conformable fractional derivative definitions over traditional fractional derivative definitions allows us to obtain analytical solutions for NPDE. Other fractional derivative definitions, for example, cannot be used to obtain analytical solutions to the time fractional generalized Rosenau-Kawahara-RLW problem. Because they don't follow the chain rule. We describe conformable fractional derivatives and integrals in this paper and explore some of their properties. The generalized Rosenau-Kawahara-RLW (GRK-RLW) equation's evolution is then discussed, followed by a brief description of the auxiliary equation technique. We provide one example that proves the method's dependability and efficiency. Figures depicting the different values of l as well as the parameters in the solutions are also provided. These solutions have never been published or reported in the literature, to the best of our knowledge.

§2 Governing equation

Let us now express the progression of the GRK-RLW equation. Many scholars describe many mathematical models to explain the wave behavior that happens in shallow waters in restricted channels such as canals [17]. For instance, KdV equation [19] has been used as a model of fluid waves, ion sound waves, and longitudinal astigmatic waves [20]. But it is understood that KdV equation can not define the wave-wave and wave-wall interactions in the context of dynamics of compact discrete systems [21]. To annihilate this deficiency Rosenau stated the Rosenau equation [22]

$$u_t + u_x + uu_x + u_{xxxxt} = 0. \tag{4}$$

u_{xxt} viscous term is included in Rosenau equation for more explanation on nonlinear waves. The newly obtained equation is called Rosenau-RLW equation [23]

$$u_t + u_x + uu_x - u_{xxt} + u_{xxxxt} = 0. \tag{5}$$

After then a term with p-th order nonlinearity is added to Rosenau-RLW equation so named generalized Rosenau-RLW equation [24]

$$u_t + u_x + (u^p)_x - u_{xxt} + u_{xxxxt} = 0, \;\; p \geq 2. \tag{6}$$

By the unification of KdV and Rosenau equation, the Rosenau-KdV equation is obtained [25]

$$u_t + u_x + uu_x + u_{xxx} + u_{xxxxt} = 0. \tag{7}$$

Again adding the p-th order nonlinear term Equation 7 is called as generalized Rosenau-KdV equation [26]

$$u_t + u_x + (u^p)_x + u_{xxx} + u_{xxxxt} = 0, \;\; p \geq 2. \tag{8}$$

By combining the Rosenau-RLW equation and the generalized Rosenau-KdV equation, one may obtain the generalized Rosenau-KdV-RLW equation defined as [27]

$$u_t + u_x + (u^p)_x - u_{xxt} + u_{xxx} + u_{xxxxt} = 0, \;\; p \geq 2. \tag{9}$$

The Kawahara equation [28]

$$u_t + u_x + uu_x + u_{xxx} - u_{xxxxx} = 0, \tag{10}$$

arises in the theory of studying shallow water waves with surface tension. By attaching the linear viscous term u_{xxxxx} to Rosenau-KdV equation Rosenau-Kawahara equation comes to light [29]

$$u_t + u_x + uu_x + u_{xxx} + u_{xxxxt} - u_{xxxxx} = 0. \tag{11}$$

Again adding the p-th order nonlinear term to Equation 11 generalized Rosenau-Kawahara equation can be expressed [30]

$$u_t + au_x + b(u^p)_x + cu_{xxx} + \lambda u_{xxxxt} - \nu\, u_{xxxxx} = 0, \;\; p \geq 2. \tag{12}$$

with power law nonlinearity.

In this study, we consider the GRK-RLW equation, which is a time fractional higher order equation for unidirectional water wave propagation as:

$$D_t^l u + \alpha\, D_x u + \beta(u^p) D_x u - \gamma\, D_{xx} D_t^l u + \epsilon D_{xxx} u + \lambda\, D_{xxxx} D_t^l u + \mu\, D_{xxxxx} u = 0, \;\; p \geq 2. \tag{13}$$

The fractional derivatives are defined in this context in a conformable sense. The Rosenau-RLW equation and the generalized Rosenau-Kawahara equation are combined to form this equation. In Equation 13 $u(x,t)$ represents a nonlinear wave profile while α, β, γ, λ, ε, are μ are arbitrary parameters.

§3 A short overview of the AEM

The AEM [13, 14] is an efficient and reliable method which is used for obtaining the exact solutions for PDEs. The approach is based on an auxiliary differential equation

$$\left(\frac{dz}{d\zeta}\right)^2 = az^2(\zeta) + bz^3(\zeta) + cz^4(\zeta). \tag{14}$$

where the solutions of these equations are expressed in the following table for changing values of a, b, c and Δ. The solution procedure can be described as follows. Considering the general form of nonlinear conformable fractional differential equation

$$P\left(\frac{\partial^l u}{\partial t^l}, \frac{\partial u}{\partial x}, \frac{\partial^{2l} u}{\partial t^{2l}}, \frac{\partial^2 u}{\partial x^2}, \cdots\right) = 0. \tag{15}$$

where $\frac{\partial^{2l} u}{\partial t^{2l}}$ indicates two times conformable derivative of function $u(x, t)$. Applying the wave transform $\zeta = x + w\frac{t^l}{l}$ where w indicates the velocity of the wave and the chain rule for conformable fractional derivatives [15], Equation 15 can be rearranged as

$$G(U, U', U'', U''', \cdot) = 0, \tag{16}$$

where the prime shows integer order derivatives with respect to ζ. Suppose that $U(\zeta)$ can be written as the sum of the following finite serial

$$U(\zeta) = \sum_{i=0}^{n} a_i z^i(\zeta). \tag{17}$$

Here, the solution of Equation 14 is defined as $z(\zeta)$. The variables a_i, w, a, b, and c, are real constants to be examined thereafter and positive integer n can be evaluated with the aid of the balancing principle [16]. Substituting Equation 17 and Equation 14 into Equation 16 an algebraic equations is aroused that includes the powers of $z(\zeta)$ and variables a_i, w, a, b, and $, c$. Collecting all the powers of $z(\zeta)$ together and equating all the coefficients of $z(\zeta)$ led to an algebraic equations system including a_i, w, , a, b, and c. So the unknown parameters can be evaluated by solving the algebraic equation system.

Table 1. Solutions of Equation 14 with $\Delta = b^2 - 4ac$ and $\varepsilon = \pm 1$.

No	$z(\zeta)$	
1	$\frac{-ab\text{sech}^2(\frac{\sqrt{a}}{2}\zeta)}{b^2-ac(1+\varepsilon\tanh(\frac{\sqrt{a}}{2}\zeta))^2}$,	$a > 0$
2	$\frac{ab\text{csch}^2(\frac{\sqrt{a}}{2}\zeta)}{b^2-ac(1+\varepsilon\coth(\frac{\sqrt{a}}{2}\zeta))^2}$,	$a > 0$
3	$\frac{2a\text{sech}(\sqrt{a}\zeta)}{\varepsilon\sqrt{\Delta}-b\text{sech}(\sqrt{a}\zeta)}$,	$a > 0$, $\Delta > 0$
4	$\frac{2a\sec(\sqrt{-a}\zeta)}{\varepsilon\sqrt{\Delta}-b\sec(\sqrt{-a}\zeta)}$,	$a < 0$, $\Delta > 0$
5	$\frac{2a\text{csch}(\sqrt{a}\zeta)}{\varepsilon\sqrt{-\Delta}-b\text{csch}(\sqrt{a}\zeta)}$,	$a > 0$, $\Delta < 0$
6	$\frac{2a\csc(\sqrt{-a}\zeta)}{\varepsilon\sqrt{\Delta}-b\csc(\sqrt{-a}\zeta)}$,	$a < 0$, $\Delta > 0$
7	$\frac{-a\text{sech}^2(\frac{\sqrt{a}}{2}\zeta)}{b+2\varepsilon\sqrt{ac}\tanh(\frac{\sqrt{a}}{2}\zeta)}$,	$a > 0$, $c > 0$
8	$\frac{-a\sec^2(\frac{\sqrt{-a}}{2}\zeta)}{b+2\varepsilon\sqrt{-ac}\tan(\frac{\sqrt{-a}}{2}\zeta)}$,	$a < 0$, $c > 0$
9	$\frac{a\text{csch}^2(\frac{\sqrt{a}}{2}\zeta)}{b+2\varepsilon\sqrt{ac}\coth(\frac{\sqrt{a}}{2}\zeta)}$,	$a > 0$, $c > 0$
10	$\frac{-a\csc^2(\frac{\sqrt{-a}}{2}\zeta)}{b+2\varepsilon\sqrt{-ac}\cot(\frac{\sqrt{-a}}{2}\zeta)}$,	$a < 0$, $c > 0$
11	$-\frac{a}{b}[1+\varepsilon\tanh(\frac{\sqrt{a}}{2}\zeta)]$,	$a > 0$, $\Delta = 0$
12	$-\frac{a}{b}[1+\varepsilon\coth(\frac{\sqrt{a}}{2}\zeta)]$,	$a > 0$, $\Delta = 0$
13	$\frac{4ae^{\varepsilon\sqrt{a}\zeta}}{(e^{\varepsilon\sqrt{a}\zeta}-b)^2-4ac}$,	$a > 0$
14	$\frac{\pm 4ae^{\varepsilon\sqrt{a}\zeta}}{1-4ac\,e^{2\varepsilon\sqrt{a}\zeta}}$,	$a > 0$, $b = 0$

§4 Implementation of the proposed method

Consider the time-fractional generalized Rosenau-Kawahara-RLW equation. Using the wave transform $\zeta = x + w\frac{t^l}{l}$ and chain rule for conformable fractional derivatives [15] the time fractional generalized Rosenau-Kawahara-RLW equation turns into a nonlinear integer order differential equation

$$(\epsilon - \gamma * w)U^{''} + (\lambda w + \mu)U^{(iv)} + \frac{\beta\, U^{p+1}}{p+1} + (w+\alpha)U = 0. \tag{18}$$

Changing the independent variable as $U(\zeta) = V(\zeta)^{(1/p)}$ in Equation 18 yields:

$$\begin{aligned} & p^4V^4\left((1+p)(w+\alpha) + \beta\left(V^{\frac{1}{p}}\right)^p\right) - (-1+p)(1+p)(-1+2p)(-1+3p)(w\lambda+\mu)(V')^4 \\ & + 6(-1+p)p(1+p)(-1+2p)(w\lambda+\mu)V(V^{'})^2V'' \\ & - p^3(1+p)V^3\left((w\gamma-\epsilon)V'' - (w\lambda+\mu)V^{(iv)}\right) \\ & + p^2\left(-1+p^2\right)V^2\left((w\gamma-\epsilon)(V')^2 - 3(w\lambda+\mu)(V'')^2 - 4(w\lambda+\mu)V'V'''\right) = 0. \end{aligned} \tag{19}$$

Now assume that Equation 19 has the solution as the sum of the following finite serial

$$V(\zeta) = \sum_{i=0}^{n} a_i z^i(\zeta). \tag{20}$$

Using the balancing principle [16], we obtain $n = 2$. So the solution of the Equation 19 can be regarded as

$$V(\zeta) = a_0 + a_1 z(\zeta) + a_2 z(\zeta)^2. \tag{21}$$

Subrogating Equation 21 into Equation 19, collecting the powers of $z(\zeta)$ together and equating the coefficients to zero an algebraic equations system arises. Solving the equation system led to following solution set

$$\begin{aligned} w =& \frac{a_1p^2\beta + 2b\epsilon + 3bp\epsilon + bp^2\epsilon}{b(2+3p+p^2)\gamma}, \;\; c = \frac{a_2 b}{2a_1}, \;\; a_0 = 0, \\ a =& \frac{a_1 b}{2a_2}, \;\; \mu = -\frac{(a_1p^2\beta + 2b\epsilon + 3bp\epsilon + bp^2\epsilon)\lambda}{b(2+3p+p^2)\gamma}, \\ \alpha =& \frac{-2a_1a_2p^2\beta + a_1^2b\beta\gamma - 4a_2b\epsilon - 6a_2bp\epsilon - 2a_2bp^2\epsilon}{2a_2b(2+3p+p^2)\gamma}. \end{aligned} \tag{22}$$

Using the obtained solution set and with the assistance of Table 1, the new wave solutions of conformable fractional generalized Rosenau-Kawahara-RLW can be given as follows:

$$\begin{aligned} & u_1(x,t) = \\ & \left(\frac{a_1^2b^4\mathrm{sech}\left(\frac{\sqrt{\frac{a_1 b}{a_2}}}{2\sqrt{2}}\zeta\right)^4}{4a_2\left(b^2 - \frac{1}{4}b^2\left(1-\tanh\left(\frac{\sqrt{\frac{a_1 b}{a_2}}}{2\sqrt{2}}\zeta\right)\right)^2\right)^2} - \frac{a_1^2b^2\mathrm{sech}\left(\frac{\sqrt{\frac{a_1 b}{a_2}}}{2\sqrt{2}}\zeta\right)^2}{2a_2\left(b^2 - \frac{1}{4}b^2\left(1-\tanh\left(\frac{\sqrt{\frac{a_1 b}{a_2}}}{2\sqrt{2}}\zeta\right)\right)^2\right)}\right)^{\frac{1}{p}}, \end{aligned} \tag{23}$$

$$u_2(x,t) =$$

$$\left(\frac{a_1^2 b^2 \operatorname{csch}\left(\frac{\sqrt{\frac{a_1 b}{a_2}}}{2\sqrt{2}}\zeta\right)^2}{2a_2\left(b^2-\frac{1}{4}b^2\left(1+\coth\left(\frac{\sqrt{\frac{a_1 b}{a_2}}}{2\sqrt{2}}\zeta\right)\right)^2\right)}+\frac{a_1^2 b^4 \operatorname{csch}\left(\frac{\sqrt{\frac{a_1 b}{a_2}}}{2\sqrt{2}}\zeta\right)^4}{4a_2\left(b^2-\frac{1}{4}b^2\left(1+\coth\left(\frac{\sqrt{\frac{a_1 b}{a_2}}}{2\sqrt{2}}\zeta\right)\right)^2\right)^2}\right)^{\frac{1}{p}}, \tag{24}$$

$$u_3(x,t)=\left(\frac{a_1^2 b^2 \operatorname{sech}\left(\frac{\sqrt{\frac{a_1 b}{a_2}}}{2\sqrt{2}}\zeta\right)^4}{4a_2\left(b+\sqrt{b^2}\tanh\left(\frac{\sqrt{\frac{a_1 b}{a_2}}}{2\sqrt{2}}\zeta\right)\right)^2}-\frac{a_1^2 b \operatorname{sech}\left(\frac{\sqrt{\frac{a_1 b}{a_2}}}{2\sqrt{2}}\zeta\right)^2}{2a_2\left(b+\sqrt{b^2}\tanh\left(\frac{\sqrt{\frac{a_1 b}{a_2}}}{2\sqrt{2}}\zeta\right)\right)}\right)^{\frac{1}{p}}, \tag{25}$$

$$u_4(x,t)=\left(\frac{a_1^2 b \operatorname{csch}\left(\frac{\sqrt{\frac{a_1 b}{a_2}}}{2\sqrt{2}}\zeta\right)^2}{2a_2\left(b+\sqrt{b^2}\coth\left(\frac{\sqrt{\frac{a_1 b}{a_2}}}{2\sqrt{2}}\zeta\right)\right)}+\frac{a_1^2 b^2 \operatorname{csch}\left(\frac{\sqrt{\frac{a_1 b}{a_2}}}{2\sqrt{2}}\zeta\right)^4}{4a_2\left(b+\sqrt{b^2}\coth\left(\frac{\sqrt{\frac{a_1 b}{a_2}}}{2\sqrt{2}}\zeta\right)\right)^2}\right)^{\frac{1}{p}}, \tag{26}$$

$$u_5(x,t)=\left(-\frac{a_1^2\left(1+\tanh\left(\frac{\sqrt{\frac{a_1 b}{a_2}}}{2\sqrt{2}}\zeta\right)\right)}{2a_2}+\frac{a_1^2\left(1+\tanh\left(\frac{\sqrt{\frac{a_1 b}{a_2}}}{2\sqrt{2}}\zeta\right)\right)^2}{4a_2}\right)^{\frac{1}{p}}, \tag{27}$$

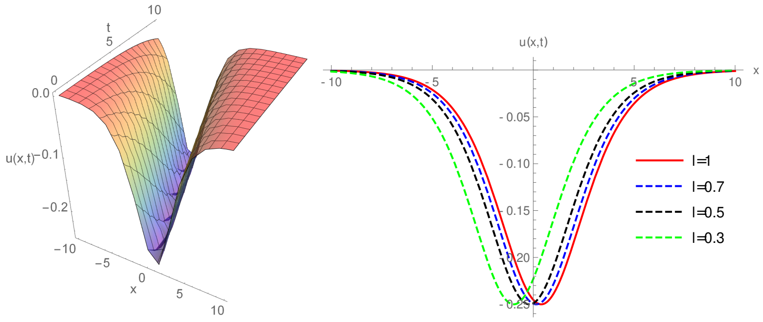
$$u_6(x,t)=\left(-\frac{a_1^2\left(1+\coth\left(\frac{\sqrt{\frac{a_1 b}{a_2}}}{2\sqrt{2}}\zeta\right)\right)}{2a_2}+\frac{a_1^2\left(1+\coth\left(\frac{\sqrt{\frac{a_1 b}{a_2}}}{2\sqrt{2}}\zeta\right)\right)^2}{4a_2}\right)^{\frac{1}{p}}, \tag{28}$$

$$u_7(x,t)=\left(\frac{4a_1^2 b^2 e^{\sqrt{2}\sqrt{\frac{a_1 b}{a_2}}\zeta}}{a_2\left(-b^2+\left(-b+e^{\frac{\sqrt{\frac{a_1 b}{a_2}}\zeta}{\sqrt{2}}}\right)^2\right)^2}+\frac{2a_1^2 b e^{\frac{\sqrt{\frac{a_1 b}{a_2}}\zeta}{\sqrt{2}}}}{a_2\left(-b^2+\left(-b+e^{\frac{\sqrt{\frac{a_1 b}{a_2}}\zeta}{\sqrt{2}}}\right)^2\right)}\right)^{\frac{1}{p}} \tag{29}$$

where

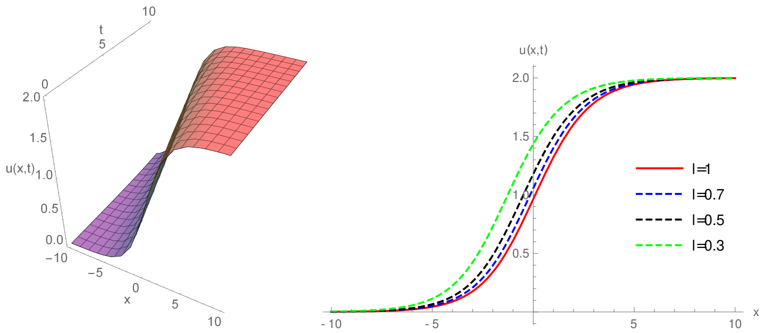
$$\zeta = x+\frac{t^l(a_1p^2\beta+2b\epsilon+3bp\epsilon+bp^2\epsilon)}{bl(2+3p+p^2)\gamma}. \tag{30}$$

Remark 1. *In the solutions presented above, we use $\varepsilon = 1$. Other solutions for $\varepsilon = -1$ can easily be found with the help of Table 1.*



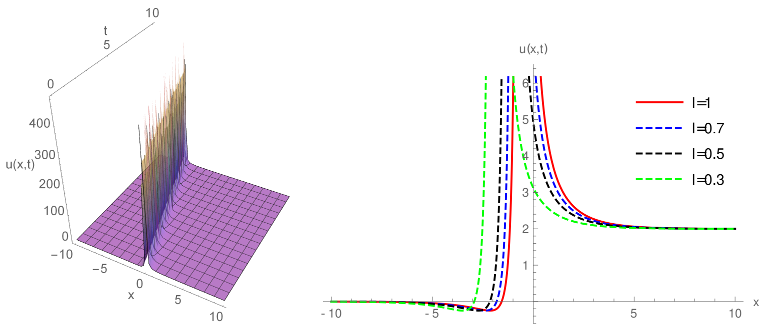
(a) (b)

Figure 1. Some plots of solution 23 (a) $l = 1$ and (b) different l when t=1 with parameters $p = a_1 = b = a_2 = \beta = 1$, $\epsilon = 1$, and $\gamma = 2$.



(a) (b)

Figure 2. Some plots of solution 27 (a) $l = 1$ and (b) different l when t=1 with parameters $p = a_1 = b = a_2 = \beta = 1$, $\epsilon = 1$, and $\gamma = 2$.



(a) (b)

Figure 3. Some plots of solution 28 (a) $l = 1$ and (b) different l when t=1 with parameters $p = a_1 = b = a_2 = \beta = 1$, $\epsilon = 1$, and $\gamma = 2$.

§5 Conclusion

In this study, we utilize the auxiliary equation method to get the new wave solutions of time fractional generalized Rosenau-Kawahara-RLW equation which involves both Rosenau equation, Kawahara equation and RLW equation and arising in the wave phenomena especially the wave-wave and wave-wall interactions in shallow and narrow channel waters. In Fig.1, Fig.2, and Fig.3 we depicted some solutions in 3D and also 2D for various values of l with parameters in the solutions. The considered solutions show that the considered method is efficient, accurate and reliable. The considered equation includes p-th order nonlinearity which makes the equation more general. This study can attract researchers to further studies that make investigations on ocean engineering and modelling.

Declarations

Conflict of interest The authors declare no conflict of interest.

References

[1] A Esen, O Tasbozan. *Numerical solution of time fractional Burgers equation*, Acta Universitatis Sapientiae Mathematica, 2015, 7(2): 167-185.

[2] M Senol, O Tasbozan, A Kurt. *Comparison of two reliable methods to solve fractional Rosenau-Hyman equation*, Math Meth Appl Sci, 2021, 44(10): 7904-7914.

[3] K M Owolabi. *Riemann-Liouville fractional derivative and application to model chaotic differential equations*, Prog Fract Differ Appl, 2018, 4: 99-110.

[4] C Celik, M Duman. *CrankNicolson method for the fractional diffusion equation with the Riesz fractional derivative*, J Computat Phys, 2012, 231(4): 1743-1750.

[5] R Khalil, M Al Horani, A Yousef, M Sababheh. *A new definition of fractional derivative*, J Comput Appl Math, 2014, 264: 65-70.

[6] X Wang, Y Liu. *All single travelling wave patterns to fractional Jimbo Miwa equation and Zakharov Kuznetsov equation*, Pramana, 2019, 92(3): 31.

[7] M Khater, R Attia, D Lu. *Modified Auxiliary Equation Method versus Three Nonlinear Fractional Biological Models in Present Explicit Wave Solutions*, Math Comput Appl, 2019, 24(1): 1.

[8] M Senol, O Tasbozan, A Kurt. *Numerical Solutions of Fractional Burgers Type Equations with Conformable Derivative*, Chin J Phys, 2019, 58: 75-84.

[9] O Tasbozan, M Senol, A Kurt, O Ozkan. *New solutions of fractional Drinfeld-Sokolov-Wilson system in shallow water waves*, Ocean Eng, 2018, 161: 62-68.

[10] S J Huang, N J Yang. *Auto-Bcklund transform and exact solutions to local conformable time-fractional viscous Burgers system*, Eur Lett, 2019, 125(1): 15003.

[11] L Akinyemi, M Senol, H Rezazadeh, H Ahmad, H Wang. *Abundant optical soliton solutions for an integrable* $(2+1)$*-dimensional nonlinear conformable Schrödinger system*, Res Phys, 2021, 25: 104177.

[12] F M Alharbi, D Baleanu, A Ebaid. *Physical properties of the projectile motion using the conformable derivative*, Chin J Phys, 2019, 58: 18-28.

[13] Sirendaoreji, S Jiong. *Auxiliary equation method for solving nonlinear partial differential equations*, Phys Lett A, 2003, 309(5-6): 387-396.

[14] L Akinyemi, H Rezazadeh, S W Yao, M A Akbar, M M Khater, A Jhangeer, M Inc, H Ahmad. *Nonlinear dispersion in parabolic law medium and its optical solitons*, Res Phys, 2021, 26, DOI: 10.1016/j.rinp.2021.104411.

[15] T Abdeljawad. *On conformable fractional calculus*, J Comput Appl Math, 2015, 279: 57-66.

[16] W Malfiet. *Solitary wave solutions of nonlinear wave equations*, Am J Phys, 1992, 6: 650-654.

[17] A Kurt. *New periodic wave solutions of a time fractional integrable shallow water equation*, Appl Ocean Res, 2019, 85: 128-135.

[18] A Atangana, D Baleanu, A Alsaedi. *New properties of conformable derivative*, Open Math, 2015, 13(1): 889-898.

[19] D Korteweg, G De Vries. *On the change of form of long waves advancing in a rectangular canal, and on a new type of long stationary waves*, Philos Mag, 2011, 91(6): 1007-1028.

[20] S Ozer, S Kutluay. *An analytical numerical method applied to Kortewegde Vries equation*, Appl Math Comput, 2005, 164: 789-797.

[21] J Hu, Y Wang. *A high accuracy linear conservative difference scheme for Rosenau-RLW equation*, Math Probl Eng, 2013, 2013(2), DOI: 10.1155/2013/870291.

[22] P Rosenau. *Dynamics of dense discrete systems*, Prog Theor Phys, 1988, 79: 1028-1042.

[23] X Pan, K Zheng, L Zhang. *Finite difference discretization of the Rosenau-RLW equation*, Appl Anal, 2013, 92(12): 2578-258.

[24] H Wang, S Li, J Wang. *A conservative weighted finite difference scheme for the generalized Rosenau-RLW equation*, Comput Appl Math, 2017, 36: 63-78.

[25] A Saha. *Topological 1-soliton solutions for the generalized Rosenau-KdV equation*, Fund J Math Phys, 2012, 2: 19-25.

[26] M Zheng, J Zhou. *An average linear difference scheme for the generalized Rosenau-KdV equation*, J Appl Math, 2014, 2014: 1-9.

[27] P Sanchez, G Ebadi, A Mojaver, M Mirzazadeh, M Eslami, A Biswas. *Solitons and other solutions to perturbed Rosenau-KdV-RLWequation with power law nonlinearity*, Acta Phys Pol A, 2015, 127: 1577-1586.

[28] T Kawahara. *Oscillatory solitary waves in dispersive media*, J Phys Soc Japan, 1972, 33: 260264.

[29] M Labidi, A Biswas. *Application of Hes principles to RosenauKawahara equation*, J MESA, 2011, 2: 183197.

[30] A Biswas, H Triki, M Labidi. *Bright and dark solitons of the RosenauKawahara equation with power law nonlinearity*, Phys Wave Phenom, 2011, 19: 2429.

[31] X Lü, H W Hui, F F Liu, Y L Bai. *Stability and optimal control strategies for a novel epidemic model of COVID-19*, Nonlinear Dyn, 2021, 106: 1491-1507.

[32] X Lü, S J Chen. *New general interaction solutions to the KPI equation via an optional decoupling condition approach*, Commun Nonlinear Sci Numer Simul, 2021, 103: 105939.

[33] M Z Yin, Q W Zhu, X Lü. *Parameter estimation of the incubation period of COVID-19 based on the doubly interval-censored data model*, Nonlinear Dyn, 2021, 106: 1347-1358.

[34] S J Chen, X Lü, M G Li, F Wang. *Derivation and simulation of the M-lump solutions to two (2+1)-dimensional nonlinear equations*, Physica Scripta, 2021, 96(9): 095201.

[35] Y H Yin, S J Chen, X Lü. *Localized characteristics of lump and interaction solutions to two extended Jimbo-Miwa equations*, Chin Phys B, 2020, 29(12): 120502.

[36] X Lü, S J Chen. *Interaction solutions to nonlinear partial differential equations via Hirota bilinear forms: One-lump-multi-stripe and one-lump-multi-soliton types*, Nonlinear Dyn, 2021, 103(1): 947-977.

[37] X Lü, Y F Hua, S J Chen, X F Tang. *Integrability characteristics of a novel (2+1)-dimensional nonlinear model: Painleve analysis, soliton solutions, Backlund transformation, Lax pair and infinitely many conservation laws*, Commun Nonlinear Sci Numer Simul, 2021, 95: 105612.

[38] X Lü, H W Hui, F F Liu, Y L Bai. *Stability and optimal control strategies for a novel epidemic model of COVID-19*, Nonlinear Dyn, 2021, 106: 1491-1507.

[39] X Lü, S J Chen. *New general interaction solutions to the KPI equation via an optional decoupling condition approach*, Commun Nonlinear Sci Numer Simul, 2020, 103: 105939.

[40] J W Xia, Y W Zhao, X Lü. *Predictability, fast calculation and simulation for the interaction solutions to the cylindrical Kadomtsev-Petviashvili equation*, Commun Nonlinear Sci Numer Simul, 2020, 90: 105260.

[41] H N Xu, W Y Ruan, Y Zhang, X Lü. *Multi-exponential wave solutions to two extended Jimbo-Miwa equations and the resonance behavior*, Appl Math Lett, 2020, 99: 105976.

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