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Uniform isochronous center of a class of higher degree polynomial differential systems

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Abstract. In this paper, we give the necessary and sufficient conditions for a class of higher degree polynomial systems to have a uniform isochronous center. At the same time, we prove that for this system the composition conjecture is correct.

§1 Introduction

Consider differential system

$$\begin{cases} x' = -y + \Phi(x,y), \\ y' = x + \Psi(x,y), \end{cases} \tag{1}$$

where Φ and Ψ are real polynomial functions with a degree of n without constant nor linear terms. The singular point $O(0,0)$ is a center if there exists an open neighborhood U of O where all the orbits contained in $U\backslash\{O\}$ are periodic. For every $p \in U\backslash\{O\}$ if the period of the periodic orbit through p is a constant, then the point $O(0,0)$ is called an **isochronous center**. In literature [24], the authors have proved that if the system (1) has a center at $O(0,0)$, then this center is a uniform isochronous center if and only if doing a linear change of variables and a scaling of the time it can be written as the rigid system:

$$\begin{cases} x' = -y + xP(x,y), \\ y' = x + yP(x,y), \end{cases} \tag{2}$$

where $P(x,y) = \sum_{k=1}^{n-1} P_k(x,y)$, $P_k(x,y)$ is a homogeneous polynomial in x and y of degree k. The interest in the isochronous centers started in the XVII century with the works of [2]-[4], [11, 13, 16, 19] and references therein. The isochronous phenomena appear in many physical problems [14].

In polar coordinates the system (2) becomes

$$\frac{dr}{d\theta} = r\sum_{k=1}^{n-1} P_k(\theta) r^k, \tag{3}$$

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where $P_k(\theta)$ $(k = 1, 2, ..., n-1)$ are 2π-periodic functions. Therefore, the system (2) has a center at $O(0, 0)$ if and only if all solutions $r(\theta)$ of equation (3) near $r = 0$ are periodic. In such case, it is said that equation (3) has a center at $r = 0$ [2, 5, 25].

The center-focus problem asks about the conditions on the coefficients of Φ and Ψ under which the origin point of system (1) is a center. This is an interesting and difficult problem and is closely related to the Hilbert's sixteenth problem. In spite of all efforts, there is no general method to solve this problem. Up to now, only for quadratic systems and some special systems the center-focus problem has been solved [2, 15, 17, 21, 25] and others. However, for the higher degree polynomial differential systems, corresponding results are few.

Alwash and Lloyd [6, 7, 9] give the following simple sufficient condition for the Abel equation

$$\frac{dr}{d\theta} = r(R_1(\theta)r + R_2(\theta)r^2) \tag{4}$$

to have a center, where $R_1(\theta)$ and $R_2(\theta)$ are continuous 2π-periodic functions.

Theorem 1.1. [7,9] *If there exists a differentiable function $u(\theta)$ of period 2π such that*

$$R_1(\theta) = u'(\theta)\check{R}_1(u(\theta)), R_2(\theta) = u'(\theta)\check{R}_2(u(\theta))$$

for some continuous functions $\check{R}_1$ and $\check{R}_2$, then the Abel equation (4) has a center at $r = 0$.

The following statement presents a generalization of Theorem 1.1.

Theorem 1.2. [6,30] *If there exists a differentiable function $u(\theta)$ of period 2π such that*

$$P_i(\theta) = u'\check{P}_i(u), \ (i = 1, 2, ..., n-1)$$

for some continuous functions $\check{P}_i$ $(i = 1, 2, ..., n-1)$, then the equation (3) has a center at $r = 0$.

The condition in Theorem 1.1 (or Theorem 1.2) is called the Composition Condition. When an Abel equation (or the polynomial equation (3)) has a center because its coefficients satisfy the composition condition we will say that this equation has a CC-center. Obviously, the composition condition is the sufficient condition for $r = 0$ to be a center. A counterexample was presented in [9, 10] to demonstrate that the composition condition is not a necessary condition of a center. Whether the composition condition is the necessary and sufficient condition for the singular point to be a center? This problem is called Composition Conjecture, which first appeared in [7]. What kind of differential system is this conjecture right? Studying this problem has attracted the interest of many scholars. In [15, 20, 23] the authors have proved that for some Abel differential equations, the composition conjecture is valid. In the literatures [26, 27] for the system (2) with $P = P_2 + P_4$ or $P = P_2 + P_6$, the composition conjecture has been proved to be correct. Later, in [2, 18, 28, 29] the authors have used different methods to demonstrate that the composition conjecture is correct for the rigid system (2) with $P = P_1 + P_m$ or $P = P_2 + P_{2m}$, m is an arbitrary natural number. In literatures [2, 8, 12, 30] the authors applied different computational techniques to prove that for system (2) with $P = P_1 + P_2 + P_3$, the composition is correct. In [31, 32] , we have proved that for system (2) with $P = P_1 + P_2 + P_5$ $(P_1 \neq 0)$ or $P = P_1 + P_m + P_{2m+1}$ $(P_1 \neq 0)$, the composition conjecture is valid under several restrictions conditions. It's natural to ask, when $P_1 \equiv 0$, for system (2) with $P = P_2 + P_5$ or $P = P_2 + P_{2m+1}$ $(m > 1)$, is the composition conjecture true?

In this case, although the system (2) is very simple in form, it is much more difficult to deduce the necessary conditions of the center than when $P_1 \neq 0$. Meanwhile, to prove the obtained conditions to be sufficient is more complicated, even with the help of the computer, it can't lighten our burden. To find some useful methods is the key to open this problem. In this paper, we use some effective calculation techniques to get the necessary and sufficient conditions for the origin point of the following higher degree polynomial differential system

$$\begin{cases} x' = -y + x(P_2(x,y) + P_5(x,y)), \\ y' = x + y(P_2(x,y) + P_5(x,y)), \end{cases} \tag{5}$$

to be a center, where $P_k(x,y) = \sum_{i+j=k} p_{ij}x^i y^j$, $k = 2, 5$, $(p_{ij} \in R)$. At the same time, we prove that the composition conjecture is correct for its corresponding periodic differential equation

$$\frac{dr}{d\theta} = r(P_2(\cos\theta, \sin\theta)r^2 + P_5(\cos\theta, \sin\theta)r^5).$$

By this, we can derive all the focal values of the system (5) and they contain exactly six relations.

In the following we denote

$$P_k = P_k(\cos\theta, \sin\theta);\ \bar{P}_k = \int_0^\theta P_k(\cos\tau, \sin\tau)d\tau;\ \bar{\alpha}_k = \int_0^\theta \alpha_k(\tau)d\tau,$$

$$\overline{\bar{P}_2^{\,k} P_5} = \int_0^\theta (\int_0^\tau P_2(\cos t, \sin t)dt)^k P_5(\cos\tau, \sin\tau)d\tau;\ \overline{\bar{\alpha}_i \alpha_j} = \int_0^\theta \int_0^\tau \alpha_i(s)ds\alpha_j(\tau)d\tau, etc.$$

§2 Several Lemmas

Lemma 2.1. *The relations:*

(I) $\int_0^{2\pi} P_2 d\theta = 0$;
(II) $\int_0^{2\pi} \bar{P}_2 P_5 \bar{P}_5 d\theta = 0$;
(III) $\int_0^{2\pi} \bar{P}_2^2 P_5 \bar{P}_5 d\theta = 0$;
(IV) $7\int_0^{2\pi} \bar{P}_2^3 P_5 \bar{P}_5 d\theta + 3\int_0^{2\pi} \bar{P}_2^2 P_5 \overline{\bar{P}_2 P_5} d\theta = 0$;
(V) $3\int_0^{2\pi} \bar{P}_2^4 P_5 \bar{P}_5 d\theta + 2\int_0^{2\pi} \bar{P}_2^3 P_5 \overline{\bar{P}_2 P_5} d\theta = 0$;
(VI) $33\int_0^{2\pi} \bar{P}_2^5 P_5 \bar{P}_5 d\theta + 27\int_0^{2\pi} \bar{P}_2^4 P_5 \overline{\bar{P}_2 P_5} d\theta + 10\int_0^{2\pi} \bar{P}_2^3 P_5 \overline{\bar{P}_2^2 P_5} d\theta = 0$.

are respectively equivalent to the following relations:

(i) $\int_0^{2\pi} P_2 d\theta = 0$;
(ii) $\int_0^{2\pi} \hat{P}_2 P_5 \bar{P}_5 d\theta = 0$;
(iii) $\int_0^{2\pi} \hat{P}_2^2 P_5 \bar{P}_5 d\theta = 0$;
(iv) $7\int_0^{2\pi} \hat{P}_2^3 P_5 \bar{P}_5 d\theta + 3\int_0^{2\pi} \hat{P}_2^2 P_5 \overline{\hat{P}_2 P_5} d\theta = 0$;
(v) $3\int_0^{2\pi} \hat{P}_2^4 P_5 \bar{P}_5 d\theta + 2\int_0^{2\pi} \hat{P}_2^3 P_5 \overline{\hat{P}_2 P_5} d\theta = 0$;
(vi) $33\int_0^{2\pi} \hat{P}_2^5 P_5 \bar{P}_5 d\theta + 27\int_0^{2\pi} \bar{P}_2^4 P_5 \overline{\hat{P}_2 P_5} d\theta + 10\int_0^{2\pi} \hat{P}_2^3 P_5 \overline{\hat{P}_2^2 P_5} d\theta = 0$.

Where $\hat{P}_2 = A_2 \sin 2\theta - B_2 \cos 2\theta$, $A_2 = p_{20}$, $B_2 = \frac{1}{2}p_{11}$.

Proof. As $\int_0^{2\pi} P_2 d\theta = 0$, then $P_2 = A_2 \cos 2\theta + B_2 \sin 2\theta$ and $2\bar{P}_2 = \hat{P}_2 + B_2$. In view of P_5 is an odd polynomial in $\cos\theta, \sin\theta$ and the definite integral from 0 to 2π of an odd degree polynomial in $\cos\theta, \sin\theta$, is equal to zero. Thus

$$2\int_0^{2\pi}\bar{P}_2P_5\bar{P}_5d\theta=\int_0^{2\pi}(\hat{P}_2+B_2)P_5\bar{P}_5d\theta=\int_0^{2\pi}\hat{P}_2P_5\bar{P}_5d\theta,$$

which implies that the relation (II) is equivalent to (ii). By this we get

$$4\int_0^{2\pi}\bar{P}_2^2P_5\bar{P}_5d\theta=\int_0^{2\pi}(\hat{P}_2+B_2)^2P_5\bar{P}_5d\theta=\int_0^{2\pi}\hat{P}_2^2P_5\bar{P}_5d\theta,$$

so, the relations (III) and (iii) are equivalent.

Similarly, we can prove that the other relations are equivalent. □

Lemma 2.2. *Suppose that* $A_2^2+B_2^2\neq 0$ *and* $\int_0^{2\pi}P_2(\cos\theta,\sin\theta)d\theta=0$,

$$P_5\bar{P}_5=\sum_{k=1}^{5}\hat{a}_{2k}\cos 2k\theta+\hat{b}_{2k}\sin 2k\theta+\varsigma_1P_5,$$

$$A_{2k}\hat{b}_{2k}-B_{2k}\hat{a}_{2k}=0,\ (k=1,2,...,5). \tag{6}$$

Then

$$\sum_{k=1}^{5}\hat{a}_{2k}\cos 2k\theta+\hat{b}_{2k}\sin 2k\theta=P_2\sum_{k=0}^{4}\mu_k\bar{P}_2^k,\ P_5=u'(\theta)\check{P}_5(u(\theta)),$$

where $u(\theta)$ *is a* 2π*-periodic function,* $\check{P}_5$ *is a continuous function,* μ_k $(k=0,1,...,4)$ *and* ς_1 *are real constants,*

$$A_2=p_{20},B_2=\frac{1}{2}p_{11},\ A_4=-A_2B_2,B_4=\frac{1}{2}(A_2^2-B_2^2),$$

$$A_6=\frac{1}{4}A_2(3B_2^2-A_2^2),\ B_6=\frac{1}{4}B_2(B_2^2-3A_2^2),\ A_8=-A_4B_4,B_8=\frac{1}{2}(A_4^2-B_4^2),$$

$$A_{10}=-\frac{1}{16}A_2(10A_2^2B_2^2-A_2^4-5B_2^4),\ B_{10}=\frac{1}{16}B_2(5A_2^4-10A_2^2B_2^2+B_2^4).$$

Proof. As $\int_0^{2\pi}P_2(\cos\theta,\sin\theta)d\theta=0$, $P_2=A_2\cos 2\theta+B_2\sin 2\theta$,

$$\bar{P}_2=\frac{1}{2}(\hat{P}_2+B_2),\ \hat{P}_2=A_2\sin 2\theta-B_2\cos 2\theta,$$

then

$$P_2\hat{P}_2=A_4\cos 4\theta+B_4\sin 4\theta, \tag{7}$$

$$\hat{P}_2^2P_2=\frac{1}{4}\rho P_2+A_6\cos 6\theta+B_6\sin 6\theta,\ \rho=A_2^2+B_2^2, \tag{8}$$

$$\hat{P}_2^3P_2=\frac{1}{2}\rho(A_4\cos 4\theta+B_4\sin 4\theta)+A_8\cos 8\theta+B_8\sin 8\theta, \tag{9}$$

$$\hat{P}_2^4P_2=\frac{1}{8}\rho^2(A_2\cos 2\theta+B_2\sin 2\theta)+\frac{3}{4}\rho(A_6\cos 6\theta+B_6\sin 6\theta)+A_{10}\cos 10\theta+B_{10}\sin 10\theta. \tag{10}$$

By (6) we get $A_2\hat{b}_2-B_2\hat{a}_2=0$.

If $A_2\neq 0$, then $\hat{b}_2=\frac{B_2}{A_2}\hat{a}_2$ and $\hat{a}_2\cos 2\theta+\hat{b}_2\sin 2\theta=\frac{\hat{a}_2}{A_2}P_2$;

If $B_2\neq 0$, then $\hat{a}_2=\frac{A_2}{B_2}\hat{b}_2$ and $\hat{a}_2\cos 2\theta+\hat{b}_2\sin 2\theta=\frac{\hat{b}_2}{B_2}P_2$.

By the Lemma 3.6 of [29], we know that if $A_2^2+B_2^2\neq 0$, then $A_{2k}^2+B_{2k}^2\neq 0$, $(k=1,2,...)$. Using $A_4\hat{b}_4-B_4\hat{a}_4=0$ and (7) we get

If $A_4\neq 0$, then $\hat{b}_4=\frac{B_4}{A_4}\hat{a}_4$ and $\hat{a}_4\cos 4\theta+\hat{b}_4\sin 4\theta=\frac{\hat{a}_4}{A_4}P_2\hat{P}_2=\frac{\hat{a}_4}{A_4}P_2(2\bar{P}_2-B_2)$;

If $B_4\neq 0$, then $\hat{a}_4=\frac{A_4}{B_4}\hat{b}_4$ and $\hat{a}_4\cos 4\theta+\hat{b}_4\sin 4\theta=\frac{\hat{b}_4}{B_4}P_2\hat{P}_2=\frac{\hat{b}_4}{B_4}P_2(2\bar{P}_2-B_2)$.

Similar to above and using (6) and (8)-(10) we can get

$$\sum_{k=1}^{5}\hat{a}_{2k}\cos 2k\theta+\hat{b}_{2k}\sin 2k\theta=P_2\sum_{k=0}^{4}\mu_k\bar{P}_2^k,$$

thus, $\bar{P}_5^2 = \sum_{k=0}^{4} \frac{2}{k+1}\mu_k \bar{P}_2^{k+1} + \varsigma_1 \bar{P}_5$.

On the other hand, by $\int_0^{2\pi} P_2(\cos\theta, \sin\theta)d\theta = 0$.

If $p_{20} \neq 0$, $P_2 = \frac{p_{20}}{\epsilon}u'(\theta)u(\theta)$, $u(\theta) = \cos\theta + \epsilon\sin\theta$, $\epsilon = \frac{p_{11}+\sqrt{p_{11}^2+4p_{20}^2}}{2p_{20}}$;

If $p_{20} = 0$, then $P_2 = p_{11}u(\theta)u'(\theta)$, $u(\theta) = \sin\theta$.

Therefore, $\bar{P}_2 = \mu(u^2(\theta) - u^2(0))$ and $P_5 = \bar{P}_5' = u'(\theta)\check{P}(u(\theta))$, $u(\theta + 2\pi) = u(\theta)$, μ is a constant. □

§3 Main Theorem

Consider 2π-periodic equation

$$\frac{dr}{d\theta} = r(P_2(\cos\theta, \sin\theta)r^2 + P_5(\cos\theta, \sin\theta)r^5), \tag{11}$$

where $P_k = \sum_{i+j=k} p_{ij}\cos^i\theta\sin^j\theta$, p_{ij} are real numbers.

In the following, we always assume that P_2 and P_5 are not equal to zero. In the case of $P_2 = 0$, by (11) we see that all its solutions are 2π-periodic, i.e., $r = 0$ is a center. In the case of $P_5 = 0$, it is easy to get that $r = 0$ is a center if and only if $\int_0^{2\pi} P_2(\theta)d\theta = 0$.

Theorem 3.1. *$r = 0$ is a center of* (11)*, if and only if the conditions (I)−(VI) of Lemma 2.1 are satisfied. Furthermore, this center is a CC-center and uniform isochronous center.*

Proof. Necessity:

Taking $\rho = \frac{r}{(1+2\bar{P}_2r^2)^{\frac{1}{2}}}$ and applying the Langrange-Bürman formula [1], the equation (11) becomes

$$\frac{d\rho}{d\theta} = P_5\rho^6\sum_{k=0}^{\infty}\alpha_k\rho^{2k}, \tag{12}$$

where $\alpha_k = \frac{(2k+1)!!}{k!}\bar{P}_2^kP_5\,(k = 1, 2, 3, ...)$, $\alpha_0 = P_5$.

Obviously, if $\int_0^{2\pi} P_2 d\theta = 0$, then $r = 0$ is a center of (11) if and only if $\rho = 0$ is a center of (12). Let $\rho(\theta, c)$ be the solution of (12) such that $\rho(0, c) = c\,(0 < c \ll 1)$. We write

$$\rho(\theta, c) = c\sum_{n=0}^{\infty} r_n(\theta)c^n,$$

where $r_0(0) = 1$ and $r_n(0) = 0$ for $n \geq 1$. The solution $\rho = 0$ of (12) is a center if and only if $\rho(\theta + 2\pi, c) = \rho(\theta, c)$, i.e., $r_0(2\pi) = 1$, $r_n(2\pi) = 0\,(n = 1, 2, 3, ...)$ [7].

Substituting $\rho(\theta, c)$ into (12) and equating the corresponding coefficients of $c^i\,(i = 0, 1, 2, 3, 4)$ we obtain $r_0(\theta) = 1$, $r_i(\theta) = 0(i = 1, 2, 3, 4)$. Thus, rewriting

$$\rho = c(1 + c^5h),\ h = \sum_{i=0}^{\infty} h_i(\theta)c^i,\ h_i(0) = 0,\ (i = 0, 1, 2...).$$

Substituting it into (12) we get

$$\sum_{i=0}^{\infty} h_i'(\theta)c^i = \sum_{i=0}^{\infty}\alpha_i c^{2i}\sum_{j=0}^{i+6} C_{i+6}^j h^j c^{5j}. \tag{13}$$

Equating the corresponding coefficients of $c^i\,(i = 0, 1, 2, ..., 15)$ of (13) yields

$$h_0 = \bar{\alpha}_0,\ h_1 = 0,\ h_2 = \bar{\alpha}_1,\ h_3 = 0,\ h_4 = \bar{\alpha}_2,\ h_5 = 3\bar{\alpha}_0^2,\ h_6 = \bar{\alpha}_3,$$

$$h_7 = 6\bar{\alpha}_0\bar{\alpha}_1 + 2\overline{\bar{\alpha}_0\alpha_1},\ h_8 = \bar{\alpha}_4,\ h_9 = 6\bar{\alpha}_0\bar{\alpha}_2 + 4\overline{\bar{\alpha}_0\alpha_2} + 4\bar{\alpha}_1^2,$$
$$h_{10} = \bar{\alpha}_5 + 11\bar{\alpha}_0^3,\ h_{11} = 6\bar{\alpha}_0\bar{\alpha}_3 + 6\overline{\bar{\alpha}_0\alpha_3} + 8\bar{\alpha}_1\bar{\alpha}_2 + 2\overline{\bar{\alpha}_1\alpha_2},$$
$$h_{12} = \bar{\alpha}_6 + 33\bar{\alpha}_0^2\bar{\alpha}_1 + 12\bar{\alpha}_0\overline{\bar{\alpha}_0\alpha_1} + 7\overline{\bar{\alpha}_0^2\alpha_1},$$
$$h_{13} = 6\bar{\alpha}_0\bar{\alpha}_4 + 8\bar{\alpha}_1\bar{\alpha}_3 + 4\overline{\bar{\alpha}_1\alpha_3} + 8\overline{\bar{\alpha}_0\alpha_4} + 5\bar{\alpha}_2^2,$$
$$h_{14} = \bar{\alpha}_7 + 33\bar{\alpha}_0^2\bar{\alpha}_2 + 24\bar{\alpha}_0\overline{\bar{\alpha}_0\alpha_2} + 18\overline{\bar{\alpha}_0^2\alpha_2} + 10\overline{\bar{\alpha}_0\bar{\alpha}_1\alpha_1} + 16\bar{\alpha}_1\overline{\bar{\alpha}_0\alpha_1} + 39\bar{\alpha}_1^2\bar{\alpha}_0,$$
$$h_{15} = 6\bar{\alpha}_0\bar{\alpha}_5 + 10\overline{\bar{\alpha}_0\alpha_5} + 44\bar{\alpha}_0^4 + 8\bar{\alpha}_1\bar{\alpha}_4 + 6\overline{\bar{\alpha}_1\alpha_4} + 10\bar{\alpha}_2\bar{\alpha}_3 + 2\overline{\bar{\alpha}_2\alpha_3}.$$

As, the solution $\rho = 0$ of (12) is a center if and only if $\rho(\theta + 2\pi, c) = \rho(\theta, c)$, i.e., $h_i(2\pi) = 0\,(i = 0, 1, 2, 3, ...)$. In view of the definite integral from 0 to 2π of an odd degree polynomial in $\cos\theta\sin\theta$ is equal to zero, from the expressions of $h_i\,(i = 0, 1, 2, ..., 15)$ above, we can deduce that if $h_i(2\pi) = 0\,(i = 0, 1, 2, ..., 15)$ then

$$\int_0^{2\pi} \bar{\alpha}_0\alpha_1 d\theta = 0;$$
$$\int_0^{2\pi} \bar{\alpha}_0\alpha_2 d\theta = 0;$$
$$3\int_0^{2\pi} \bar{\alpha}_0\alpha_3 d\theta + \int_0^{2\pi} \bar{\alpha}_1\alpha_2 d\theta = 0;$$
$$\int_0^{2\pi} \bar{\alpha}_1\alpha_3 d\theta + 2\int_0^{2\pi} \bar{\alpha}_0\alpha_4 d\theta = 0;$$
$$5\int_0^{2\pi} \bar{\alpha}_0\alpha_5 d\theta + 3\int_0^{2\pi} \bar{\alpha}_1\alpha_4 d\theta + \int_0^{2\pi} \bar{\alpha}_2\alpha_3 d\theta = 0.$$

Simplifying these relations we get that if the solution $r = 0$ of (11) is a center then the conditions (I)−(VI) are satisfied. Therefore, the conditions (I)-(VI) are necessary for $r = 0$ to be the center of equation (11).

Sufficiency:

In the following we denote:

$$X_k = A_{2k}\hat{b}_{2k} - B_{2k}\hat{a}_{2k}\,(k = 1, 2, .., 5), X = A_2 I_1 + B_2 I_2,\ I_1 = a_3a_5 + b_3b_5, I_2 = a_3b_5 - a_5b_3.$$

By Theorem 1.2 and Lemma 2.2, if $X_k = 0\,(k = 1, 2, 3, 4, 5)$, then $r = 0$ is a center of (11). In the following, we will check that $X_k = 0\,(k = 1, 2, 3, 4, 5)$ under conditions (I)-(VI).

As $\int_0^{2\pi} P_2 d\theta = 0$, $p_{20} + p_{02} = 0$ and $P_2 = A_2\cos 2\theta + B_2\sin 2\theta$ and

$$\hat{P}_2 = A_2\sin 2\theta - B_2\cos 2\theta, \tag{14}$$
$$\hat{P}_2^2 = A_4\sin 4\theta - B_4\cos 4\theta + \frac{1}{2}\rho, \tag{15}$$
$$\hat{P}_2^3 = \frac{3}{4}\rho\hat{P}_2 + A_6\sin 6\theta - B_6\cos 6\theta, \tag{16}$$
$$\hat{P}_2^4 = \rho(A_4\sin 4\theta - B_4\cos 4\theta) + A_8\sin 8\theta - B_8\cos 8\theta + \frac{3}{8}\rho^2, \tag{17}$$
$$\hat{P}_2^5 = \frac{5}{8}\rho^2(A_2\sin 2\theta - B_2\cos 2\theta) + \frac{5}{4}\rho(A_6\sin 6\theta - B_6\cos 6\theta) + A_{10}\sin 10\theta - B_{10}\cos 10\theta, \tag{18}$$

where $A_{2k}, B_{2k}(k = 1, 2, ..., 5)$ and ρ are the same as they in Lemma 2.2.

Expanding P_5 to the Fourier series

$$P_5 = \sum_{k=0}^{2}(a_{2k+1}\cos(2k+1)\theta + b_{2k+1}\sin(2k+1)\theta),$$

where $a_{2k+1} = \frac{1}{\pi}\int_0^{2\pi} P_5 \cos(2k+1)\theta d\theta$, $b_{2k+1} = \frac{1}{\pi}\int_0^{2\pi} P_5 \sin(2k+1)\theta d\theta$, then

$$\bar{P}_5 = \sum_{k=0}^{2} \frac{1}{2k+1}(a_{2k+1}\sin(2k+1)\theta - b_{2k+1}\cos(2k+1)\theta) + \varsigma_1,$$

$$P_5\bar{P}_5 = \sum_{k=0}^{5}(\hat{a}_{2k}\cos 2k\theta + \hat{b}_{2k}\sin 2k\theta) + \varsigma_1 P_5, \tag{19}$$

where ς_1 is a constant,

$\hat{a}_2 = -a_1b_1 + \frac{1}{3}(a_1b_3 - a_3b_1) + \frac{1}{15}I_2$, $\hat{b}_2 = \frac{1}{2}(a_1^2 - b_1^2) - \frac{1}{3}(a_1a_3 + b_1b_3) - \frac{1}{15}I_1$,
$\hat{a}_4 = -\frac{2}{3}(a_1b_3 + a_3b_1) + \frac{2}{5}(a_1b_5 - a_5b_1)$, $\hat{b}_4 = \frac{2}{3}(a_1a_3 - b_1b_3) - \frac{2}{5}(a_1a_5 + b_1b_5)$,
$\hat{a}_6 = -\frac{1}{3}a_3b_3 - \frac{3}{5}(a_1b_5 + a_5b_1)$, $\hat{b}_6 = \frac{1}{6}(a_3^2 - b_3^2) - \frac{3}{5}(b_1b_5 - a_1a_5)$,
$\hat{a}_8 = -\frac{4}{15}(a_3b_5 + a_5b_3)$, $\hat{b}_8 = \frac{4}{15}(a_3a_5 - b_3b_5)$, $\hat{a}_{10} = -\frac{1}{5}a_5b_5$, $\hat{b}_{10} = \frac{1}{10}(a_5^2 - b_5^2)$.

By Lemma 2.1, the conditions (I)-(VI) are equivalent to conditions (i)$-$(vi). Using (14) and (19) and condition (ii) we get

$$A_2\hat{b}_2 - B_2\hat{a}_2 = 0. \tag{20}$$

Applying (15) and (19) and condition (iii) we obtain

$$A_4\hat{b}_4 - B_4\hat{a}_4 = 0. \tag{21}$$

Using (14) we have

$$\hat{P}_2P_5 = \sum_{k=0}^{3}(d_{2k+1}\cos(2k+1)\theta + e_{2k+1}\sin(2k+1)\theta),$$

$$\overline{\hat{P}_2P_5} = \sum_{k=0}^{3}\frac{1}{2k+1}(d_{2k+1}\sin(2k+1)\theta - e_{2k+1}\cos(2k+1)\theta) + \varsigma_2,$$

where ς_2 is a constant and

$d_1 = \frac{1}{2}(A_2(b_1 + b_3) - B_2(a_1 + a_3)), e_1 = \frac{1}{2}(A_2(a_1 - a_3) - B_2(b_3 - b_1)),$
$d_3 = \frac{1}{2}(A_2(-b_1 + b_5) - B_2(a_1 + a_5)), e_3 = \frac{1}{2}(A_2(a_1 - a_5) - B_2(b_1 + b_5),$
$d_5 = \frac{1}{2}(-A_2b_3 - B_2a_3)$, $e_5 = \frac{1}{2}(A_2a_3 - B_2b_3)$, $d_7 = \frac{1}{2}(-A_2b_5 - B_2a_5)$, $e_7 = \frac{1}{2}(a_5A_2 - b_5B_2)$.

So,

$$\hat{P}_2P_5\overline{\hat{P}_2P_5} = \sum_{k=2}^{7}(\hat{d}_{2k}\cos 2k\theta + \hat{e}_{2k}\sin 2k\theta) + \varsigma_2\hat{P}_2P_5, \tag{22}$$

where

$\hat{d}_2 = \rho(\frac{1}{6}a_1b_1 - \frac{7}{30}(a_1b_3 - a_3b_1) + \frac{19}{210}I_2) + A_4(\frac{1}{4}(b_1^2 - a_1^2 + a_3^2 - b_3^2) + \frac{1}{6}(a_1a_3 + b_1b_3 + b_1b_5 - a_1a_5) + \frac{1}{30}I_1) + B_4(\frac{1}{2}(a_3b_3 - a_1b_1) + \frac{1}{6}(a_1b_3 - a_3b_1 - a_1b_5 - a_5b_1) + \frac{1}{30}I_2),$

$\hat{e}_2 = \rho(\frac{1}{12}(b_1^2 - a_1^2) + \frac{7}{30}(a_1a_3 + b_1b_3) - \frac{19}{210}I_1) + A_4(\frac{1}{2}(a_1b_1 + a_3b_3) - \frac{1}{6}(a_5b_1 + a_1b_5 + a_1b_3 - a_3b_1) - \frac{1}{30}I_2) + B_4(\frac{1}{4}(b_1^2 + b_3^2 - a_1^2 - a_3^2) - \frac{1}{6}(b_1b_5 - a_1a_5 - a_1a_3 - b_1b_3) + \frac{1}{30}I_1),$

$\hat{d}_4 = -\rho(\frac{1}{15}(a_1b_3 + a_3b_1) + \frac{1}{7}(a_1b_5 - a_5b_1)) + \frac{1}{3}A_4(a_3a_5 - b_3b_5 - a_1^2 - b_1^2 + \frac{3}{5}(a_3^2 + b_3^2) + \frac{1}{7}(a_5^2 + b_5^2)) + \frac{1}{3}B_4(a_3b_5 + a_5b_3),$

$\hat{e}_4 = \rho(\frac{1}{7}(a_1a_5 + b_1b_5) + \frac{1}{15}(a_1a_3 - b_1b_3)) + \frac{1}{3}A_4(a_5b_3 + a_3b_5) + \frac{1}{3}B_4(b_3b_5 - a_3a_5 - a_1^2 - b_1^2 + \frac{3}{5}(a_3^2 + b_3^2) + \frac{1}{7}(a_5^2 + b_5^2)),$

$\hat{d}_6 = \frac{1}{2}\rho(\frac{1}{21}(a_5b_1 + a_1b_5) - \frac{3}{5}a_3b_3) + A_4(\frac{1}{12}(b_1^2 - a_1^2 + a_5^2 - b_5^2) - \frac{3}{10}(a_1a_3 + b_1b_3) + \frac{3}{14}I_1) + B_4(\frac{1}{6}(a_1b_1 + a_5b_5) + \frac{3}{10}(a_1b_3 - a_3b_1) - \frac{3}{14}I_2),$

$\hat{e}_6 = \frac{1}{2}\rho(\frac{1}{21}(b_1b_5 - a_1a_5) + \frac{3}{10}(a_3^2 - b_3^2)) + \frac{1}{2}A_4(\frac{1}{3}(a_5b_5 - a_1b_1) + \frac{3}{7}I_2 + \frac{3}{5}(a_3b_1 - a_1b_3) + +\frac{1}{4}B_4(\frac{1}{3}(b_1^2 - a_1^2 + b_5^2 - a_5^2) + \frac{6}{7}I_1) - \frac{6}{5}(a_1a_3 + b_1b_3)).$

Using (14) and (22), (16) and (19) and condition (iv) we get

$$7(A_6\hat{b}_6 - B_6\hat{a}_6) + 3(A_2\hat{e}_2 - B_2\hat{d}_2) = 0. \tag{23}$$

By (17) and (19), (15) and (22) and condition (v) we obtain

$$3(A_8\hat{b}_8 - B_8\hat{a}_8) + 2(A_4\hat{e}_4 - B_4\hat{d}_4) = 0. \tag{24}$$

Applying (15) we have

$\hat{P}_2^2P_5 = (A_4\sin 4\theta - B_4\cos 4\theta + \frac{1}{2}\rho)\sum_{k=0}^{2}(a_{2k+1}\cos(2k+1)\theta + b_{2k+1}\sin(2k+1)\theta)$
$= \sum_{k=0}^{4} s_{2k+1}\cos(2k+1)\theta + t_{2k+1}\sin(2k+1)\theta,$

$\overline{\hat{P}_2^2P_5} = \sum_{k=0}^{4}\frac{1}{2k+1}(s_{2k+1}\cos(2k+1)\theta + t_{2k+1}\sin(2k+1)\theta) + \varsigma_3,$

where ς_3 is a constant and

$s_1 = \frac{1}{2}(A_4(b_3 + b_5) - B_4(a_3 + a_5) + \rho a_1),\ t_1 = \frac{1}{2}(A_4(a_3 - a_5) - B_4(b_5 - b_3) + \rho b_1),$
$s_3 = \frac{1}{2}(A_4b_1 - B_4a_1 + \rho a_3),\ t_3 = \frac{1}{2}(A_4a_1 + B_4b_1 + \rho b_3),$
$s_5 = \frac{1}{2}(-A_4b_1 - a_1B_4 + \rho a_5),\ t_5 = \frac{1}{2}(A_4a_1 - B_4b_1 + \rho b_5),$
$s_7 = \frac{1}{2}(-A_4b_3 - B_4a_3),\ t_7 = \frac{1}{2}(A_4a_3 - B_4b_3),\ s_9 = \frac{1}{2}(-A_4b_5 - B_4a_5),\ t_5 = \frac{1}{2}(A_4a_5 - B_4b_5).$

Thus,

$$\hat{P}_2^2P_5\overline{\hat{P}_2^2P_5} = \sum_{k=1}^{9}(\hat{s}_{2k}\cos 2k\theta + \hat{t}_{2k}\sin 2k\theta) + \varsigma_3\hat{P}_2^2P_5, \tag{25}$$

where

$\hat{s}_2 = -s_1t_1 + \frac{1}{3}(s_1t_3 - s_3t_1) + \frac{1}{15}(s_3t_5 - s_5t_3) + \frac{1}{35}(s_5t_7 - s_7t_5) + \frac{1}{63}(s_7t_9 - t_7s_9),$
$\hat{t}_2 = \frac{1}{2}(s_1^2 - t_1^2) - \frac{1}{3}(s_1s_3 + t_1t_3) - \frac{1}{15}(s_3s_5 + t_3t_5) - \frac{1}{35}(s_5s_7 + t_5t_7) - \frac{1}{63}(s_7s_9 + t_7t_9).$

After calculating and simplifying we get

$\hat{s}_2 = A_8(-\frac{1}{4}(a_3^2 + b_5^2 - b_3^2 - a_5^2) + \frac{1}{6}(a_1a_5 - b_1b_5)) + B_8(\frac{1}{2}(a_5b_5 - a_3b_3) + \frac{1}{6}(a_1b_5 + a_5b_1)) + \rho A_4(-\frac{7}{30}(b_1b_3 + a_1a_3 + b_1b_5 - a_1a_5) + \frac{1}{12}(b_3^2 - a_3^2 + a_1^2 - b_1^2) + \frac{19}{210}I_1) + \rho B_4(-\frac{7}{30}(a_1b_3 - a_3b_1 - a_5b_1 - a_1b_5) + \frac{1}{6}(a_1b_1 - a_3b_3) + \frac{19}{210}I_2) + \rho^2(-\frac{113}{2520}I_2 - \frac{29}{120}a_1b_1 + \frac{89}{840}(a_1b_3 - a_3b_1)),$

$\hat{t}_2 = A_8(\frac{1}{2}(a_3b_3 + a_5b_5) - \frac{1}{6}(a_1b_5 + a_5b_1)) + B_8(\frac{1}{4}(b_3^2 - a_3^2 + b_5^2 - a_5^2) - \frac{1}{6}(b_1b_5 - a_1a_5)) + \rho A_4(\frac{7}{30}(a_1b_3 - a_3b_1 + a_1b_5 + a_5b_1) - \frac{1}{6}(a_3b_3 + a_1b_1) - \frac{19}{210}I_2) + \rho B_4(\frac{7}{30}(b_5b_1 - a_1a_5 - a_1a_3 - b_1b_3) - \frac{1}{12}(b_1^2 - a_1^2 + b_3^2 - a_3^2) + \frac{19}{210}I_1) + \rho^2(\frac{113}{2520}I_1 + \frac{29}{240}(a_1^2 - b_1^2) - \frac{89}{840}(a_3a_1 + b_3b_1)).$

Using (18) and (19), (16) and (22) , (14) and (25) and condition (vi) and (23) we obtain

$$-6\rho(A_6\hat{b}_6 - B_6\hat{a}_6) + 33(A_{10}\hat{b}_{10} - B_{10}\hat{a}_{10}) + 27(A_6\hat{e}_6 - B_6\hat{d}_6) + 10(A_2\hat{t}_2 - B_2\hat{s}_2) = 0. \tag{26}$$

According to Lemma 3.6 of [29] we see that if $A_2^2+B_2^2\neq 0$, then $A_{2k}^2+B_{2k}^2\neq 0$. By (21) we obtain

$$\frac{\hat{a}_4}{A_4}=\frac{\hat{b}_4}{B_4}=\frac{2}{15}k_1, \tag{27}$$

where k_1 is a constant.

Case I. If $\delta=25(a_3^2+b_3^2)-9(a_5^2+b_5^2)\neq 0$.

From (27) follows that

$$a_1=\lambda_1(-A_4(5b_3+3b_5)+B_4(3a_5+5a_3)),\ b_1=\lambda_1(-A_4(5a_3-3a_5)+B_4(3b_5-5b_3)), \tag{28}$$

where$\lambda_1=\frac{k_1}{\delta}$. Using (28) rewriting $\hat{a}_2$ and $\hat{b}_2$ as follow

$$\hat{a}_2=(\frac{1}{15}-\frac{45}{4}\rho^2\lambda_1^2)I_2-450\lambda_1^2(B_8\hat{a}_{10}-A_8\hat{b}_{10})-150\lambda_1^2(A_8\hat{b}_6-B_8\hat{a}_6)+10\lambda_1(A_4\hat{b}_6-B_4\hat{a}_6)-$$
$$(\frac{225}{4}\lambda_1^3\rho^2+\lambda_1)(A_4I_1+B_4I_2)+2700\lambda_1^3(B_{12}\hat{a}_{10}-A_{12}\hat{b}_{10}), \tag{29}$$

$$\hat{b}_2=-(\frac{1}{15}-\frac{45}{4}\rho^2\lambda_1^2)I_1-450\lambda_1^2(B_8\hat{b}_{10}+A_8\hat{a}_{10})-150\lambda_1^2(B_8\hat{b}_6+A_8\hat{a}_6)-10\lambda_1(A_4\hat{a}_6+B_4\hat{b}_6)+$$
$$(\frac{225}{4}\lambda_1^3\rho^2+\lambda_1)(A_4I_2-B_4I_1)-2700\lambda_1^3(A_{12}\hat{a}_{10}+B_{12}\hat{b}_{10}). \tag{30}$$

Substituting (29) and (30) into (20), we obtain

$$5\lambda_1(4-15\lambda_1\rho)X_3+900\lambda_1^2(1-3\lambda_1\rho)X_5+(\frac{45}{4}\lambda_1^2\rho^2-\frac{225}{8}\lambda_1^3\rho^3-\frac{1}{2}\rho\lambda_1-\frac{1}{15})X=0. \tag{31}$$

Substituting (28) into the previous expressions of $\hat{d}_2$ and $\hat{e}_2$ we get

$$\hat{d}_2=-\tfrac{1}{6}\rho\hat{a}_2+\tfrac{32}{315}\rho I_2+\tfrac{1}{2}(B_4\hat{a}_2-A_4\hat{b}_2)+\tfrac{3}{2}(A_4\hat{b}_6-B_4\hat{a}_6)+\tfrac{8}{15}\rho\lambda_1(A_4I_1+B_4I_2)-$$
$$\tfrac{16}{3}\rho\lambda_1(A_4\hat{b}_6-B_4\hat{a}_6)+4\rho^3\lambda_1^2I_2+192\rho\lambda_1^2(B_8\hat{a}_{10}-A_8\hat{b}_{10})-\tfrac{4}{3}\lambda_1\rho^2I_2-64\lambda_1(B_8\hat{a}_{10}-A_8\hat{b}_{10}),$$

$$\hat{e}_2=-\tfrac{1}{6}\rho\hat{b}_2-\tfrac{32}{315}\rho I_1-\tfrac{1}{2}(A_4\hat{a}_2+B_4\hat{b}_2)-\tfrac{3}{2}(A_4\hat{a}_6+B_4\hat{b}_6)+\tfrac{8}{15}\rho\lambda_1(B_4I_1-A_4I_2)+$$
$$\tfrac{16}{3}\rho\lambda_1(A_4\hat{a}_6+B_4\hat{b}_6)-4\rho^3\lambda_1^2I_1+192\rho\lambda_1^2(B_8\hat{b}_{10}+A_8\hat{a}_{10})+\tfrac{4}{3}\lambda_1\rho^2I_1-64\lambda_1(A_8\hat{a}_{10}+B_8\hat{b}_{10}).$$

Applying these relations we get

$$A_2\hat{e}_2-B_2\hat{d}_2=(3-\frac{32}{3}\rho\lambda_1)X_3+128\lambda_1(1-3\rho\lambda_1)X_5+4\rho(-\frac{8}{315}+\frac{2}{5}\lambda_1\rho-\lambda_1^2\rho^2)X. \tag{32}$$

Substituting (32) into (23) we get

$$4(1-2\rho\lambda_1)X_3+96\lambda_1(1-3\rho\lambda_1)X_5+3\rho(-\frac{8}{315}+\frac{2}{5}\lambda_1-\lambda_1^2\rho^2)X=0. \tag{33}$$

Applying (28) and rewriting $\hat{d}_4$ and $\hat{e}_4$ as follows

$$\hat{d}_4=(A_4\hat{b}_8-B_4\hat{a}_8)(\tfrac{5}{4}-\tfrac{24}{7}\rho\lambda_1)+\rho\lambda_1A_4(\tfrac{1}{3}(a_3^2+b_3^2)+\tfrac{3}{7}(a_5^2+b_5^2))+A_4(-\tfrac{1}{3}(a_1^2+b_1^2)+\tfrac{1}{5}(a_3^2+b_3^2)+\tfrac{1}{21}(a_5^2+b_5^2)),$$

$$\hat{e}_4=(A_4\hat{a}_8+B_4\hat{b}_8)(-\tfrac{5}{4}+\tfrac{24}{7}\rho\lambda_1)+\rho\lambda_1B_4(\tfrac{1}{3}(a_3^2+b_3^2)+\tfrac{3}{7}(a_5^2+b_5^2))+B_4(-\tfrac{1}{3}(a_1^2+b_1^2)+\tfrac{1}{5}(a_3^2+b_3^2)+\tfrac{1}{21}(a_5^2+b_5^2)).$$

By this we get

$$A_4\hat{e}_4-B_4\hat{d}_4=(\frac{48}{7}\rho\lambda_1-\frac{5}{2})X_4.$$

Substituting it into (24) we get

$$(7-12\rho\lambda_1)X_4=0. \tag{34}$$

Substituting (28) into the previous expressions of $\hat{d}_6$ and $\hat{e}_6$ we get

$\hat{d}_6 = \hat{a}_6(\frac{9}{10}\rho - \frac{8}{3}\rho^2\lambda_1) + (A_4\hat{b}_{10} - B_4\hat{a}_{10})(\frac{5}{6} + \frac{592}{35}\rho\lambda_1 - 48\rho^2\lambda_1^2) - \frac{1}{6}(A_4\hat{b}_2 + B_4\hat{a}_2) + +(A_4I_1 - B_4I_2)(\frac{64}{315} - \frac{296}{105}\rho\lambda_1 + 8\rho^2\lambda_1^2) + \frac{32}{15}\lambda_1(A_8I_1 + B_8I_2),$

$\hat{e}_6 = \hat{b}_6(\frac{9}{10}\rho - \frac{8}{3}\rho^2\lambda_1) + (A_4\hat{a}_{10} + B_4\hat{b}_{10})(-\frac{5}{6} - \frac{592}{35}\rho\lambda_1 + 48\rho^2\lambda_1^2) + \frac{1}{6}(A_4\hat{a}_2 - B_4\hat{b}_2) + +(A_4I_2 + B_4I_1)(\frac{64}{315} - \frac{296}{105}\rho\lambda_1 + 8\rho^2\lambda_1^2) + \frac{32}{15}\lambda_1(-A_8I_2 + B_8I_1).$

By this we get

$$A_6\hat{e}_6 - B_6\hat{d}_6 = (\frac{9}{10}\rho - \frac{8}{3}\rho^2\lambda_1)X_3 + (\frac{5}{3} + \frac{1184}{35}\rho\lambda_1 - 96\rho^2\lambda_1^2)X_5 + \rho^2(\frac{44}{105}\rho\lambda_1 - \rho^2\lambda_1^2 - \frac{8}{315})X. \quad (35)$$

Using (28) we get

$\hat{s}_2 = (A_8\hat{b}_{10} - B_8\hat{a}_{10})(\frac{5}{2} + \frac{192}{7}\rho^2\lambda_1^2 - 32\rho\lambda_1) + (A_8\hat{b}_6 - B_8\hat{a}_6)(-\frac{3}{2} + \frac{32}{3}\rho\lambda_1) + \frac{1}{6}\rho(A_4\hat{b}_2 - B_4\hat{a}_2) + (A_4\hat{b}_6 - B_4\hat{a}_6)(-\frac{1}{2}\rho + \frac{16}{21}\rho^2\lambda_1) + (A_4I_1 + B_4I_2)(\frac{32}{315}\rho - \frac{26}{35}\rho^2\lambda_1 + 4\rho^3\lambda_1^2) + (A_{12}\hat{b}_{10} - B_{12}\hat{a}_{10})(-32\lambda_1 + 192\rho\lambda_1^2) + \frac{29}{120}\rho^2\hat{a}_2 + \rho^2I_2(-\frac{32}{525} + \frac{4}{5}\rho\lambda_1 - \frac{4}{7}\rho^2\lambda_1^2),$

$\hat{t}_2 = -(A_8\hat{a}_{10} + B_8\hat{b}_{10})(\frac{5}{2} + \frac{192}{7}\rho^2\lambda_1^2 - 32\rho\lambda_1) + (A_8\hat{a}_6 + B_8\hat{b}_6)(-\frac{3}{2} + \frac{32}{3}\rho\lambda_1) + \frac{1}{6}\rho(A_4\hat{a}_2 + B_4\hat{b}_2) - (A_4\hat{a}_6 + B_4\hat{b}_6)(-\frac{1}{2}\rho + \frac{16}{21}\rho^2\lambda_1) - (A_4I_2 - B_4I_1)(\frac{32}{315}\rho - \frac{26}{35}\rho^2\lambda_1 + 4\rho^3\lambda_1^2) + (B_{12}\hat{b}_{10} + A_{12}\hat{a}_{10})(-32\lambda_1 + 192\rho\lambda_1^2) + \frac{29}{120}\rho^2\hat{b}_2 - \rho^2I_1(-\frac{32}{525} + \frac{4}{5}\rho\lambda_1 - \frac{4}{7}\rho^2\lambda_1^2),$

where $A_{12} = \frac{1}{2}A_4(3B_4^2 - A_4^2)$, $B_{12} = \frac{1}{2}B_4(B_4^2 - 3A_4^2)$. Applying these relations we obtain

$$A_2\hat{t}_2 - B_2\hat{s}_2 = (\frac{48}{7}\rho^2\lambda_1 - \frac{7}{4}\rho)X_3 + (5 - 96\rho\lambda_1 + \frac{1728}{7}\rho^2\lambda_1^2)X_5 + (\frac{176}{1575}\rho^2 - \frac{41}{35}\rho^3\lambda_1 + \frac{18}{7}\rho^4\lambda_1^2)X. \quad (36)$$

Substituting (35) and (36) into (26) we get

$$4\rho(\frac{1}{5} - \frac{6}{7}\rho\lambda_1)X_3 + 32(4 - \frac{51}{35}\rho\lambda_1 - \frac{27}{7}\rho^2\lambda_1^2)X_5 + \frac{1}{7}\rho^2(\frac{136}{45} - \frac{14}{5}\rho\lambda_1 - 9\rho^2\lambda_1^2)X = 0. \quad (37)$$

By (20) and (21) we see that $X_1 = X_2 = 0$. According to Lemma 2.2, to prove that conditions (I)-(VI) are sufficient for the origin to be a center of (11), only need to check that $X_k = 0$, $(k = 3, 4, 5)$.

Case i. If $7 - 12\rho\lambda_1 = 0$. As the determinant of the coefficients matrix of the equations (31) and (33) and (37) is

$$\Delta(\lambda_1\rho) = 64(-\frac{17581}{245}\lambda_1^3\rho^3 + \frac{3950}{147}\lambda_1^2\rho^2 + \frac{68}{175}\lambda_1\rho - \frac{8}{15}),$$

and $\Delta(\frac{7}{12}) \neq 0$, which implies that these linear equations only have zero solution: $X_3 = X_5 = X = 0$.

If $a_5^2 + b_5^2 \neq 0$, by $X = 0$ we get

$$a_3 = \lambda_2(a_5B_2 - b_5A_2),\ b_3 = \lambda_2(a_5A_2 + B_2b_5),$$

where $\lambda_2 = \frac{a_5b_3 - a_3b_5}{A_2(a_5^2 + b_5^2)}$ or $\lambda_2 = \frac{a_5a_3 + a_3a_5}{B_2(a_5^2 + b_5^2)}$. By this we get

$$X_4 = -\frac{16}{3}\lambda_2X_5 = 0.$$

If $a_5^2 + b_5^2 = 0$, then $\hat{a}_8 = \hat{b}_8 = 0$ and $X_4 = 0$.

Therefore, in the case of $7 - 12\rho\lambda_1 = 0$, we get $A_{2k}\hat{b}_{2k} - B_{2k}\hat{a}_{2k} = 0, (k = 1, 2, ..., 5)$. By Lemma 2.2 and Theorem 1.2, $r = 0$ is a center and CC-center of (11).

Case ii. If $7 - 12\rho\lambda_1 \neq 0$, then from (34) follows that

$$X_4 = A_8\hat{b}_8 - B_4\hat{a}_8 = 0. \quad (38)$$

Case (1)**.** If $\Delta(\lambda_1\rho)\neq 0$, then there exists only zero solution for the homogeneous linear equations (31) and (33) and (37), i.e., $X_3=X_5=X=0$.

Case (2)**.** If $\Delta(\lambda_1\rho)=0$, i.e.,

$$-\frac{17581}{245}\lambda_1^3\rho^3+\frac{3950}{147}\lambda_1^2\rho^2+\frac{68}{175}\lambda_1\rho-\frac{8}{15}=0, \tag{39}$$

then $\rho\lambda_1\approx 0.3180,\ 0.1837,\ -0.1272$.

Case ($\dot{1}$)**.** If $a_5^2+b_5^2\neq 0$. Using (38) we get

$$a_3=\lambda_3(A_8b_5-B_8a_5),\ b_3=\lambda_3(b_5B_8+A_8a_5), \tag{40}$$

where $\lambda_3=\frac{a_3b_5+a_5b_3}{A_8(a_5^2+b_5^2)}$ or $\lambda_3=\frac{a_3b_5-a_5a_3}{B_8(a_5^2+b_5^2)}$. By this and (21) we get

$$X_2=(\frac{2}{5}+\frac{1}{12}\rho^2\lambda_3)(A_4(a_1a_5+b_1b_5)+B_4(a_1b_5-a_5b_1))=0 \tag{41}$$

and (24) can be written as

$$A_4\hat{e}_4-B_4\hat{d}_4=(\frac{1}{7}\rho-\frac{1}{120}\rho^3\lambda_3)(A_4(a_1a_5+b_1b_5)+B_4(a_1b_5-a_5b_1))=0. \tag{42}$$

From (41) and (42) follow that

$$A_4(a_1a_5+b_1b_5)+B_4(a_1b_5-a_5b_1)=0,$$

which implies that

$$a_1=\lambda_4(a_5B_4-b_5A_4),\ b_1=\lambda_4(a_5A_4+b_5B_4), \tag{43}$$

where $\lambda_4=\frac{a_5b_1-a_1b_5}{A_4(a_5^2+b_5^2)}$ or $\lambda_4=\frac{a_5a_1+b_1b_5}{B_4(a_5^2+b_5^2)}$. Applying (43), the relations (20) and (23) and (26) can be rewritten as follows

$$20\lambda_4^2X_5-(\frac{1}{15}+\frac{1}{6}\rho\lambda_4)X=0, \tag{44}$$

$$4X_3+32\lambda_4X_5+(\frac{1}{15}\rho^2\lambda_4-\frac{8}{105}\rho)X=0, \tag{45}$$

$$\frac{4}{5}\rho X_3+(128-\frac{544}{35}\rho\lambda_4)X_5+(\frac{1}{35}\lambda_4\rho^3+\frac{136}{315}\rho^2)X=0. \tag{46}$$

The determinant of the coefficients matrix of the equations (44) -(46) is

$$W(\lambda_4\rho)=-\frac{128}{1575}(15\rho^3\lambda_4^3+260\lambda_4^2\rho^2+978\lambda_4\rho+420). \tag{47}$$

1^*. If $W\neq 0$, then there exists only zero solution for the above homogeneous linear equations, so, $X_3=X_5=X=0$.

2^*. If $W=0$, i.e.,

$$15\rho^3\lambda_4^3+260\lambda_4^2\rho^2+978\lambda_4\rho+420=0, \tag{48}$$

then $\rho\lambda_4\approx -12.161444094087901, -4.679924423645147, -0.491964815600285$.

Using (40) and calculating we get

$$X_3=-(\frac{5}{48}\rho^3\lambda_3^2+12\lambda_4)X_5,\ X=20\lambda_3X_5.$$

Substituting them into (44) we obtain

$$(30\lambda_4^2-\lambda_3(2+5\lambda_4\rho))X_5=0. \tag{49}$$

Now, we show that $30\lambda_4^2-\lambda_3(2+5\lambda_4\rho)\neq 0$. Otherwise, if it is equal to zero, by (48) we see that $\rho\lambda_4\neq -0.4$, so

$$\lambda_3=\frac{30\lambda_4^2}{2+5\rho\lambda_4}. \tag{50}$$

On the other hand, by substituting (40) into (28) we get

$$a_1 = (3\lambda_1 + \frac{5}{8}\rho^2\lambda_1\lambda_3)(a_5B_4 - b_5A_4),\ b_1 = (3\lambda_1 + \frac{5}{8}\rho^2\lambda_1\lambda_3)(a_5A_4 + b_5B_4),$$

as $a_5^2 + b_5^2 \neq 0$, $A_4^2 + B_4^2 \neq 0$, by the above relations and (43) we get

$$\lambda_4 = 3\lambda_1 + \frac{5}{8}\rho^2\lambda_1\lambda_3. \tag{51}$$

Substituting (50) into (51) we obtain

$$\rho\lambda_1 = \frac{8\rho\lambda_4 + 20\rho^2\lambda_4^2}{75\rho^2\lambda_4^2 + 60\rho\lambda_4 + 24},$$

putting it into (39) and using (48) we get

$$\frac{28575808611488}{19845}(\rho\lambda_4)^2 + \frac{48881032663424}{6615}\rho\lambda_4 + \frac{651385833609232357}{198180864} = 0, \tag{52}$$

solving (52), we get $\rho\lambda_4 \approx -4.639757046976126, -0.491964682547174$, this means that the equations (48) and (52) can't hold at the same time. Therefore, $30\lambda_4^2 - \lambda_3(2 + 5\lambda_4\rho) \neq 0$, by (49) we get $X_5 = 0$ and $X_3 = 0$. By Lemma 2.2 and Theorem 1.2, $r = 0$ is a CC-center of (11).

Case $(\dot{2})$**.** If $a_5^2 + b_5^2 = 0$, then $X_5 = X = 0$. Substituting them into (33) we get $(1 - 2\rho\lambda_1)X_3 = 0$. In view of $\Delta(\rho\lambda_1) = 0$, so $\rho\lambda_1 \neq \frac{1}{2}$ and $X_3 = 0$.

In summary, in the case of $7 - 12\rho\lambda_1 \neq 0$, we have verified that $X_{2k} = 0$, $(k = 1, 2, .., 5)$. By Lemma 2.2 and Theorem 1.2, $r = 0$ is a center and CC-center of (11).

Case II. If $\delta = 25(a_3^2 + b_3^2) - 9(a_5^2 + b_5^2) = 0$. By $A_4\hat{b}_4 - B_4\hat{a}_4 = 0$, we get

$$\frac{3b_5 - 5b_3}{5a_3 - 3a_5} = \frac{3a_5 + 5a_3}{5b_3 + 3b_5} = \frac{A_4}{B_4},$$

which implies that the relation (38) is valid, i.e. $X_4 = 0$.

If $a_5^2 + b_5^2 \neq 0$, similar to the previous **Case** $(\dot{1})$ we can get (40)-(47). If $W(\rho\lambda_4) \neq 0$, then $X_{2k} = 0\,(k = 1, 2, ..., 5)$. If $W(\rho\lambda_4) = 0$, it implies that (48) and (49) are valid. Now, we show that $30\lambda_4^2 - \lambda_3(2 + 5\lambda_4\rho) \neq 0$. Otherwise, suppose that

$$30\lambda_4^2 - \lambda_3(2 + 5\lambda_4\rho) = 0. \tag{53}$$

Applying (40) and $25(a_3^2+b_3^2)-9(a_5^2+b_5^2) = 0$, we obtain $25\lambda_3^2\rho^4 = 576$. Substituting $\lambda_3\rho^2 = \pm\frac{24}{5}$ into (53) we get

$$25(\rho\lambda_4)^2 = \pm 4(2 + 5\rho\lambda_4). \tag{54}$$

It's not difficult to verify that the equations (48) and (54) can't be held at the same time. Therefore, $30\lambda_4^2 - \lambda_3(2 + 5\lambda_4\rho) \neq 0$. By (49) we get $X_5 = 0$ and $X_3 = 0$. By Lemma 2.2 and Theorem 1.2, $r = 0$ is a CC-center of (11).

If $a_5^2 + b_5^2 = 0$, from $\delta = 0$ implies that $a_3^2 + b_3^2 = 0$, so, $\hat{a}_{2k} = \hat{b}_{2k} = 0\,(k = 3, 4, 5)$ and $X_3 = X_4 = X_5 = 0$. By Lemma 2.2 and Theorem 1.2, $r = 0$ is a CC-center of (11).

Therefore, the conditions (I)$-$(VI) are sufficient for the $r = 0$ to be a center of (11). □

Remark. The relations (I)-(VI) are derived from the six values focus of the system (5), are equal to zero.

Example 3.1. *For differential system* (5) *with*

$P_2 = 2x^2 + 3xy - 2y^2$, $P_5 = (2x - y)((m_0 + m_1 + m_2)x^4 + 4(m_1 + 2m_2)x^3y + (2m_0 + 5m_1 + 24m_2)x^2y^2 + 4(m_1 + 8m_2)xy^3 + (m_0 + 4m_1 + 16m_2)y^4)$, *its origin point is a center and CC-center and uniform isochronous center, where* m_0, m_1, m_2 *are arbitrary real numbers.*

Declarations

Conflict of interest The authors declare no conflict of interest.

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