Appl. Math. J. Chinese Univ.
2024, 39(4): 604-613

On the global L^1-boundedness of Fourier integral operators with rough amplitude and phase functions

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Abstract. Let $T_{\phi,a}$ be a Fourier integral operator with amplitude a and phase functions ϕ. In this paper, we study the boundedness of Fourier integral operator of rough amplitude $a \in L^\infty S^m_\rho$ and rough phase functions $\phi \in L^\infty \Phi^2$ with some measure condition. We prove the global L^1 boundedness for $T_{\phi,a}$ when $\frac{1}{2} < \rho \leqslant 1$ and $m < \rho - \frac{n+1}{2}$. Our theorem improves some known results.

§1 Introduction and main result

The Fourier integral operator (FIO) is defined by

$$T_{\phi,a}f(x) = \frac{1}{2\pi}\int_{\mathbb{R}^n} e^{i\phi(x,\xi)}a(x,\xi)\hat{f}(x,\xi)\mathrm{d}\xi.$$

Here $a(x,\xi)$ is the amplitude and $\phi(x,\xi)$ is the phase function, and $\hat{f}$ denotes the Fourier transform of f. When $\phi(x,\xi) = x.\xi$, the FIO is said to be a pseudo-differential operator. This operator is very important in harmonic analysis and is related with many problems arising in partial differential equations(see [6][7]). We say that the amplitude $a(x,\xi)$ is in the Hörmander class $S^m_{\rho,\delta}$ if $a \in C^\infty(R^n \times R^n)$ with

$$\sup_{\xi\in\mathbb{R}^n}(1+|\xi|)^{-m+\rho|\alpha|-\delta|\beta|}\left|\partial^\alpha_\xi\partial^\beta_x a(x,\xi)\right| < +\infty.$$

for $m \in R, 0 \le \rho, \delta \le 1$ and all multi-indices α, β. For the phase function $\phi(x,\xi)$, one usually assumes that it is real-valued, in $C^\infty(R^n \times R^n/\{0\})$ with homogeneous of degree 1 in the frequency variable ξ, and satisfies nondegeneracy condition, that is

$$\det\left(\frac{\partial^2\phi}{\partial x_i \partial\xi_j}\right) \neq 0$$

Received: 2021-05-13. Revised: 2021-12-04.
MR Subject Classification: 42B20, 42B15.
Keywords: Fourier integral operators, L^1-Boundedness, rough amplitude, rough phase functions.
Digital Object Identifier(DOI): https://doi.org/10.1007/s11766-024-4452-6.
Supported by the National Natural Science Foundation of China(11801518), the Natural Science Foundation of Zhejiang Province(LQ18A010005), the Science Foundation of Zhejiang Education Department(Y201738640).
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for all $\xi \neq 0$. The theory of Fourier integral operators (FIOs) on $\mathbb{R}^n$ has been initiated, developed and studied extensively by Hörmander in [10]. After then, there were many studies on the L^p boundedness of FIOs. For $p = 2$, Eskin [8] firstly proved the local L^2 boundedness when $a \in S^0_{1,0}$, then Hörmander [10] show the same conclusion with $a \in S^0_{\rho,1-\rho}$, $\frac{1}{2} < \rho \le 1$, and the case $\rho = \frac{1}{2}$ was proved by Beals [2]. For the global L^2 boundedness of FIOs, one can see [1][3][15][17] and their references. For the L^p boundedness, Seeger, Sogge and Stein[20] showed the local L^p boundedness for $a \in S^m_{1,0}$ with $m \le (1-n)|\frac{1}{p} - \frac{1}{2}|$. In [20], they also proved that the Fourier integral operator was locally bounded from H^1 to L^1 when $m = -\frac{n-1}{2}$. The results of global L^p boundedness can be found in [4][5][16][18] and so on.

Later, Kenig and Staubach [13] studied the L^p boundedness of pseudodifferential operators with rough amplitude and rough phase functions which behave in the variable x like L^∞ functions and variable ξ like that in the Hörmander class $S^m_{\rho,0}$. Precisely, the definitions of rough amplitude and rough phase functions are as follows.

Definition 1.1: Let $m \in R$,$0 \le \rho \le 1$, a function $a(x,\xi)$ belongs to the class $L^\infty S^m_\rho$, if it satisfies

$$\sup_{\xi \in R^n} (1+|\xi|)^{-m+\rho|\alpha|} \|\partial^\alpha_\xi a(\cdot,\xi)\|_{L^\infty(R^n)} < \infty$$

for all multi-indices α.

Definition 1.2: A rough class $L^\infty\Phi^2$ is the set of all function ϕ which is homogeneous of degree 1 and smooth on $\mathbb{R}^n \setminus \{0\}$ in frequency variable ξ and for all multi-indices $|\alpha| \geqslant 2$ satisfies

$$\sup_{\xi \in \mathbb{R}^n \setminus \{0\}} |\xi|^{-1+|\alpha|} \|\nabla^\alpha_\xi \phi(\cdot,\xi)\|_{L^\infty(\mathbb{R}^n)} < +\infty. \tag{1}$$

In [13], Kenig and Staubach established the L^1,L^∞ boundedness of pseudo-differential operators with $a \in L^\infty S^m_\rho$. Motivate by this work, Dos Santos Ferreira and Staubach [9] studied the global and local L^p-boundedness of FIOs when the $a \in L^\infty S^m_\rho$ and $\phi \in L^\infty\Phi^2$ satisfying the rough non-degeneration condition. More results about the L^p boundedness of FIOs with rough amplitude and rough phase functions can be found in [11][12][14][21] and so on.

Recently, Dos Santos Ferreira and W. Staubach [9] studied the L^p boundedness of FIOs with rough amplitude and rough phase functions. When $p = 1$, they obtained the following result by using a semiclassical version of Seeger-Sogge-Stein decomposition.

Theorem A: ([9]) Suppose that amplitude $a \in L^\infty S^m_\rho$ and phase function $\phi \in L^\infty\Phi^2$ satisfying the rough non-degeneracy condition. Then the Fourier integral operator $T_{\phi,a}$ is bounded in $L^1(R^n)$ provided $m < -\frac{n-1}{2} + n(\rho - 1)$ and $0 \le \rho \le 1$.

In this paper, we also study the L^1 boundedness of FIOs with rough amplitude and rough phase functions. By replacing the rough non-degeneracy condition with other conditions, we improve the result of Theorem A. Specifically, we assume that there exists a constant $A > 0$ such that for any $x, \xi \in \mathbb{R}^n$ and $E \subset \mathbb{R}^n$ such that

$$\left|\{x : \nabla_\xi \phi(x,\xi) \in E\}\right| \leqslant A|E|. \tag{2}$$

The following theorem is our main result in this paper.

Theorem 1.1: Suppose that $a \in L^\infty S^m_\rho$ and $\phi \in L^\infty\Phi^2$ satisfies **(2)**.When $\frac{1}{2} < \rho \leqslant 1$ and

$m < \rho - \frac{n+1}{2}$, the operator $T_{\phi,a}$ is bounded from L^1 to itself.

Remark 1.1: When $\frac{1}{2} < \rho \leqslant 1$ and $n > 1$, it is easy to see that $-\frac{n-1}{2} + n(\rho - 1) \leqslant \rho - \frac{n+1}{2}$. So, theorem 1.1 improves the conclusion of Theorem A.

Remark 1.2: When $n = 1$, it is easy to remark that our theorem is similar to Theorem A.

This paper is organized as follows: In Section 2, we will introduce some preliminary knowledge which includes some useful lemmas. The proof of Theorem 1.1 will be presented in Section 3.

Throughout this paper, we use the inequality $A \preceq B$ to mean that there is a positive number C independent of all main variables such that $A \leq CB$, and use the notation $A \simeq B$ to mean $A \preceq B$ and $B \preceq A$. We denote by B_r the ball in R^n with center 0 and radius r.

§2 Preliminaries and Lemmas

In this section, we will state some useful tools and prove some lemmas that will be used in the proof of our results.

Definition 2.1: (Littlewood-Paley decomposition)[19]

Let $\psi \in C^\infty(R^n)$ be supported in $\{\xi : \frac{1}{2} \leq |\xi| \leq 2\}$, $0 \leq \psi(\xi) \leq 1$ and $\psi(\xi) > c > 0$ if $\frac{3}{5} \leq |\xi| \leq \frac{5}{3}$. Let $\psi_j(\xi) = \psi(2^{-j}\xi)$ and require that ψ satisfies

$$\sum_{j=-\infty}^{j=+\infty} \psi_j(\xi) \equiv 1 \quad \text{for all} \quad \xi \in R^n \setminus \{0\}$$

Let $\eta(\xi) = 1 - \sum_{j=1}^{\infty} \psi_j(\xi)$ and it is easy to check that

$$\psi_j \in C_c^\infty(B_{2^{j+1}} \setminus B_{2^{j-1}}), |\nabla^k \psi_j(\xi)| \leq C_k 2^{-jk}, k \in \mathbb{N}.$$

Definition 2.2: (Seeger-Sogge-Stein decomposition)[20]

For $j \in \mathbb{N}$, there exist no more than $C2^{\frac{j(n-1)}{2}}$ points $\xi_j^v \in S^{n-1}$ and functions $\varphi_j^v \in C^\infty(S^{n-1})$ such that

$$\begin{aligned}
&|\xi_j^{v_1} - \xi_j^{v_2}| \geq 2^{-\frac{j}{2}}, v_1 \neq v_2;\\
&\inf_v |\xi_j^v - \xi| \leq 2^{-\frac{j}{2}}, \forall \xi \in S^{n-1};\\
&\sum_v \varphi_j^v \equiv 1, supp(\varphi_j^v) \subset \{\theta \in S^{n-1} : |\theta - \xi_j^v| \leq 2^{1-\frac{j}{2}}\};\\
&|\nabla^k \varphi_j^v| \leq C_k 2^{\frac{jk}{2}}, k \geq 0.
\end{aligned}$$

Lemma 2.1: Suppose that u is supported in B_1 and satisfies that

$$|\nabla^k u(x)| \leq A|x|^{1-k}, k = 0, 1, ... \tag{3}$$

Then for any $0 < \mu < 1$, we have

$$|\int_{B_1} e^{-iyx} u(x) dx| \leq C(1 + |y|)^{-n-\mu}$$

where C depends only on n, μ, A.

Proof. When $|y| \leq 1$, it is easy to check the desired estimation, so we only need to consider $|y| \geq 1$. Let $\chi(x)$ be a $C_0^\infty(R^n)$ which satisfies $supp\chi(x) \subset B_1$ and $\chi(x) = 1$ for $x \in B_{\frac{1}{2}}$.

Choosing $0<\varepsilon\le 1$ and integrations by parts yield that

$$\begin{aligned}\left|\int_{B_1} e^{-iy\cdot x}u(x)dx\right| &\preceq |y|^{-n}\left|\int_{B_1} e^{-iy\cdot x}\partial_x^n u(x)dx\right|\\ &\preceq |y|^{-n}\left(\left|\int_{B_1} e^{-iy\cdot x}\partial_x^n u(x)\chi(\frac{x}{\varepsilon})dx\right|+\left|\int_{B_1} e^{-iy\cdot x}\partial_x^n u(x)(1-\chi(\frac{x}{\varepsilon}))dx\right|\right)\\ &= |y|^{-n}(I+II)\end{aligned}$$

By the condition (3), the term I is bounded by ε^μ for any $0<\mu<1$. For the term II, by the integration by parts, we obtain

$$\begin{aligned}II &\preceq |y|^{-1}(\varepsilon^{-1}|\int_{B_1} e^{-iy\cdot x}\partial_x\chi(\frac{x}{\varepsilon})\partial_x^n u(x)dx|+|\int_{B_1} e^{-iy\cdot x}(1-\chi(\frac{x}{\varepsilon}))\partial_x^{n+1}u(x)dx|)\\ &\preceq |y|^{-1}(\varepsilon^{-1+\mu}+1)\end{aligned}$$

Choosing $\varepsilon=|y|^{-1}$, we have

$$|\int_{B_1} e^{-iyx}u(x)dx|\le C(1+|y|)^{-n-\mu} \tag{4}$$

for all $0<\mu<1$.

§3 Proof of main result

We first show the L^1 boundedness of FIOs in low frequency.

Lemma 3.1: If $a\in L^\infty S^m_\rho$ and $\phi\in L^\infty\Phi^2$ satisfies (2). Then for any $\eta\in C_c^\infty(B_1)$, the following operator

$$T_{0,\phi,a}f(x)=\int_{\mathbb{R}^n} e^{i\phi(x,\xi)}a(x,\xi)\eta(\xi)\hat{f}(\xi)d\xi$$

is bounded from L^1 to itself.

Proof. We first localize the amplitude in ξ by introducing a finite open convex covering $\{U_l\}_{l=1}^M$, with A maximum of diameters $d<\frac{1}{10}$, of the unit sphere S^{n-1}. Let κ_l be a smooth partition of unity subordinate to the covering U_l and set

$$a_l(x,\xi)=a(x,\xi)\kappa_l(\frac{\xi}{|\xi|}) \tag{5}$$

Some simple computations imply that

$$\begin{aligned}&T_{0,\phi,a}f(x)\\ =&\int_{\mathbb{R}^n} e^{i\phi(x,\xi)}a(x,\xi)\eta(\xi)\hat{f}(\xi)d\xi\\ =&\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} e^{i(\phi(x,\xi)-y\cdot\xi)}a(x,\xi)\eta(\xi)f(y)dyd\xi\\ =&\sum_{l=1}^M\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} e^{i(\phi(x,\xi)-y\cdot\xi)}a_l(x,\xi)\eta(\xi)f(y)dyd\xi\\ =&\sum_{l=1}^M T_l f(x)\end{aligned}$$

where $T_lf(x)=\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} e^{i(\phi(x,\xi)-y\cdot\xi)}a_l(x,\xi)\eta(\xi)f(y)dyd\xi$.

Thus, we only need to show the L^1 boundedness for every $T_lf(x)$. For $1\le l\le M$, by choosing some $\xi_l \in supp\kappa_l$ and setting $w_l(x,\xi) = \phi(x,\xi) - \nabla_\xi \phi(x,\xi_l)\cdot\xi$, we have

$$\begin{aligned} T_l f(x) &= \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} e^{i(\nabla_\xi\phi(x,\xi_i)-y)\cdot\xi} e^{iw_i(x,\xi)} a_l(x,\xi)\eta(\xi)f(y)dyd\xi \\ &= \int_{\mathbb{R}^n} k_l(x,y)f(y)dy \end{aligned}$$

where $k_l(x,y) = \int_{\mathbb{R}^n} e^{i(\nabla_\xi\phi(x,\xi_l)-y)\cdot\xi} e^{iw_l(x,\xi)} a_l(x,\xi)\eta(\xi)d\xi$.

As $\phi(x,\xi)$ is homogeneous of degree 1 in ξ variable and satisfy (1), by the mean value theorem, we have

$$\begin{aligned} |\nabla_\xi w_l(x,\xi)| &= |\nabla_\xi\phi(x,\xi) - \nabla_\xi\phi(x,\xi_l)| \\ &= |\nabla_\xi\phi(x,\frac{\xi}{|\xi|}) - \nabla_\xi\phi(x,\xi_l)| \\ &\le \|\nabla_\xi^2\phi(\cdot,\zeta_l)\|_{L^\infty(R^n)}|\frac{\xi}{|\xi|} - \xi_l| \\ &\preceq 2|\zeta_l|^{-1} \end{aligned}$$

where $\frac{\xi}{|\xi|} \in U_l$ and $\zeta_l = \theta\xi_l + (1-\theta)\frac{\xi}{|\xi|}$ for some $\theta \in (0,1)$. For the diameter of U_l is less then $\frac{1}{10}$, we have $|\zeta_l| \ge \frac{1}{2}$ which implies

$$|\nabla_\xi w_l(x,\xi)| \le C.$$

On the other hand, when $k \ge 2$, one have

$$|\nabla_\xi^k w_l(x,\xi)| = |\nabla_\xi^k \phi(x,\xi)| \le A|\xi|^{1-k}.$$

So, for any $\xi \in B_1, k \in \mathbb{N}$, there holds

$$|\nabla_\xi^k w_l(x,\xi)| \le C|\xi|^{1-k}.$$

Now for $k \ge 0$, some direct computations yield that

$$\begin{aligned} &|\nabla_\xi^k[e^{iw_l(x,\xi)}a_l(x,\xi)\eta(\xi)]| \\ \le\ & C\sum_{k_0=0}^{k}\sum_{t=1}^{n+1-k_0} |\nabla_\xi^{k_0}[a_l(x,\xi)\eta(\xi)]| \sum_{k_1+\cdots+k_t=k-k_0,k_s>0}\prod_{s=1}^{t}|\nabla_\xi^{k_s} w_l(x,\xi)| \\ \le\ & C\sum_{k_0=0}^{k}\sum_{t=1}^{k-k_0}\sum_{k_1+\cdots+k_t=k-k_0,k_s>0}\prod_{s=1}^{t}|\xi|^{1-k_s} \\ \le\ & C\sum_{k_0=0}^{k}\sum_{t=1}^{k-k_0}|\xi|^{t+k_0-k} \\ \le\ & C|\xi|^{1-k}. \end{aligned}$$

For any $0<\mu<1$, by Lemma 2.1, we obtain that

$$|k_l(x,y)| = |\int_{\mathbb{R}^n} e^{i(\nabla_\xi\phi(x,\xi_l)-y)\cdot\xi}e^{iw_l(x,\xi)}a_l(x,\xi)\eta(\xi)d\xi| \le C(1+|\nabla_\xi\phi(x,\xi_l)-y|)^{-n-\mu}. \tag{6}$$

On the other hand, the assumption (2) yields that for $y \in \mathbb{R}^n$,

$$|\{x : |\nabla_\xi\phi(x,\xi_l) - y| < r\}| \le Cr^n. \tag{7}$$

Now, for $y \in R^n$, (6) and (7) imply that

$$\begin{aligned}
&\int_{\mathbb{R}^n} |k_l(x,y)|dx \\
\leq\ & C\int_{\mathbb{R}^n} (1+|\nabla_\xi\phi(x,\xi_l)-y|)^{-n-\mu}dx \\
\leq\ & C(\int_{|\nabla_\xi\phi(x,\xi_l)-y|<1} (1+|\nabla_\xi\phi(x,\xi_l)-y|)^{-n-\mu}dx \\
&+\sum_{j=1}^{\infty}\int_{2^{j-1}<|\nabla_\xi\phi(x,\xi_l)-y|<2^j} (1+2^{j-1})^{-n-\mu}dx) \\
\leq\ & C(|\{x:|\nabla_\xi\phi(x,\xi_l)-y|<1\}| \\
&+\sum_{j=1}^{\infty} 2^{-j(\mu+n)}|\{x:|\nabla_\xi\phi(x,\xi_l)-y|<2^j\}|) \\
\leq\ & C(1+\sum_{j=1}^{\infty}2^{-j\mu})<\infty.
\end{aligned}$$

So T_l is bounded on L^1 for every $1 \leq l \leq M$ which means $T_{0,\phi,a}$ is also bounded on L^1 and we prove this lemma. $\square$

Next, we will prove the L^1 boundedness in high frequency. By the standard Littlewood-Paley decomposition and Seeger-Sogge-Stein decomposition. The operator $T_{\phi,a}$ can be divided as

$$\begin{aligned}
T_{\phi,a}f(x) &= \int_{\mathbb{R}^n} e^{i\phi(x,\xi)}a(x,\xi)\hat{f}(\xi)d\xi \\
&= \int_{\mathbb{R}^n} e^{i\phi(x,\xi)}a(x,\xi)(\eta_0+\sum_{j=1}^{\infty}\sum_{v=1}\psi_j\varphi_j^v(\frac{\xi}{|\xi|})\hat{f}(\xi)d\xi \\
&= T_{0,\phi,a}f(x)+\sum_{j=1}^{\infty}\sum_{v=1}T_j^v f(x).
\end{aligned}$$

The L^1 boundedness of $T_{0,\phi,a}$ has been proved in lemma 3.1, we only need to focus on the high frequency. Set $w_j^v(x,\xi)=\phi(x,\xi)-\nabla_\xi\phi(x,\xi_j^v)\cdot\xi$. Then some simple computations yield

$$\begin{aligned}
T_j^v f(x) &= \int_{\mathbb{R}^n} e^{i\phi(x,\xi)}a(x,\xi)\psi_j(\xi)\varphi_j^v(\frac{\xi}{|\xi|})\hat{f}(\xi)d\xi \\
&= \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} e^{i(\nabla_\xi\phi(x,\xi_j^v)-y)\cdot\xi}e^{iw_j^v(x,\xi)}a(x,\xi)\psi_j\varphi_j^v(\frac{\xi}{|\xi|})d\xi f(y)dy \\
&= \int_{\mathbb{R}^n} k_j^v(x,y)f(y)dy
\end{aligned}$$

where $k_j^v(x,y)=\int_{\mathbb{R}^n} e^{i(\nabla_\xi\phi(x,\xi_j^v)-y)\cdot\xi}e^{iw_j^v(x,\xi)}a(x,\xi)\psi_j\varphi_j^v(\frac{\xi}{|\xi|})d\xi$.

Without loss of generality, we estimate $k_j^v(x,y)$ for $\xi_j^v=(1,0,\cdots,0)$. Set $\xi'=(0,\xi_2,...,\xi_n)$ which is perpendicular to ξ_j^v. In this case, we define the operator $L_jf=1-2^{2j\rho}\partial_{\xi_1}^2 f - 2^j\nabla_{\xi'}^2 f$. Then it is self-adjoint and

$$L_j(e^{i(\nabla_\xi\phi(x,\xi_j^v)-y)\cdot\xi})=(1+2^{2j\rho}|\partial_{\xi_1}\phi(x,\xi_j^v)-y_1|^2+2^j|\nabla_{\xi'}\phi(x,\xi_j^v)-y'|^2)e^{i(\nabla_\xi\phi(x,\xi_j^v)-y)\cdot\xi}.$$

Now we use the same arguments in [19] p407. In the domain $\Omega=\{\xi:2^{j-1}<|\xi|<2^{j+1},|\frac{\xi}{|\xi|}-\xi_j^v|\preceq 2^{-\frac{j}{2}}\}$, there holds

$$\partial_{\xi_1}=\partial_r+O(2^{-\frac{j}{2}})\cdot\nabla_\xi. \tag{8}$$

As $\varphi_j^v(\frac{\xi}{|\xi|})$ is homogeneous of order 0, for $\xi\in\Omega, k\in\mathbb{N}$ and a multi-index α we have

$$|\partial_{\xi_1}^k\partial_{\xi'}^\alpha\varphi_j^v(\xi)|=|(\partial_r+O(2^{-\frac{j}{2}})\cdot\nabla_\xi)^k\partial_{\xi'}^\alpha\varphi_j^v(\xi)|\preceq\sum_{k_1+k_2=k}2^{-\frac{jk_2}{2}}|\partial_r^{k_1}\nabla_\xi^{k_2}\partial_{\xi'}^\alpha\varphi_j^v(\xi)|$$

Since $\partial_r^N\varphi_j^v(\xi)=0$ for all $N\geq 1$, we have

$$\begin{aligned}|\partial_{\xi_1}^k\partial_{\xi'}^\alpha\varphi_j^0(\frac{\xi}{|\xi|})| &\leq C2^{-\frac{jk}{2}}|\nabla_\xi^{k_2}\partial_{\xi'}^\alpha\varphi_j^v(\xi)|\\ &\leq C2^{-\frac{jk}{2}}|\xi|^{-k-|\alpha|}|\nabla_\theta^{k+|\alpha|}\varphi_j^v(\theta)|\\ &\leq C2^{-\frac{jk}{2}}2^{-j(k+|\alpha|)}2^{\frac{j(k+|\alpha|)}{2}}\\ &\leq C2^{-j(k+\frac{|\alpha|}{2})}.\end{aligned} \tag{9}$$

Since $a\in L^\infty S_\rho^m$, some simple computations and (9) imply that

$$\begin{aligned}&|\partial_{\xi_1}^k\partial_{\xi'}^\alpha[a(x,\xi)\psi_j(\xi)\varphi_j^v(\frac{\xi}{|\xi|})]|\\ \leq\ & C\sum_{k_1+k_2=k}\sum_{\alpha_1+\alpha_2=\alpha}|\partial_{\xi_1}^{k_1}\partial_{\xi'}^{\alpha_1}[a(x,\xi)\psi_j(\xi)]||\partial_{\xi_1}^{k_2}\partial_{\xi'}^{\alpha_2}\varphi_j^v(\frac{\xi}{|\xi|})]|\\ \leq\ & C\sum_{k_1+k_2=k}\sum_{\alpha_1+\alpha_2=\alpha}2^{j(m-\rho k_1-\rho|\alpha_1|)}2^{-j(k_2+\frac{|\alpha_2|}{2})}\\ \leq\ & C2^{j(m-\rho k-\frac{|\alpha|}{2})}.\end{aligned} \tag{10}$$

As $\phi(x,\xi)$ is homogeneous of order 1 in ξ, when $N\geq 2$ and $k\geq 1$, there holds

$$\partial_r^k\nabla_\xi^{N-k}\phi(x,\xi)=\nabla_\xi^{N-k}\partial_r^{k-1}\partial_r\phi(x,\xi)(k\geq 2)\text{ or }\nabla_\xi^{N-k-1}\partial_r\nabla_\xi\phi(x,\xi)(k=1)=0. \tag{11}$$

When $k+|\alpha|\geq 2$ and $\xi\in\Omega$, by virtue of (8), (11) and the fact $\phi\in L^\infty\Phi^2$, we can obtain that

$$\begin{aligned}|\partial_{\xi_1}^k\partial_{\xi'}^\alpha w_j^v(x,\xi)| &= |\partial_{\xi_1}^k\partial_{\xi'}^\alpha\phi(x,\xi)|\\ &= |(\partial_r+O(2^{-\frac{j}{2}})\cdot\nabla_\xi)^k\partial_{\xi'}^\alpha\phi(x,\xi)|\\ &\leq C\sum_{k_1=0}^k 2^{-\frac{j(k-k_1)}{2}}|\partial_r^{k_1}\nabla_\xi^{k-k_1+|\alpha|}\phi(x,\xi)|\\ &= C2^{-\frac{jk}{2}}|\nabla_\xi^{k+|\alpha|}\phi(x,\xi)|\\ &\leq C2^{j(-\frac{k}{2}-k-|\alpha|+1)}\leq C2^{-j(k+\frac{|\alpha|}{2})}.\end{aligned} \tag{12}$$

When $\xi\in\Omega$, from (12) we have $|\partial_{\xi_1}\partial_{\xi'}\phi(x,\xi)|\leq C2^{-\frac{3j}{2}}$. By the mean value theorem and the homogeneity of ϕ, one can get that

$$\begin{aligned}|\partial_{\xi_1}w_j^v(x,\xi)| &= |\partial_{\xi_1}[\phi(x,\xi)-\phi(x,\xi_j^v)]|\\ &= |\partial_{\xi_1}\phi(x,\xi)-\partial_{\xi_1}\phi(x,|\xi|\xi_j^v)|\end{aligned}$$

$$
\begin{aligned}
&\leq C|\xi-|\xi|\xi_j^v|\sup_{\zeta\in\Omega}|\partial_{\xi_1}\partial_{\xi'}\phi(x,\zeta)| \\
&\leq C2^{\frac{j}{2}}2^{-\frac{3j}{2}}=C2^{-j}. \qquad (13)
\end{aligned}
$$

On the other hand, it is easy to see that

$$
|\partial_{\xi'}w_j^v(x,\xi)|=|\partial_{\xi'}\phi(x,\xi)-\partial_{\xi'}\phi(x,\xi_j^v)|\leq C2^{-\frac{j}{2}}. \qquad (14)
$$

It can be derived from (12), (13) and (14) that

$$
|\partial_{\xi_1}^k\partial_{\xi'}^\alpha e^{iw_j^v(x,\xi)}|\leq C2^{-j(k+\frac{|\alpha|}{2})}. \qquad (15)
$$

Now by virtue of (10) and (15), for any $M\in\mathbb{N}$, we have

$$
\begin{aligned}
&|(L_j)^M[e^{iw_j^v(x,\xi)}a(x,\xi)\psi_j(\xi)\varphi_j^v(\frac{\xi}{|\xi|})]| \\
\leq\ & C\sum_{k+|\alpha|\leq 2M}2^{jk\rho}2^{\frac{j|\alpha|}{2}}|\partial_{\xi_1}^k\partial_{\xi'}^\alpha[e^{iw_j^v(x,\xi)}a(x,\xi)\psi_j\varphi_j^v(\frac{\xi}{|\xi|})]| \\
\leq\ & C\sum_{k+|\alpha|\leq 2M}\sum_{k_1+k_2=k}\sum_{\alpha_1+\alpha_2=\alpha}2^{jk\rho}2^{\frac{j|\alpha|}{2}}|\partial_{\xi_1}^{k_1}\partial_{\xi'}^{\alpha_1}e^{iw_j^v(x,\xi)}||\partial_{\xi_1}^{k_2}\partial_{\xi'}^{\alpha_2}[a(x,\xi)\psi_j(\xi)\varphi_j^v(\frac{\xi}{|\xi|})]| \\
\leq\ & C\sum_{k+|\alpha|\leq 2M}\sum_{k_1+k_2=k}\sum_{\alpha_1+\alpha_2=\alpha}2^{jk\rho}2^{\frac{j|\alpha|}{2}}2^{-j(k_1+\frac{|\alpha_1|}{2})}2^{j(m-k_2\rho-\frac{|\alpha_2|}{2})} \\
\leq\ & C2^{jm}. \qquad (16)
\end{aligned}
$$

From (15), for any $x,y\in\mathbb{R}^n$ on can get that

$$
\begin{aligned}
&|k_j^v(x,y)| \\
=\ & |\int_{\mathbb{R}^n}e^{i(\nabla_\xi\phi(x,\xi_j^v)-y)\cdot\xi}e^{iw_j^v(x,\xi)}a(x,\xi)\psi_j\varphi_j^v(\frac{\xi}{|\xi|})d\xi| \\
=\ & |\int_{\mathbb{R}^n}(1+2^{2j\rho}|\partial_{\xi_1}\phi(x,\xi_j^v)-y_1|^2+2^j|\nabla_{\xi'}\phi(x,\xi_j^v)-y'|^2)^{-M} \\
& \times L_j^M(e^{iw_j^v(x,\xi)})a(x,\xi)\psi_j(\xi)\varphi_j^v(\frac{\xi}{|\xi|})d\xi| \\
=\ & (1+2^{2j\rho}|\partial_{\xi_1}\phi(x,\xi_j^v)-y_1|^2+2^j|\nabla_{\xi'}\phi(x,\xi_j^v)-y'|^2)^{-M}| \\
& \times\int_\Omega e^{iw_j^v(x,\xi)}L_j^M[a(x,\xi)\psi_j(\xi)\varphi_j^v(\frac{\xi}{|\xi|})]d\xi| \\
\leq\ & C2^{jm}|\Omega|(1+2^{2j\rho}|\partial_{\xi_1}\phi(x,\xi_j^v)-y_1|^2+2^j|\nabla_{\xi'}\phi(x,\xi_j^v)-y'|^2)^{-M} \\
\leq\ & C2^{j(m+\frac{n+1}{2})}(1+2^{2j\rho}|\partial_{\xi_1}\phi(x,\xi_j^v)-y_1|^2+2^j|\nabla_{\xi'}\phi(x,\xi_j^v)-y'|^2)^{-M}.
\end{aligned}
$$

By virtue of the assumption (2) and the similar computations in section 2, for any $y\in\mathbb{R}^n$ and $M>\frac{n}{2}$ we have

$$
\begin{aligned}
\int_{\mathbb{R}^n}|k_j^v(x,y)|dx &\leq C2^{j(m+\frac{n+1}{2})}\int_{\mathbb{R}^n}(1+2^{2j\rho}|\partial_{\xi_1}\phi(x,\xi_j^v)-y_1|^2+2^j|\nabla_{\xi'}\phi(x,\xi_j^v)-y'|^2)^{-M}dx \\
&\leq C2^{j(m+\frac{n+1}{2})}\Big(\int_{2^{j\rho}|\partial_{\xi_1}\phi(x,\xi_j^v)-y_1|+2^{\frac{j}{2}}|\nabla_{\xi'}\phi(x,\xi_j^v)-y'|<1} \\
&\quad +\sum_{s=1}^\infty\int_{2^{s-1}<2^{j\rho}|\partial_{\xi_1}\phi(x,\xi_j^v)-y_1|+2^{\frac{j}{2}}|\nabla_{\xi'}\phi(x,\xi_j^v)-y'|<2^s}\Big)
\end{aligned}
$$

$$
\begin{aligned}
& (1+2^{2j\rho}|\partial_{\xi_1}\phi(x,\xi_j^v)-y_1|^2+2^j|\nabla_{\xi'}\phi(x,\xi_j^v)-y'|^2)^{-M}dx \\
\leq\ & C2^{j(m+\frac{n+1}{2})}(|\{x: 2^{j\rho}|\partial_{\xi_1}\phi(x,\xi_j^v)-y_1|+2^{\frac{j}{2}}|\nabla_{\xi'}\phi(x,\xi_j^v)-y'|<1\}| \\
& +\sum_{s=1}^{\infty}|\{x: 2^{j\rho}|\partial_{\xi_1}\phi(x,\xi_j^v)-y_1|+2^{\frac{j}{2}}|\nabla_{\xi'}\phi(x,\xi_j^v)-y'|<2^s\}|2^{-2sM}) \\
\leq\ & C2^{j(m+\frac{n+1}{2})}(|\{x: |\partial_{\xi_1}\phi(x,\xi_j^v)-y_1|<2^{-j\rho}, |\nabla_{\xi'}\phi(x,\xi_j^v)-y'|<2^{-\frac{j}{2}}\}| \\
& +\sum_{s=1}^{\infty}|\{x: |\partial_{\xi_1}\phi(x,\xi_j^v)-y_1|<2^{s-j\rho}, |\nabla_{\xi'}\phi(x,\xi_j^v)-y'|<2^{s-\frac{j}{2}}\}|2^{-2sM}) \\
\leq\ & C2^{j(m+\frac{n+1}{2})}(2^{-j(\rho+\frac{n-1}{2})}+\sum_{s=1}^{\infty}2^{-j(\rho+\frac{n-1}{2})}2^{s(n-2M)}) \\
\leq\ & C2^{j(m-\rho+1)}.
\end{aligned}
$$

Obviously, it is also true for any v. So we have

$$
\begin{aligned}
\int_{\mathbb{R}^n}|k_1(x,y)|dx &\leq \sum_{j=1}^{\infty}\sum_{v}\int_{\mathbb{R}^n}|k_j^v(x,y)|dx \\
&\leq C\sum_{j=1}^{\infty}\sum_{v}2^{j(m-\rho+1)} \\
&\leq C\sum_{j=1}^{\infty}2^{j(m-\rho+\frac{n+1}{2})}\leq C
\end{aligned}
$$

if $m<\rho-\frac{n+1}{2}$. Thus we have proved the main theorem when $\frac{1}{2}<\rho\leq 1$. □

Declarations

Conflict of interest The authors declare no conflict of interest.

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