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A convergent flow in warped product spaces

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Abstract. In this paper, we study a kind of curvature flow in warped product spaces. We obtain convergence results under barrier conditions and restrictions on prescribed function. We also obtain the asymptotic behavior of a kind of inverse curvature flow in Schwarzschild manifold.

§1 Introduction

Define $N = [a, b] \times E^n$ with warped product metric

$$\bar{g} = d\rho^2 + \phi(\rho)^2 g_0,$$

where (E^n, g_0) is a compact Riemannian manifold and $\phi > 0$ is a smooth convex function of ρ. Let M be a closed hypersurface in N and ν be the unit outer normal. As in [2], we call function $u = \langle \phi \frac{\partial}{\partial \rho}, \nu \rangle$ the generalized support function of the hypersurface. If N is replaced by Euclidean space $\mathbb{R}^{n+1}$, then u is the usual support function.

In this paper, we considered the prescribed curvature problem in N. The approach to the existence of solutions of the prescribed curvature problem is twofold: one is to study the very elliptic equation directly, c.f.[9, 10, 16, 12] and the reference therein, another is to solve the problem with the help of curvature flows in the smooth category, see [13, 3, 18, 1, 17], for examples.

In [13], Li, Sheng and Wang studied a contracting flow of closed, convex hypersurfaces in the Euclidean space and resolved the dual Minkowski problem by parabolic approach. In [3], Bryan et al. employed curvature flows to seek strictly convex, spacelike solutions of a broad class of prescribed curvature problems in simply connected Riemannian space forms and the Lorentzian de Sitter space, respectively, where the prescribed function may depend on the position and the normal vector. As for [17], Sheng and Yi considered a more general curvature flow with variational structure compared to the flow in [13]. In [2], Brendle et al. introduced an inverse curvature type hypersurface flow in space forms which is related to quermassintegrals in space forms. While in general warped product spaces, their method may fail due to that the

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second fundamental form is not a Codazzi tensor. Still, we can find a convergent flow without variational structure, which is inspired by [3].

Let $\kappa = (\kappa_1, \cdots, \kappa_n)$ be the vector of n principal curvatures of the hypersurface M and we denote the k-th elementary symmetric function of κ by $\sigma_k(\kappa)$. Consider the following parabolic evolution equation of a family of smooth embeddings $X : E^n \times [0, T) \to (N, \bar{g})$ satisfying

$$\partial_t X = \big(\frac{1}{F(\kappa)} - f(X, \nu)\big)\nu, \tag{1}$$

where X is the position vector, F is a curvature function defined by κ, ν is the unit outer normal, $f : TN \to \mathbb{R}^+$ is a smooth function defined on the tangent bundle of N. More precisely, let (x, ξ) be a point in TN, i.e. $x \in N$ and $\xi \in T_x N$, then f maps (x, ξ) to $\mathbb{R}^+$. In the following, we denote the second partial derivatives of f with respect to x(resp. ξ) as f_{xx}(resp. $f_{\xi\xi}$). Furthermore, assume F satisfies the following conditions:

- $F(\kappa_1, \kappa_2, \cdots, \kappa_n)$ is concave, monotonically increasing, 1-homogeneous symmetric function defined on the positive cone $\Gamma_+ := \{(x_1, x_2, \cdots, x_n) \in \mathbb{R}^n | x_i > 0, \forall i = 1, 2, \cdots, n\}$.
- $\sum_i \frac{\partial F}{\partial \kappa_i}\kappa_i^2 \le C_0 F \sigma_1$ for some constant C_0.
- $F|_{\partial\Gamma_+} = 0$.

Special curvature functions satisfying the above conditions include $\sigma_n^{\frac{1}{n}}$ and $\sigma_n^{\frac{1}{2n}}\sigma_k^{\frac{1}{2k}}$ for $1 \le k \le n-1$.

Gerhardt in [6] introduced the following barrier hypersurface to (1) as an analogue of subsolutions or supersolutions to partial differential equations. We say that a closed, strictly convex hypersurface $M \subset N$ is a lower barrier for the pair (F, f) if

$$\big(\frac{1}{F} - f(X, \nu)\big)\Big|_M \ge 0,$$

and an upper barrier if

$$\big(\frac{1}{F} - f(X, \nu)\big)\Big|_M \le 0.$$

To explain the barrier condition more clearly, we take a planar curve for example. Suppose $x(t)$ is a convex closed curve on the plane and the curvature is $k(t)$. Let $F(\kappa) = k$. If $x(t)$ is a lower barrier for the pair (F, f) which means the curve should satisfy $\frac{1}{k(t)} - f(x(t), \nu(t)) \ge 0$ where $\nu(t)$ is the unit normal vector at $x(t)$. If a circle of radius r is a lower barrier for (k, f), then equivalently we have a restriction on f, i.e.

$$r \ge f(r\theta, \theta), \quad \forall \theta \in \mathbb{S}^1.$$

Our first result is the following:

Theorem 1. *Suppose $M_a = \{a\} \times E^n \subset N$ is a lower barrier and $M_b = \{b\} \times E^n \subset N$ is an upper barrier for the pair (F, f), respectively. Let $X(\cdot, 0) = M_a$ in the flow (1). Assume the Riemannian curvature of N, denoted as $\bar{R}m$, satisfies*

$$|\bar{R}m| \le K, \quad |\bar{\nabla}\bar{R}m| \le K.$$

If $f_{xx} < -(|f_\xi| + C)\bar{g}$, and $f_{\xi\xi} < -(|f_\xi| + C)g_0$, where C is a constant depending only on C_0, K, $|f|$ and $|f_\xi|$, then there exists a time-dependent family of embeddings $X : E^n \times [0, T) \to N$

satisfying (1). As $t \to T$, the embeddings $X(\cdot, t)$ converge to a strictly convex solution of the generalized Minkowski problem

$$F(\kappa) = f^{-1}(X, \nu). \tag{2}$$

Remark 1.1. This theorem can be viewed as a generalisation of the result in [3] where Bryan et al. solved the Minkowski problem in spaceforms under the similar assumption. Li and Sheng solved a wide range of prescribing curvature problems in [12] including prescribed Gaussian curvature problem where the prescribed function is independent of normal direction and the assumption is quite different.

If $f \equiv 0$ and drop the upper barrier in (1), then we get an expanding curvature flow. Gerhardt [5], Urbas [18] proved that the inverse mean curvature flow starting from mean-convex star-shaped smooth hypersurfaces of Euclidean space converges to an infinite large sphere. Gerhardt [7] also proved a convergence result in hyperbolic space. Huisken and Ilmanen [11] studied the inverse mean curvature flow in asymptotically flat 3-manifold and proved the Riemannian Penrose inequality. Guan and Li [8] studied the inverse mean curvature flow and proved the Alexandrov-Fenchel inequalities of quermassintegrals.

Given a hypersurface Σ in $(N, \bar{g})$, we say that it is strictly k-convex if its principal curvature vector is in the k-th Gårding cone Γ_k everywhere, and $\Gamma_k := \{\kappa \in \mathbb{R}^n | \sigma_1(\kappa) > 0, \cdots, \sigma_k(\kappa) > 0\}$. We say that Σ is star-shaped if it can be represented as a radial graph over the sphere $\mathbb{S}^n$. Let $F = \frac{\sigma_{k+1}}{\sigma_k}(\kappa)$. The inverse F-curvature flow of Σ is a family of smooth embeddings $X : \Sigma \times [0, T) \to (N, \bar{g})$ satisfying

$$X_t = \frac{1}{F}\nu, \tag{3}$$

where ν is the unit outward normal vector of $\Sigma_t = X(\Sigma, t)$.

We will obtain the following long-time existence and convergence result of the inverse F-curvature flow (3) for strictly $(k+1)$-convex and star-shaped hypersurface in the Schwarzschild manifold. The metric of Schwarzschild manifold is a special warped product metric, where $\phi(\rho)$ satisfies an ODE, see Section 4 for more details.

Theorem 2. *The inverse F-curvature flow (3) starting from a strictly $(k+1)$-convex and star-shaped hypersurface in the Schwarzschild manifold $(N, \bar{g})$ exists for all time. The flow hypersurface Σ_t converges to infinity while preserving $(k+1)$-convexity and star-shapedness. Denote the radial function of Σ_t as $\rho(\cdot, t)$, then there exist two positive constants c_0 and $\bar{\phi}$ such that $e^{-c_0 t}\phi(\rho) \to \bar{\phi}$ in C^∞-catagory as $t \to +\infty$. The flow Σ_t converges to a large coordinate sphere as $t \to +\infty$.*

Remark 1.2. If $k = 0$, then $F = H$ is the mean curvature. Theorem 2 is a generalisation of [14]. Above theorem holds for all $k = 0, 1, 2, \cdots, n-1$.

The rest of the paper is organized as follows. In section 2, we list some basic facts for hypersurfaces in N. In section 3 we prove the a priori estimates and prove Theorem 1. In section 4 we give some examples to the prescribed curvature problems in Schwarzschild manifold. In section 5 we show the asymptotic behavior of the inverse F-curvature flow and conclude Theorem 2.

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§2 Preliminaries

Let the ambient space $N = [a, b] \times E^n$ with warped product metric

$$\bar{g} = d\rho^2 + \phi(\rho)^2 g_0.$$

g_0 is a Riemannian metric on closed manifold E. $\phi > 0$ is a smooth convex function of ρ. Let $\Phi(\rho) = \int_a^\rho \phi(s)ds$.

Lemma 1. *Let $M \subset N$ be a closed hypersurface with induced metric g. Then $\Phi|_M$ satisfies*

$$\nabla_i \nabla_j \Phi = \phi'(\rho) g_{ij} - h_{ij} u, \tag{4}$$

where ∇ is the covariant derivative with respect to g, h_{ij} is the second fundamental form of the hypersurface.

Proof. Let $V = \phi \frac{\partial}{\partial \rho}$. It is well-known that V is a conformal Killing vector field, i.e. $\mathcal{L}_V \bar{g} = 2\phi' \bar{g}$ where $\mathcal{L}_V$ is the Lie derivative. Let X, Y be smooth vector fields on M, and f be a smooth function on N, then we have

$$\mathrm{Hess}_M f(X, Y) = Y(X(f)) - (\nabla_X Y) f = \mathrm{Hess}_N f(X, Y) + (\bar{\nabla}_X Y - \nabla_X Y) f,$$

where $\bar{\nabla}$ is the Levi-Civita connection on TN. Let $\{e_i\}(1 \leq i \leq n)$ be an orthonormal frame on M. Since $\bar{\nabla}\Phi = V$,

$$\nabla_{e_i} \nabla_{e_j} \Phi|_M = \langle \bar{\nabla}_{e_i} V, e_j \rangle - h_{ij} \langle V, \nu \rangle = \phi' g_{ij} - h_{ij} u.$$

□

Next, we derive the gradient and Hessian of the support function u under the induced metric g on M. The main difference between the following equations and those in [2] is that the second fundamental form of M is not a Codazzi tensor in general warped product spaces.

Lemma 2. *The support function u satisfies*

$$\begin{aligned} \nabla_i u = &\ g^{kl} h_{ik} \nabla_l \Phi, \\ \nabla_i \nabla_j u = &\ g^{kl} \nabla_k h_{ij} \nabla_l \Phi + \phi' h_{ij} - (h^2)_{ij} u + \bar{R}_{\nu jki} \nabla_k \Phi, \end{aligned} \tag{5}$$

where $(h^2)_{ij} := g^{kl} h_{ik} h_{jl}$.

Proof. Choose an orthonormal frame, then $g_{ij} = \delta_{ij}$ and $\nabla_i u = \bar{\nabla}_i \langle V, \nu \rangle = \langle \bar{\nabla}\Phi, \bar{\nabla}_i \nu \rangle = h_{ik} \bar{\nabla}_k \Phi$. Thus the first identity holds.

Differentiating ∇u again and applying (4), we have

$$\begin{aligned} \nabla_i \nabla_j u = &\ \nabla_i h_{jk} \nabla_k \Phi + h_{jk} \nabla_i \nabla_k \Phi \\ = &\ \nabla_k h_{ij} \nabla_k \Phi + \bar{R}_{\nu jki} \nabla_k \Phi + h_{jk}(\phi' g_{ik} - h_{ik} u) \\ = &\ \nabla_k h_{ij} \nabla_k \Phi + \bar{R}_{\nu jki} \nabla_k \Phi + \phi' h_{ij} - (h^2)_{ij} u, \end{aligned}$$

where we use the Codazzi equation:

$$\nabla_i h_{jk} - \nabla_k h_{ij} = \bar{R}_{\nu jki} = \bar{R}(\nu, e_j, e_k, e_i).$$

□

Let M_t be a smooth family of closed hypersurfaces in N. Let $X(t, \cdot)$ denote a point on M_t.

Lemma 3. *Under flow $\partial_t X = G\nu$ on the hypersurface in a Riemannian manifold, we have the following evolution equations*

$$
\begin{aligned}
\partial_t g_{ij} =& \quad 2Gh_{ij},\\
\partial_t h_{ij} =& \quad -\nabla_i\nabla_j G + G(h^2)_{ij} - G\bar{R}_{\nu i\nu j},\\
\partial_t h_i^j =& \quad -g^{jk}\nabla_k\nabla_i G - g^{jk}G(h^2)_{ki} - Gg^{jk}\bar{R}_{\nu i\nu k}.
\end{aligned}
\tag{6}
$$

These equations are well-known, so we omit the proof.

Let (M,g) be a hypersurface in $(N,\bar{g})$ with induced metric g. Suppose M is a graph of a smooth, positive function $\rho(z)$ on E^n. We have the following local expressions of the support function, induced metric, second fundamental form, Weingarten curvatures [15].

$$
\begin{aligned}
u =& \quad \frac{\phi^2}{\sqrt{\phi^2+|\nabla\rho|^2}},\\
g_{ij} =& \quad \phi^2 e_{ij} + \rho_i\rho_j, \quad g^{ij} = \frac{1}{\phi^2}\left(e^{ij} - \frac{\rho_i\rho_j}{\phi^2+|\nabla\rho|^2}\right),\\
h_{ij} =& \quad \left(\sqrt{\phi^2+|\nabla\rho|^2}\right)^{-1}\left(-\phi\nabla_i\nabla_j\rho + 2\phi'\rho_i\rho_j + \phi^2\phi' e_{ij}\right),\\
h_j^i =& \quad \left(\phi^2\sqrt{\phi^2+|\nabla\rho|^2}\right)^{-1}\left(e^{ik} - \frac{\rho_i\rho_k}{\phi^2+|\nabla\rho|^2}\right)\left(-\phi\nabla_k\nabla_j\rho + 2\phi'\rho_k\rho_j + \phi^2\phi' e_{kj}\right).
\end{aligned}
\tag{7}
$$

where all the covariant derivatives ∇ and ρ_i are taken with respect to g_0. For the sake of simplicity, we denote ∇ the covariant derivative with respect to g_0 if no confusion arises.

We now consider the flow (1) of radial graphs over E^n in N. It is known that if a family of closed hypersurfaces defined by radial graphs and satisfies $X_t = G\nu$, then the evolution of the scalar function $\rho(t,z)$ satisfies $\rho_t = G\omega$, where $\omega = \sqrt{1+\phi^{-2}|\nabla\rho|^2}$. Thus we only need to consider following scalar equation on E^n,

$$
\begin{cases}
\partial_t\rho = & \left(\frac{1}{F} - f\right)\omega,\\
\rho(0,\cdot) = & \rho_0.
\end{cases}
\tag{8}
$$

Recall $u = \phi\langle\frac{\partial}{\partial\rho},\nu\rangle$. Thus

$$
\begin{aligned}
u_t =& \quad \phi'\rho_t\langle\frac{\partial}{\partial\rho},\nu\rangle + \phi\langle\frac{\partial}{\partial\rho},\nu_t\rangle\\
=& \quad G\phi' - \nabla\Phi\cdot\nabla G.
\end{aligned}
\tag{9}
$$

For convenience, we introduce a new variable γ satisfying

$$
\frac{d\gamma}{d\rho} = \frac{1}{\phi}.
$$

Then the equation for γ is

$$
\begin{cases}
\partial_t\gamma = & \frac{\omega^4}{\tilde{F}} - \frac{f}{u},\\
\gamma(0,\cdot) = & \gamma_0.
\end{cases}
\tag{10}
$$

where $\tilde{F} = \phi\omega^3 F$. The induced metric and the second fundamental forms in (7) can be rewritten in terms of γ as:

$$
\begin{aligned}
g_{ij} =& \quad \phi^2(\delta_{ij}+\gamma_i\gamma_j), \quad g^{ij} = \phi^{-2}\left(\delta_{ij} - \frac{\gamma_i\gamma_j}{1+|\nabla\gamma|^2}\right),\\
h_{ij} =& \quad \frac{\phi'}{\phi\sqrt{1+|\nabla\gamma|^2}} g_{ij} - \frac{\phi}{\sqrt{1+|\nabla\gamma|^2}}\gamma_{ij}.
\end{aligned}
\tag{11}
$$

Let

$$
h_j^i = \frac{1}{\phi\omega^3} b_j^i, \tag{12}
$$

$$
b_j^i = \omega^2(\phi' e_{ij} - \gamma_{ij} + \gamma_i\gamma_{lj}\gamma_l). \tag{13}
$$

By 1-homogeneity of F, we have $\frac{\partial F}{\partial h_j^i} = \frac{\partial\tilde{F}}{\partial b_j^i}$.

§3 A priori estimates

Since we have barrier conditions, it is easy to see that the flow remains between the two barriers (c.f. Theorem 2.7.10 in [6]). In the following, we turn to prove the C^1 and C^2 estimates.

Lemma 4. *Along flow (8),* $|\nabla\rho| \le C$.

Proof. We consider the test function $\log\frac{|\nabla\rho|^2}{\varphi}$ to obtain a gradient estimate. Here $\varphi = \varphi(\rho)$ to be determined later. Note that at the critical points of $\log\frac{|\nabla\rho|^2}{\varphi}$, we have $\nabla|\nabla\rho|^2 = \frac{\varphi'}{\varphi}|\nabla\rho|^2\nabla\rho$. We may assume $|\nabla\rho| = \rho_1$, $\rho_{1j} = 0$ for $j \ge 2$, and $\rho_{11} = \frac{1}{2}\frac{\varphi'}{\varphi}|\nabla\rho|^2$. Since the computations are done at one point, we take an orthonormal frame so that $e_{ij} = \delta_{ij}$.

$$h_1^1 = (\phi^2 + |\nabla\rho|^2)^{-\frac{3}{2}} \cdot \Big(-\phi \cdot \frac{\varphi'}{2\varphi} \cdot \rho_1^2 + 2\phi'\rho_1^2 + \phi^2\phi' \Big).$$

Set $\varphi = \phi^8$, by convexity we have $|\nabla\rho|^2 = \rho_1^2 \le C$. □

Remark 3.1. The convexity of the flow holds as long as the flow exists, since if there is $t_0 > 0$, principal curvature vector $\kappa \to \partial\Gamma_+$ as $t \to t_0$, then $\frac{1}{F} \to +\infty$ and the flow will blow up at $t = t_0$. In the following, we will prove that principal curvatures are bounded away from 0, so that the flow is strictly convex.

For simplicity we denote $G = \frac{1}{F} - f(X,\nu)$, $\mathcal{L} = \partial_t - \frac{1}{F^2}F^{ij}\nabla_i\nabla_j$. Direct computation shows that

$$\mathcal{L}G = G\frac{F^{ij}(h^2)_{ij}}{F^2} + G\frac{F^{ij}\bar{R}_{\nu i\nu j}}{F^2} - f_\nu(\nabla G) + f_X(\nu)G.$$

Lemma 5. $G(t,\cdot) \ge 0$ *for all* $t \ge 0$ *as long as* $X(t,\cdot)$ *exists.*

Proof. Suppose the flow (1) exists on $[0,T)$ and $0 < t_0 < T$. The curvatures of $M(t)$ on $[0,t_0]$ are all bounded. Thus

$$\mathcal{L}G = c \cdot G + \text{gradient terms}. \tag{14}$$

where c is bounded. By the weak maximum principle (c.f. Lemma 2.7.2 in [6]), $\min G(t,\cdot) \ge 0$ on $[0,t_0]$. Since t_0 is arbitrary, $G \ge 0$ as long as the flow exists. □

Lemma 6. $F \ge C_0 > 0$ *for all* t.

Proof. The evolution of the radial function ρ is (8). We calculate at a critical point P of the function γ_t. At P, $\nabla\gamma_t = 0$ and $\partial_t\omega^2 = 0$. Differentiate (10), we have

$$\partial_t\gamma_t = -\frac{\omega^4}{\tilde{F}^2}\tilde{F}_t - (\frac{f}{u})_t.$$

At P,

$$\begin{aligned}\tilde{F}_t &= F^{ij}\partial_t(\omega^2(-\gamma_{ij} + \phi'\delta_{ij}) + \gamma_i\gamma_{jl}\gamma_l)\\ &= F^{ij}(\omega^2(-(\gamma_t)_{ij} + \phi\phi''\gamma_t\delta_{ij}) + \gamma_i\gamma_{tjl}\gamma_l)\\ &= -F^{ij}(\omega^2\delta_{il} - \gamma_i\gamma_l)(\gamma_t)_{lj} + \phi\phi''(\sum F^{ii})\omega^2\gamma_t.\end{aligned}$$

Putting the above equations together, we have

$$\partial_t\gamma_t - \frac{\omega^4 F^{ij}}{\tilde{F}^2}(\omega^2\delta_{il} - \gamma_i\gamma_l)(\gamma_t)_{jl} = -\frac{\omega^6\phi\phi'' F^{ii}}{\tilde{F}^2}\gamma_t - \frac{f_t}{u} + \frac{f u_t}{u^2}, \tag{15}$$

and $f_t = f_\xi(\nabla u)\gamma_t + f_x(\nu)u\gamma_t +$ gradient terms,$u_t = (u\phi' - \nabla\Phi\cdot\nabla u)\gamma_t +$ gradient terms. Thus

$$\partial_t\gamma_t - \frac{\omega^4 F^{ij}}{\tilde{F}^2}(\omega^2\delta_{il} - \gamma_i\gamma_l)(\gamma_t)_{jl} = -\frac{\omega^6\phi\phi'' F^{ii}}{\tilde{F}^2}\gamma_t + c\gamma_t + \text{gradient terms}.$$

where $c = \frac{f}{u^2}\cdot(u\phi' - \nabla\Phi\nabla u) - \frac{f_\xi(\nabla u)}{u} - f_x(\nu)$ is bounded above by C^0 and C^1 estimates. If $\gamma_t \to \infty$, then $\frac{1}{\tilde{F}} \to \infty$, while $\partial_t\gamma_t < 0$. Thus γ_t is bounded and $\frac{1}{F}$ is also bounded above. $\square$

Lemma 7. *The principal curvatures are bounded, i.e.*

$$0 < \frac{1}{C} \le \kappa_i \le C, \quad \forall i.$$

Proof. Without loss of generality, we may choose an orthonormal frame locally on the flow at time t. We need some equalities about Weingarten matrix first.

$$(h^i_j)_t = -G_{ij} - G(h^2)_{ij} - G\bar{R}_{\nu i\nu j}. \tag{16}$$

Recall that $G = \frac{1}{F} - f(X,\nu)$. Take derivatives on both sides,

$$G_{ij} = \frac{2F_iF_j}{F^3} - \frac{F^{kl,pq}h_{kli}h_{pqj}}{F^2} - \frac{1}{F^2}F^{kl}h_{klij} - f_{ij}. \tag{17}$$

Using Ricci identity and the Codazzi equation, we have

$$\begin{aligned} h_{klij} = h_{ijkl} + \bar{R}_{\nu ikj,l} + \bar{R}_{\nu kli,j} + \bar{R}_{pilj}h_{pk} + \bar{R}_{pklj}h_{pi} + (h^2)_{kl}h_{ij} \\ - (h^2)_{kj}h_{il} + (h^2)_{il}h_{kj} - (h^2)_{ij}h_{kl}. \end{aligned} \tag{18}$$

Combining (16),(17),(18) with $\mathcal{L} = \partial_t - \frac{1}{F^2}F^{ij}\nabla_i\nabla_j$, we have

$$\begin{aligned} \mathcal{L}h^i_j = \frac{F^{kl,pq}h_{kli}h_{pqj}}{F^2} - \frac{2F_iF_j}{F^3} + f_{ij} - (\frac{2}{F} - f)(h^2)_{ij} - (\frac{1}{F} - f)\bar{R}_{\nu i\nu j} \\ + \frac{F^{kl}}{F^2}(h^2)_{kl}h_{ij} + \frac{F^{kl}}{F^2}(\bar{R}_{\nu ikj,l} + \bar{R}_{\nu kli,j} + \bar{R}_{pilj}h_{pk} + \bar{R}_{pklj}h_{pi}). \end{aligned} \tag{19}$$

Consequently,

$$\begin{aligned} \mathcal{L}H =& \frac{F^{kl,rs}h_{kli}h_{rsi}}{F^2} - \frac{2F_i^2}{F^3} + f_{ii} - (\frac{2}{F} - f)|A|^2 - (\frac{1}{F} - f)\bar{R}_{\nu\nu} + \frac{F^{kl}(h^2)_{kl}}{F^2}H \\ &+ \frac{F^{kl}}{F^2}(\bar{R}_{\nu iki,l} + \bar{R}_{\nu kli,i} + \bar{R}_{pili}h_{pk} + \bar{R}_{pkli}h_{pi}). \end{aligned} \tag{20}$$

Recall the Hessian of u in (5),

$$u_{ij} = h_{ijk}\Phi_k + \phi' h_{ij} - (h^2)_{ij}u + \bar{R}_{\nu jki}\Phi_k.$$

By Weingarten equations and Codazzi equation, we have

$$\nu_{ij} = h_{ijk}e_k + \bar{R}_{\nu ikj}e_k - (h^2)_{ij}\nu.$$

Direct computation shows that

$$\begin{aligned} f_{ij} =& f_{\xi\xi}\nu_i\nu_j + f_\xi\nu_{ij} + 2f_{x\xi}(e_i)\nu_j + f_{xx}(e_i,e_j) - f_x(\nu)h_{ij} \\ =& f_{\xi\xi}(e_k,e_l)h_{ik}h_{jl} - f_\xi(\nu)(h^2)_{ij} + f_\xi(e_k)h_{ijk} + 2f_{x\xi}(e_i,e_k)h_{jk} \\ &- f_x(\nu)h_{ij} + f_{xx}(e_i,e_j) + f_\xi(e_k)\bar{R}_{\nu ikj}. \end{aligned} \tag{21}$$

Let $Q = \log H + \beta F$, where $\beta > 0$ to be determined. Compute at the spatial maximum point of Q, then $H_i/H + \beta F_i = 0$.

$$\begin{aligned}\mathcal{L}F =&F^{ij}\mathcal{L}h_{ij} - \frac{F^{ij}}{F^2}F^{pq,rs}h_{pqi}h_{rsj} - \frac{F^{ij}}{F^2}F^{kl}(\bar{R}_{\nu ikj,l} + \bar{R}_{\nu kli,j} + \bar{R}_{pilj}h_{pk}\\ &+ \bar{R}_{pklj}h_{pi} + (h^2)_{kl}h_{ij} - (h^2)_{kj}h_{il} + (h^2)_{il}h_{kj} - (h^2)_{ij}h_{kl})\\ =&F^{ij}\mathcal{L}h_{ij} - \frac{F^{ij}}{F^2}F^{pq,rs}h_{pqi}h_{rsj} - \frac{F^{ij}}{F^2}F^{kl}(\bar{R}_{\nu ikj,l} + \bar{R}_{\nu kli,j}).\end{aligned}$$

By the second Bianchi identity, $\bar{R}_{\nu ikj,l} = -\bar{R}_{\nu ijl,k} - \bar{R}_{\nu ilk,j}$, we have

$$\mathcal{L}F = F^{ij}\mathcal{L}h_{ij} - \frac{F^{ij}}{F^2}F^{pq,rs}h_{pqi}h_{rsj}. \tag{22}$$

Using (19) again, we compute

$$\begin{aligned}F^{ij}\mathcal{L}h_{ij} = &\frac{F^{ij}}{F^2}F^{kl,pq}h_{kli}h_{pqj} - \frac{2F^{ij}F_iF_j}{F^3} + F^{ij}f_{ij} - (\frac{2}{F} - f)F^{ij}(h^2)_{ij}\\ &- (\frac{1}{F} - f)F^{ij}\bar{R}_{\nu i\nu j} + \frac{F^{kl}(h^2)_{kl}}{F}.\end{aligned} \tag{23}$$

Plugging (23) into (22), we have

$$\mathcal{L}F = -\frac{2F^{ij}F_iF_j}{F^3} + F^{ij}f_{ij} - (\frac{1}{F} - f)F^{ij}(h^2)_{ij} - (\frac{1}{F} - f)F^{ij}\bar{R}_{\nu i\nu j}. \tag{24}$$

Note that $F^{kl}(h^2)_{kl} \le C_0FH$. By (20), (24), (21), we have

$$\begin{aligned}\mathcal{L}Q =& \frac{\mathcal{L}H}{H} + \beta\mathcal{L}F + \frac{F^{ij}}{F^2}\cdot\frac{H_iH_j}{H^2}\\ =& \frac{F^{kl,rs}h_{kli}h_{rsi}}{HF^2} - \frac{2F_i^2}{HF^3} + \frac{1}{H}f_{ii} - (\frac{2}{F} - f)\frac{|A|^2}{H} - \frac{\bar{R}_{\nu\nu}}{H}(\frac{1}{F} - f)\\ &+ \frac{F^{kl}(h^2)_{kl}}{F^2} + \frac{F^{ij}H_iH_j}{H^2F^2} + \frac{F^{kl}}{HF^2}(\bar{R}_{\nu iki,l} + \bar{R}_{\nu kli,i} + \bar{R}_{pili}h_{pk} + \bar{R}_{pkli}h_{pi})\\ &+ \beta(-\frac{2F^{ij}F_iF_j}{F^3} + F^{ij}f_{ij} - (\frac{1}{F} - f)F^{ij}(h^2)_{ij} - (\frac{1}{F} - f)F^{ij}\bar{R}_{\nu i\nu j})\\ \le& \frac{f_{ii}}{H} + C\frac{|A|^2}{H} + \frac{C}{H} + C(\sum F^{ii}) + (\beta^2 - 2\frac{\beta}{F})F^{ij}F_iF_j\\ &+ \beta F^{ij}f_{ij} - \beta(\frac{1}{F} - f)\frac{FH}{n} + \beta C\cdot(\sum F^{ii}).\end{aligned}$$

Since F is bounded, we first choose β, s. t. $0 < \beta < \min\frac{2}{F}$. By (21),

$$f_{ij} \le (\max_i f_{\xi\xi}(e_i, e_i) + |f_\xi|)(h^2)_{ij} + Ch_{ij} + \max_i f_{xx}(e_i, e_i) + |f_\xi| + f_\xi(e_k)h_{ijk}.$$

If $f_{\xi\xi} < -(|f_\xi| + C)g_0$ and $f_{xx} < -(|f_\xi| + C)\bar{g}$, where C is depends on K, n, f, then $\mathcal{L}Q \le -\epsilon\frac{|A|^2}{H} + CH^{-1} + C$. Therefore H as well as Q is bounded above. Since $F|_{\partial\Gamma_+} = 0$, all principal curvatures are bounded away from 0. □

Proof of Theorem 1. Since we have shown that the principal curvatures are bounded above, by Evans-Krylov's theorem,

$$||\rho||_{C^{2,\alpha}} < C.$$

Note that the flow (1) is monotone and $\{\rho(t)\}$ is bounded above. Therefore $\{\rho(t)\}$ converges in C^0 with a unique limit, say ρ_∞. Since we have $||\nabla^k\rho|| \le C$, there exists a subsequence $\{\rho(t_k)\}$ converges in C^k-category. Since the limit ρ_∞ is unique, $\{\rho(t)\}$ also converges in C^k, for every $k \ge 1$. □

§4 Applications

In this section, we show a concrete example. Consider the prescribed Gaussian curvature problem in Schwarzschild manifold for a kind of function f. The Schwarzschild manifold is an $(n+1)$-dimensional ($n \geq 2$) Riemannian manifold $N = [s_0, +\infty) \times \mathbb{S}^n$ equipped with the metric

$$\bar{g} = \frac{1}{1 - 2ms^{1-n}} ds^2 + s^2 g_{\mathbb{S}^n}, \tag{25}$$

where $m > 0$ is a constant which is the mass of the black hole centered at $s = 0$, and $g_{\mathbb{S}^n}$ the standard metric on the unit sphere $\mathbb{S}^n$. Denote $s_0 = (2m)^{1/(n-1)}$, which is called as Schwarzschild radius. We only consider the case $s > s_0$. By a change of variables, $\bar{g}$ can be rewritten as

$$\bar{g} = d\rho^2 + \phi(\rho)^2 g_{\mathbb{S}^n},$$

where $\phi(\rho)$ is the solution of the following ODE:

$$\begin{cases} \frac{ds}{d\rho} & = \sqrt{1 - 2ms^{1-n}}, \\ s(0) & = s_0 + \epsilon \quad \text{for small}\, \epsilon. \end{cases}$$

Clearly, $\phi(\rho)$ is a monotonic increasing convex function on $[0, +\infty)$ satisfying $\phi''(\rho) = m(n-1)\phi(\rho)^{-n}$. The principal curvatures of the geodesic sphere with radius ρ are all $\frac{\phi'}{\phi}$, which is monotonically increasing while $\phi(0) \leq \phi(\rho) \leq (m(n+1))^{1/(n-1)}$ and is monotonically decreasing while $\phi(\rho) > (m(n+1))^{1/(n-1)}$. Let $\max_\rho \frac{\phi'}{\phi} = c_0$ and $\rho_0 = \phi^{-1}((m(n+1))^{1/(n-1)})$. Let $f(x, \xi) = -C_{ij}x_i x_j - D_{ij}\xi_i\xi_j + A_1||C|| + A_2||D||$, where $(C_{ij}), (D_{ij})$ are two symmtric positive definite matrices with constant entries. Denote $x_{n+1} = \rho$ and $x_1, \cdots, x_n$ be the (local) coordinates on $\mathbb{S}^n$ which are the polar coordinates in $\mathbb{R}^{n+1}$. There is a diffeomorphism $\varphi : \mathbb{R}^{n+1} \setminus B_{s_0}(0) \to N$, therefore on the Schwarzschild manifold N there are also global well-defined coordinates induced by φ. Naturally $\frac{\partial}{\partial x_i}(1 \leq i \leq n)$ and $\frac{\partial}{\partial \rho}$ are basis vector fields. In general $(\frac{\partial^2 f}{\partial x_i \partial x_j})$ is not the covariant Hessian of $f(x)$ with respect to a Riemannian metric. Here for simplicity, let $C_{i,n+1} = C_{n+1,i} = 0$ and $C_{n+1,n+1}$ is greater than any absolute values of eigenvalues of the $n \times n$ matrix that results from deleting row $n+1$ and column $n+1$ of (C_{ij}). Directly the second order covariant derivatives are calculated as below,

$$\begin{aligned} \bar{\nabla}_i \bar{\nabla}_j f &= \frac{\partial^2 f}{\partial x_i \partial x_j} - \sum_{k=1}^{n} \bar{\Gamma}_{ij}^k \frac{\partial f}{\partial x_k} - \bar{\Gamma}_{ij}^{n+1} \frac{\partial f}{\partial x_{n+1}} \\ &= -C_{ij} - \sum_{k=1}^{n} \Gamma_{ij}^k(\mathbb{S}^n) \frac{\partial f}{\partial x_k} - 2\phi\phi' C_{n+1,n+1}\rho\delta_{ij}, \\ \bar{\nabla}_{n+1} \bar{\nabla}_{n+1} f &= \frac{\partial^2 f}{\partial \rho^2} = -2C_{n+1,n+1}, \\ \bar{\nabla}_i \bar{\nabla}_{n+1} f &= -C_{i,n+1} - (\log\phi)' \frac{\partial f}{\partial x_i}. \end{aligned}$$

Therefore we may choose an appropriate C_{ij} so that f_{xx} satisfies the requirement in the main theorem. Things are similar to choose an appropriate (D_{ij}) so that $f_{\xi\xi}$ satisfies the requirement in Theorem 1. Next we choose $A_1 >> 1$ so that

$$||C||(A_1 - r_0^2) > c_0^{-1}.$$

Let $\tilde{f}(\rho) = -||C||\rho^2 + A_1||C|| + (A_2 - 1)||D||$. Obviously the graph of $\tilde{f}(\rho)$ and the graph of $\frac{\phi}{\phi'}(\rho)$ have two intersections. Let $\rho_1 < \rho_2$ be the two solutions of $\tilde{f}(\rho) = \frac{\phi}{\phi'}(\rho)$. Choose $a < \rho_1$ small and $\rho_1 < b < \rho_2$ so that the geodesic sphere of $\rho = a$ is a lower barrier to the pair $(\sigma_n^{1/n}, f(x,\xi))$, and the geodesic sphere of $\rho = b$ is an upper barrier to the pair $(\sigma_n^{1/n}, f(x,\xi))$. Thus by Theorem 1 we have at least one convex solution of

$$\sigma_n(\kappa_i)^{\frac{1}{n}} = f(x,\xi)^{-1}.$$

in the annulus region $\{a \le \rho \le b\}$ in the Schwarzschild manifold.

§5 An expanding curvature flow in Schwarzschild manifold

In this section, we prove Theorem 2.

5.1 A priori estimates

Lemma 8. *Suppose the radial function of Σ_t is $\rho(\cdot,t)$. Letρ_1, ρ_2 be the constants such that $0 < \rho_1 \le \rho(\cdot,0) \le \rho_2$, then on X_t we have*

$$\phi(\rho_1)e^{c_0 t} \le \phi(\rho(\cdot,t)) \le \phi(\rho_2)e^{c_0 t},$$

as long as $X(\cdot,t)$ exists.

Proof. We follow the idea in [14]. The flow (3) is equivalent to (8) if we let $f \equiv 0$,

$$\begin{cases} \partial_t \rho = & \frac{\omega}{F} \\ \rho(\cdot,0) = & \rho_0. \end{cases} \tag{26}$$

If the initial hypersurface is a coordinate sphere, i.e. ρ_0 is a constant, then (26) becomes an ODE:

$$\frac{d\rho}{dt} = \frac{c_0\phi}{\phi'},$$

where $c_0 = C_n^k / C_n^{k+1}$.

Thus $\phi(\rho(t)) = \phi(\rho(0))e^{c_0 t}$. Now by the parabolic maximum principle, the Lemma is proved. □

Next, we consider the gradient estimate. We use the same change of variables introduced in the proof of Lemma 6. Recall that $\gamma(\rho)$ satisfies

$$\frac{d\gamma}{d\rho} = \frac{1}{\phi}.$$

Then $\gamma_t = \phi^{-1}\rho_t$. We have the following gradient estimate.

Lemma 9. *Let ρ be a positive admissible solution of (26) on $\mathbb{S}^n \times [0,T)$. Then for all $t \in [0,T)$ we have*

$$\max_{\mathbb{S}^n} |\nabla\gamma|(\cdot,t) \le \max_{\mathbb{S}^n} |\nabla\gamma|(\cdot,0).$$

We remark that this gradient estimate can be improved if we prove the uniform parabolicity of (26).

Proof. Assume $w = \frac{1}{2}|\nabla\gamma|^2$ attains its spatial maximum at $\theta \in \mathbb{S}^n$. For simplicity, we choose an orthonormal frame on $\mathbb{S}^n$. At θ, $\gamma_{li}\gamma_l = 0$ for all i and $\gamma_{lij}\gamma_l + \gamma_{li}\gamma_{lj} \le 0$. Directly,

$$\begin{aligned} w_t = \gamma_l\gamma_{lt} &= \gamma_l\big(\frac{\sqrt{1+|\nabla\gamma|^2}}{\phi F}\big)_{,l} \\ &= -\gamma_l \cdot \frac{\sqrt{1+|\nabla\gamma|^2}}{\phi^2 F^2} \cdot (\phi'\rho_l F + \phi F^{pq}(h_p^q)_{,l}). \end{aligned}$$

Recall that $\omega = \sqrt{1+|\nabla\gamma|^2}$ also attains its maximum at θ. In the following, we calculate $h^q_{p,l}$. By (11), $(h_{pi}g^{qi})_{,l} = h_{pi,l}g^{iq} - h_{pi}g^{ir}g_{rs,l}g^{sq}$.

$$\begin{aligned} h_{pi,l} &= \frac{1}{\omega}(\frac{\phi'}{\phi})_{,l}g_{pi} + \frac{\phi'}{\omega\phi}g_{pi,l} - \frac{\phi_l}{\omega}\gamma_{pi} - \frac{\phi}{\omega}\gamma_{pil}, \\ g_{rs,l} &= \frac{(\phi^2)_{,l}}{\phi^2}g_{rs} + \phi^2(\gamma_{rl}\gamma_s + \gamma_r\gamma_{sl}). \end{aligned}$$

Thus

$$h_{pi,l}g^{iq} = \frac{1}{\omega}\big(\frac{\phi''}{\phi} + (\frac{\phi'}{\phi})^2\big)\rho_l\delta_p^q + \frac{\phi\phi'}{\omega}(\gamma_p\gamma_i)_l g^{iq} - \frac{\phi_l}{\omega}\gamma_{pi}g^{iq} - \frac{\phi}{\omega}\gamma_{pil}g^{iq}.$$

And

$$h_{pi}g^{ir}g_{rs,l}g^{sq} = \frac{(\phi^2)_{,l}}{\phi^2}h_p^q + \phi^2 h_{pi}g^{ir}g^{sq}(\gamma_r\gamma_s)_l.$$

Note that $(\gamma_r\gamma_s)_l\gamma_l = 0$ for any r, s, thus

$$h^q_{p,l}\gamma_l = \frac{1}{\omega}\big(\frac{\phi''}{\phi} + (\frac{\phi'}{\phi})^2\big)\delta_p^q\rho_l\gamma_l - \frac{\phi_l\gamma_l}{\omega}\gamma_{pi}g^{iq} - \frac{\phi}{\omega}\gamma_{pil}g^{iq}\gamma_l + \frac{(\phi^2)_{,l}\gamma_l}{\phi^2}h_p^q.$$

Since $\rho_l = \phi\gamma_l$ and $\gamma_{ip} = \frac{\phi'}{\phi^2}g_{ip} - \frac{\omega}{\phi}h_{ip}$, we have

$$w_t = -\frac{\phi''(\sum F^{ii})}{\phi F}|\nabla\gamma|^2 + \frac{1}{F^2}F^{pq}\gamma_{pil}\gamma_l g^{iq}. \tag{27}$$

By the Ricci identity on $\mathbb{S}^n$, we deduce that

$$\gamma_{pil}\gamma_l = \gamma_{lpi}\gamma_l + \gamma_i\gamma_p - \delta_{ip}|\nabla\gamma|^2 = w_{pi} - \gamma_{li}\gamma_{lp} + (\gamma_i\gamma_p - \delta_{ip}|\nabla\gamma|^2). \tag{28}$$

Therefore we have $(\gamma_{pil}\gamma_l) \le 0$. Since ϕ is a positive convex function, $w_t \le 0$ and the lemma is proved. □

Next, we turn to estimate the curvature function F.

Lemma 10. *There exist two positive constants $C_1 < C_2$, such that $C_1e^{-c_0t} \le F \le C_2e^{-c_0t}$.*

Proof. We can proceed as in the proof of Lemma 6. Note that $\gamma_t = \frac{\omega^4}{\tilde{F}}$ with initial data γ_0, where $\tilde{F} = \phi\omega^3 F$. By letting $f \equiv 0$ in (10), from (15) we have

$$\partial_t\gamma_t - \frac{\omega^4 F^{ij}}{\tilde{F}^2}(\omega^2\delta_{il} - \gamma_i\gamma_l)(\gamma_t)_{jl} = -\frac{\omega^6\phi\phi''(\sum F^{ii})}{\tilde{F}^2}\gamma_t + \text{gradient terms}.$$

Since F is a 1-homogeneous concave function, $\sum F^{ii} \ge C > 0$. Therefore $\gamma_t \le C$. $\omega \ge 1$, so we have $\tilde{F} \ge C > 0$. Using the C^0-estimate, we conclude that $F \ge C_1e^{-c_0t}$. In the following, we prove that $\tilde{F}$ is bounded above. At a minimal point of γ_t, by (15) again, we have

$$\partial_t(\gamma_t)_{\min} \ge -Ce^{-nc_0t}(\gamma_t)^3_{\min}.$$

Solving this ordinary differential inequality, we get

$$(\gamma_t)^2_{\min}(t) \ge \frac{2}{\frac{1}{(\gamma_t)^2_{\min}(0)} - \frac{C}{nc_0}e^{-nc_0t} + \frac{C}{nc_0}} \ge \frac{1}{C} > 0.$$

As long as the flow exists, $\gamma_t > 0$. Therefore $\tilde{F}$ is bounded above and $\tilde{F} = \phi^3\omega F$, which shows that $F \le C_2 e^{-c_0 t}$. □

For $(k+1)$-convex admissible solution ρ for (26), by Newton-Maclaurin inequalities, we have

$$F = \frac{\sigma_{k+1}}{\sigma_k} \le CH.$$

We show that

Lemma 11. *The mean curvature $H \le Ce^{-c_0 t}$.*

Proof. Let $f \equiv 0$ in (20), then we have

$$\begin{aligned}\mathcal{L}H =& \frac{F^{kl,rs}h_{kli}h_{rsi}}{F^2} - \frac{2F_i^2}{F^3} - \frac{2|A|^2}{F} - \frac{\bar{R}_{\nu\nu}}{F} + \frac{F^{kl}(h^2)_{kl}}{F^2}H \\ &+ \frac{F^{kl}}{F^2}(\bar{R}_{\nu iki,l} + \bar{R}_{\nu kli,i} + \bar{R}_{pili}h_{pk} + \bar{R}_{pkli}h_{pi}). \end{aligned} \tag{29}$$

By Lemma 4.1 in [2], we have

$$c_0 F^2 \le F^{kl}(h^2)_{kl} \le (k+1)F^2.$$

Direct computation shows that in Schwarzschild manifold, the Riemannian curvature tensor $\bar{R}m$ decays as $O(\phi(\rho)^{-n-1})$, see Section 2.1 in [14]. While $\bar{\nabla}\bar{R}m$ decays as $O(\phi(\rho)^{-n-2})$. Thus

$$\mathcal{L}H \le -\frac{2|A|^2}{F} + CH + O(e^{-nc_0 t}). \tag{30}$$

Since the solution is $(k+1)$-convex, $H^2 \ge |A|^2 \ge \frac{H^2}{n}$ for $k \ge 1$, and for $k = 0$, $F = H$ and we have proved that $F = O(e^{-c_0 t})$. Therefore for all k, H is bounded above. Indeed, we can show that H also decays to 0 as $t \to +\infty$. Consider $v := e^{c_0 t}H$.

$$\mathcal{L}v \le Cv - \frac{2}{C_2}v^2 + o(1).$$

Therefore v is bounded above, which means that H also decays as $O(e^{-c_0 t})$. □

Since the principal curvature vector $\kappa(\kappa_1, \cdots, \kappa_n) \in \Gamma_{k+1}$, if $k \ge 1$, then for all i,

$$|\kappa_i| \le CH \le Ce^{-c_0 t}.$$

In case $k = 0$, i.e. the flow (3) is the inverse mean curvature flow, we can bound the maximum principal curvature directly from (19).

$$\mathcal{L}(h_i^i)_{\max} \le -\frac{|A|^2}{H} + o(e^{-c_0 t}).$$

Therefore all principal curvatures are bounded. Thus the eigenvalues of F^{ij} are uniformly bounded. The flow is uniformly parabolic at any finite time t. We then have the longtime existence of the inverse F-curvature flow for all $0 \le k \le n-1$.

Now we can get a more accurate estimate for the gradient of γ. By (27) and (28), assuming $1/C\delta_{ij} \le F^{ij} \le C\delta_{ij}$, we have

$$\begin{aligned} w_t &\le \frac{1}{F^2}F^{pq}(\gamma_i\gamma_p - \delta_{ip}|\nabla\gamma|^2)g^{iq} \\ &\le -\frac{(n-2)|\nabla\gamma|^2}{C\phi^2 F^2}. \end{aligned}$$

There exists a positive constant β, such that

$$|\nabla\gamma| \le Ce^{-\beta t}.$$

5.2 Proof of Theorem 2

Proof. It remains to show that $\{\phi(\rho)e^{-c_0t}\}$ converges to a constant in C^∞-topology. Now things go in the same way as [14]. Let $\tilde{\phi} = \phi(\rho)e^{-c_0t}$ and $\tilde{F} = Fe^{c_0t}$, then

$$\partial_t\tilde{\phi} = \frac{\phi'\omega}{\tilde{F}} - c_0\tilde{\phi}. \tag{31}$$

Obviously, $|\partial_t\tilde{\phi}| \leq C$ independent on t. (31) is a uniformly parabolic equation. From previous discussion, $|\nabla\tilde{\phi}|$ decays exponentially fast:

$$|\nabla\tilde{\phi}| = e^{-c_0t}\phi'|\nabla\rho| = e^{-c_0t}\phi\phi'|\nabla\gamma| = O(e^{-\beta t}).$$

Thus $\{\tilde{\phi}\}$ uniformly converges to a constant, say $\bar{\phi}$. By interpolation inequality,

$$\int_{\mathbb{S}^n} |\nabla^m\tilde{\phi}|^2 \leq Ce^{-\beta' t}.$$

where β' depends only on β, n, m. Next we use the Sobolev embedding theorem on $\mathbb{S}^n$, for any $m > l + \frac{n}{2}$,

$$||\tilde{\phi} - \bar{\phi}||_{C^l} \leq C(m,l)\Big(\int_{\mathbb{S}^n} |\nabla^m\tilde{\phi}|^2 + \int_{\mathbb{S}^n} |\tilde{\phi} - \bar{\phi}|^2\Big)^{\frac{1}{2}} \leq Ce^{-\beta'' t},$$

where β'' depends on β, β'. Also we have the metric of Σ_t satisfies

$$e^{-2c_0t}g_{ij} \to \bar{\phi}^2 g_{\mathbb{S}^n}.$$

exponentially fast. From (12) we know that

$$|\frac{\phi}{\phi'}h_i^j - \delta_i^j| = O(e^{-\beta_0 t}).$$

for some positive constant β_0 □

Declarations

Conflict of interest The authors declare no conflict of interest.

References

[1] B Andrews. *Motion of hypersurfaces by Gauss curvature*, Pacific Journal of Mathematics, 2000, 195(1): 1-34.

[2] S Brendle, P Guan, J Li. *An inverse curvature type hypersurface flow in space forms*, Manuscript.

[3] P Bryan, M N Ivaki, J Scheuer. *Parabolic approaches to curvature equations*, Nonlinear Analysis, 2020, 203: 112174.

[4] L C Evans. *Partial differential equations*, Graduate Studies in Mathematics, 1998.

[5] C Gerhardt. *Flow of nonconvex hypersurfaces into spheres*, Journal of Differential Geometry, 1990, 32(1): 299-314.

[6] C Gerhardt. *Curvature problems*, Series in Geometry and Topology, Internat Press, Somerville, MA, 2006.

[7] C Gerhardt. *Inverse curvature flows in hyperbolic space*, Journal of Differential Geometry, 2011, 89(3): 487-527.

[8] P Guan, J Li. *A mean curvature type flow in space forms*, International Mathematics Research Notices, 2015, 2015(13): 4716-4740.

[9] P Guan, C Lin, X Ma. *The existence of convex body with prescribed curvature measures*, International Mathematics Research Notices, 2009, 2009(11): 1947-1975.

[10] P Guan, C Ren, Z Wang. *Global C^2-Estimates for Convex Solutions of Curvature Equations*, Communications on Pure and Applied Mathematics, 2015, LXVI: 1287-1325.

[11] G Huisken, T Ilmanen. *The inverse mean curvature flow and the Riemannian Penrose inequality*, Journal of Differential Geometry, 2001, 59(3): 353-437.

[12] Q Li, W Sheng. *Closed hypersurfaces with prescribed Weingarten curvature in Riemannian manifolds*, Calculus of Variations and Partial Differential Equations, 2013, 48(1-2): 41-66.

[13] Q Li, W Sheng, X Wang. *Flow by Gauss curvature to the Aleksandrov and dual Minkowski problems*, Journal of the European Mathematical Society, 2017, 22(3): 893-923.

[14] H Li, Y Wei. *On inverse mean curvature flow in Schwarzschild space and Kottler space*, Calculus of Variations and Partial Differential Equations, 2017, 56(3): 62-82.

[15] V I Oliker. *The Gauss curvature and Minkowski problems in space forms*, Contemporary Mathematics, 1989, 101: 107-123.

[16] W Sheng, J Urbas, X Wang. *Interior curvature bounds for a class of curvature equations*, Duke Mathematical Journal, 2004, 123(2): 235-264.

[17] W Sheng, C Yi. *A class of anisotropic expanding curvature flows*, Discrete and Continuous Dynamical Systems, 2020, 40(4): 2017-2035.

[18] J Urbas. *An expansion of convex hypersurfaces*, Journal of Differential Geometry, 1991, 33(1): 91-125.

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