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Partial sums of generalized harmonic starlike univalent functions generated by a $(p, q)-$Ruscheweyh-type harmonic differential operator

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Abstract. Let $\mathcal{H}$ denote the class of complex-valued harmonic functions f defined in the open unit disc $\mathbb{D}$ and normalized by $f(0) = f_z(0) - 1 = 0$. In this paper, we define a new generalized subclass of $\mathcal{H}$ associated with the $(p, q)-$Ruscheweyh-type harmonic differential operator in $\mathbb{D}$. We first obtain a sufficient coefficient condition that guarantees that a function f in $\mathcal{H}$ is sense-preserving harmonic univalent in $\mathbb{D}$ and belongs to the aforementioned class. Using this coefficient condition, we then examine ratios of partial sums of f in $\mathcal{H}$. In all cases the results are sharp. In addition, the results so obtained generalize the related works of some authors, and many other new results are obtained.

§1 Introduction

Let $\mathcal{A}$ be the class of functions h that are analytic in the open unit disc $\mathbb{D} := \{z : |z| < 1\}$ with the normalization $h(0) = h'(0) - 1 = 0$. A function $h \in \mathcal{A}$ can be expressed in the form

$$h(z) = z + \sum_{n=2}^{\infty} a_n z^n, \quad (z \in \mathbb{D}). \tag{1.1}$$

For $0 < q < p \leq 1$, a modified $(p, q)-$derivative operator $D_{p,q}$ of a function $h \in \mathcal{A}$ is shown by

$$(D_{p,q}h)(z) = 1 + \sum_{n=2}^{\infty} [n]_{p,q} a_n z^{n-1},$$

where $[n]_{p,q}-$bracket or twin basic number is defined, for any natural number n, by

$$[n]_{p,q} = \frac{p^n - q^n}{p - q}, \quad (q \neq p);$$

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(see [6]). For a function $h \in \mathcal{A}$, it is straightforward to observe that

$$(D_{p,q}h)(z) = \frac{h(pz) - h(qz)}{(p-q)z}, \quad (q \neq p; z \neq 0)$$

and $(D_{p,q}h)(0) = h'(0)$ provided that the function h is differentiable at $z = 0$. Note that for $p = 1$, the $(p,q)-$derivative operator of a function $h \in \mathcal{A}$ reduces to the $q-$derivative operator

$$(D_q h)(z) = \frac{h(z) - h(qz)}{(1-q)z} = 1 + \sum_{n=2}^{\infty} [n]_q a_n z^{n-1}, \quad (z \in \mathbb{D})$$

where

$$[n]_q = \frac{1-q^n}{1-q}, \quad (0 < q < 1).$$

The $(p,q)-$gamma function and the $(p,q)-$factorial are, respectively, defined by

$$\Gamma_{p,q}(n+1) = [n]_{p,q}\Gamma_{p,q}(n),$$

$$[n]_{p,q}! = [1]_{p,q}[2]_{p,q}...[n-1]_{p,q}[n]_{p,q}, \ (n > 0) \quad \text{and} \quad [0]_{p,q}! = 1.$$

The $(p,q)-$shifted factorial is defined as

$$([n]_{p,q})_m = [n]_{p,q}[n+1]_{p,q}...[n+m-1]_{p,q}, \ (m \geq 1) \quad \text{and} \quad ([n]_{p,q})_0 = 1.$$

For definitions and properties of $(p,q)-$calculus or post quantum calculus, one may refer to [13], and references therein. Also, for definitions and properties of $q-$derivative operator, one may refer to [4, 10, 11, 12].

In 1975, Ruscheweyh [19] defined the operator $\mathcal{R}^\lambda : \mathcal{A} \to \mathcal{A}$ given by

$$\mathcal{R}^\lambda h(z) = h(z) * \frac{z}{(1-z)^{\lambda+1}}, \quad (\lambda > -1; z \in \mathbb{D}).$$

The operator $\mathcal{R}^\lambda$ is called λ-Ruscheweyh differential operator. In the last four decades, several researchers defined and studied many Ruscheweyh-type differential operators; see for example [3, 5, 7]. Motivated by these researchers, we define $(p,q)-$Ruscheweyh-type differential operator $\mathcal{R}^\lambda_{p,q}$ in the following.

Definition 1. *Let $0 < q < p \leq 1$ and $\lambda > -1$. For an analytic function $h \in \mathcal{A}$ given by (1.1), the $(p,q)-$Ruscheweyh-type differential operator is defined by*

$$\begin{aligned} \mathcal{R}^\lambda_{p,q}h(z) &= h(z) * F_{p,q,\lambda+1}(z) \\ &= z + \sum_{n=2}^{\infty} \frac{\Gamma_{p,q}(\lambda+n)}{[n-1]_{p,q}!\Gamma_{p,q}(1+\lambda)} a_n z^n, \quad (z \in \mathbb{D}) \end{aligned} \tag{1.2}$$

where $*$ denotes the convolution and

$$F_{p,q,\lambda+1}(z) = z + \sum_{n=2}^{\infty} \frac{\Gamma_{p,q}(\lambda+n)}{[n-1]_{p,q}!\Gamma_{p,q}(1+\lambda)} z^n = z + \sum_{n=2}^{\infty} \frac{([\lambda+1]_{p,q})_{n-1}}{[n-1]_{p,q}!} z^n.$$

Remark 2. *From (1.2), we obtain*

$$\begin{aligned} \mathcal{R}^0_{p,q}h(z) &= h(z) \\ \mathcal{R}^1_{p,q}h(z) &= zD_{p,q}(h(z)) \\ &\vdots \\ \mathcal{R}^m_{p,q}h(z) &= \frac{zD^m_{p,q}(z^{m-1}h(z))}{[m]_{p,q}!}, \quad (\lambda = m, m \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}). \end{aligned}$$

Remark 3. *In particular, for $p = 1$, we obtain $q-$Ruscheweyh differential operator which was defined by Kanas and Raducanu [15];*

$$\mathcal{R}_q^\lambda h(z) = z + \sum_{n=2}^{\infty} \frac{\Gamma_q(\lambda+n)}{[n-1]_q!\Gamma_q(1+\lambda)} a_n z^n = z + \sum_{n=2}^{\infty} \frac{([\lambda+1]_q)_{n-1}}{[n-1]_q!} a_n z^n.$$

Remark 4. *Note that for $p = 1$ and $q \to 1^-$, we obtain*

$$\lim_{q\to 1^-} \mathcal{R}_{1,q}^\lambda h(z) = z + \sum_{n=2}^{\infty} \frac{\Gamma(\lambda+n)}{[n-1]!\Gamma(1+\lambda)} a_n z^n = z + \sum_{n=2}^{\infty} \frac{(\lambda+1)_{n-1}}{(n-1)!} a_n z^n$$

$$= h(z) * \frac{z}{(1-z)^{\lambda+1}} \equiv \mathcal{R}^\lambda h(z).$$

This shows that the differential operator $\mathcal{R}^\lambda h(z)$ defined by Ruscheweyh in [19] is a special case of the operator $\mathcal{R}_{p,q}^\lambda h(z)$ defined in (1.2).

Let $\mathcal{H}$ be the family of complex-valued harmonic functions $f = h + \overline{g}$, where h and g are analytic and have the following power series expansions

$$h(z) = z + \sum_{n=2}^{\infty} a_n z^n \quad \text{and} \quad g(z) = \sum_{n=1}^{\infty} b_n z^n, \quad |b_1| < 1. \tag{1.3}$$

Note that $f = h + \overline{g}$ is sense-preserving in $\mathbb{D}$ if and only if $h'(z) \neq 0$ in $\mathbb{D}$ and the second dilatation of f satisfies the condition $|w(z)| = |g'(z)/h'(z)| < 1$ in $\mathbb{D}$. Let $\mathcal{S}_{\mathcal{H}}$ be a subclass of functions f in $\mathcal{H}$ that are sense-preserving and univalent in $\mathbb{D}$. Clunie and Sheil-Small studied the class $\mathcal{S}_{\mathcal{H}}$ in their remarkable paper [8].

A function $f \in \mathcal{S}_{\mathcal{H}}$ is harmonic starlike in $\mathbb{D}_r = \{z : |z| < r\}$ if $\frac{\partial}{\partial t}(\arg f(r^{eit})) > 0, (t \in [0, 2\pi])$ i.e., f maps the circle $\partial(\mathbb{D}_r)$ onto a closed curve that is starlike with respect to the origin. It is easy to verify that the above condition is equivalent to

$$\text{Re}\left(\frac{D_{\mathcal{H}} f(z)}{f(z)}\right) > 0, \ (|z| = r) \ \text{ where } \ D_{\mathcal{H}} f(z) = zh'(z) - zg'(z).$$

We also recall that the convolution of two complex-valued harmonic functions

$$f_1(z) = z + \sum_{n=2}^{\infty} a_{1n} z^n + \sum_{n=1}^{\infty} \overline{b_{1n} z^n} \quad \text{and} \quad f_2(z) = z + \sum_{n=2}^{\infty} a_{2n} z^n + \sum_{n=1}^{\infty} \overline{b_{2n} z^n}$$

is defined by

$$f_1(z) * f_2(z) = (f_1 * f_2)(z) = z + \sum_{n=2}^{\infty} a_{1n} a_{2n} z^n + \sum_{n=1}^{\infty} \overline{b_{1n} b_{2n} z^n}, \quad (z \in \mathbb{D}).$$

For a comprehensive study of the theory of harmonic univalent functions, one may refer to [1, 2, 9] and references therein.

Using the $(p, q)-$Ruscheweyh-type differential operator $\mathcal{R}_{p,q}^\lambda$ defined by (1.2), for the function $f = h + \overline{g}$ given by (1.3), we define a harmonic differential operator by

$$\begin{aligned} \mathcal{H}_{p,q}^\lambda f(z) &= \mathcal{R}_{p,q}^\lambda h(z) + \overline{\mathcal{R}_{p,q}^\lambda g(z)} \\ &= z + \sum_{n=2}^{\infty} \Phi_{p,q}(n,\lambda) a_n z^n + \sum_{n=1}^{\infty} \Phi_{p,q}(n,\lambda) \overline{b_n z^n}, \quad |b_1| < 1 \end{aligned} \tag{1.4}$$

where

$$\Phi_{p,q}(n,\lambda) = \frac{\Gamma_{p,q}(\lambda+n)}{[n-1]_{p,q}!\Gamma_{p,q}(1+\lambda)}, \quad (\lambda > -1). \tag{1.5}$$

Motivated by the above-discussed literature, we introduce a new subclass of harmonic univalent functions of complex order τ ($\tau \in \mathbb{C} \setminus \{0\}$) and type α ($0 \leq \alpha < 1$) by using the operator $\mathcal{H}^{\lambda}_{p,q} f(z)$ defined by (1.4).

Definition 5. *A function $f = h + \overline{g} \in \mathcal{H}$ is said to belong to the class $(p, q)-$Ruscheweyh-type harmonic starlike of complex order τ and type α, denoted by $\mathcal{SH}^{\lambda}_{p,q}(\tau, \alpha)$, if*

$$\text{Re}\left[1 + \frac{1}{\tau}\left(\frac{zD_{p,q}(\mathcal{H}^{\lambda}_{p,q} f(z))}{\mathcal{H}^{\lambda}_{p,q} f(z)} - 1\right)\right] \geq \alpha, \ (z \in \mathbb{D}) \tag{1.6}$$

where $0 < q < p \leq 1$, $\tau \in \mathbb{C} \setminus \{0\}$, $0 \leq \alpha < 1$, $\lambda > -1$ *and*

$$zD_{p,q}(\mathcal{H}^{\lambda}_{p,q} f(z)) = zD_{p,q}(\mathcal{R}^{\lambda}_{p,q} h(z)) - \overline{zD_{p,q}\mathcal{R}^{\lambda}_{p,q} g(z)}.$$

In view of (1.3), we observe that

$$\begin{aligned} zD_{p,q}(\mathcal{H}^{\lambda}_{p,q} f(z)) &= zD_{p,q}(\mathcal{R}^{\lambda}_{p,q} h(z)) - \overline{zD_{p,q}(\mathcal{R}^{\lambda}_{p,q} g(z))} \\ &= z + \sum_{n=2}^{\infty} [n]_{p,q} \Phi_{p,q}(n, \lambda) a_n z^n - \sum_{n=1}^{\infty} [n]_{p,q} \Phi_{p,q}(n, \lambda) \overline{b_n z^n}. \end{aligned} \tag{1.7}$$

For special values of λ, p, q, τ and α, we obtain the corresponding results for several known subclasses as special cases:

(i) If $\tau = 1$, $\lambda = 0$, $p = 1$ and $q \to 1^-$, then $\mathcal{SH}^{\lambda}_{p,q}(\tau, \alpha) \equiv \mathcal{S}^*_{\mathcal{H}}(\alpha)$, [14].

(ii) If $\tau = 1$, $\lambda = 0$, $\alpha = 0$, $p = 1$ and $q \to 1^-$, then $\mathcal{SH}^{\lambda}_{p,q}(\tau, \alpha) \equiv \mathcal{S}^*_{\mathcal{H}}$, [21].

(iii) If $\tau = 1$, $p = 1$, $q \to 1^-$ and $g(z) \equiv 0$, then $\mathcal{SH}^{\lambda}_{p,q}(\tau, \alpha) \equiv \mathcal{S}^*(\lambda, \alpha)$, [3].

(iv) If $\lambda = 0$, $p = 1$, $q \to 1^-$, $\alpha = 0$ and $g(z) \equiv 0$, then $\mathcal{SH}^{\lambda}_{p,q}(\tau, \alpha) \equiv \mathcal{S}^*(\tau)$, [16].

We may also get several new subclasses as special cases, for example

(v) If $p = 1$, then $\mathcal{SH}^{\lambda}_{p,q}(\tau, \alpha) \equiv \mathcal{SH}^{\lambda}_{q}(\tau, \alpha)$.

(vi) If $g(z) \equiv 0$, then $\mathcal{SH}^{\lambda}_{p,q}(\tau, \alpha) \equiv \mathcal{S}^{*,\lambda}_{p,q}(\tau, \alpha)$.

(vii) If $p = 1$ and $g(z) \equiv 0$, then $\mathcal{SH}^{\lambda}_{p,q}(\tau, \alpha) \equiv \mathcal{S}^{*,\lambda}_{q}(\tau, \alpha)$.

In Section 2, we first obtain a sufficient coefficient condition that shows that a function of the form (1.3) is sense-preserving univalent in $\mathbb{D}$ and belongs to the class $\mathcal{SH}^{\lambda}_{p,q}(\tau, \alpha)$. We then obtain the sharp lower bounds of the real parts of different ratios of a harmonic function f of the form (1.3) to certain sequences of partial sums, where the coefficients $\{a_n\}$ and $\{b_n\}$ satisfy a sufficient condition for a function f in the family $\mathcal{SH}^{\lambda}_{p,q}(\tau, \alpha)$.

§2 Partial Sums of Functions Related to The Class $\mathcal{SH}^{\lambda}_{p,q}(\tau, \alpha)$

Let a function $f = h + \overline{g} \in \mathcal{H}$ of the form (1.3) with $b_1 = 0$. Then the sequences of partial sums of functions f are defined by

$$S_l(f) = z + \sum_{n=2}^{l} a_n z^n + \sum_{n=2}^{\infty} \overline{b_n z^n} := S_l(h) + \overline{g},$$

$$S_m(f) = z + \sum_{n=2}^{\infty} a_n z^n + \sum_{n=2}^{m} \overline{b_n z^n} := h + S_m(\overline{g}),$$
$$S_{l,m}(f) = z + \sum_{n=2}^{l} a_n z^n + \sum_{n=2}^{m} \overline{b_n z^n} := S_l(h) + S_m(\overline{g}).$$

Silverman [20] and Silvia [22] studied partial sums for starlike and convex functions. Recently, Porwal [17], Porwal and Dixit [18] studied some results on the partial sums of certain harmonic univalent functions. In order to obtain partial sums related to the class $\mathcal{SH}^{\lambda}_{p,q}(\tau,\alpha)$, we need the following sufficient condition for a function to be in this class.

Lemma 6. *Let $0 < q < p \leq 1$, $\tau \in \mathbb{C} \setminus \{0\}$, $0 \leq \alpha < 1$, $\lambda > -1$, and $z \in \mathbb{D}$. If a harmonic function $f = h + \overline{g}$ of the form (1.3) holds the condition*

$$\sum_{n=2}^{\infty} \xi_n |a_n| + \sum_{n=1}^{\infty} \mu_n |b_n| \leq |\tau|(1-\alpha), \tag{2.1}$$

where

$$\xi_n = ([n]_{p,q} - 1 + |\tau|(1-\alpha))\Phi_{p,q}(n,\lambda), \ (n \geq 2) \tag{2.2}$$

$$\mu_n = ([n]_{p,q} + 1 - |\tau|(1-\alpha))\Phi_{p,q}(n,\lambda), \ (n \geq 1) \tag{2.3}$$

and $\Phi_{p,q}(n,\lambda)$ is given by (1.5), then the function f is sense-preserving harmonic univalent in $\mathbb{D}$ and $f \in \mathcal{SH}^{\lambda}_{p,q}(\tau,\alpha)$.

Proof. Let $f = h + \overline{g}$ be of the form (1.3), and assume that there exists $n \in \{1,2,3,\dots\}$ such that $a_n \neq 0$ or $b_n \neq 0$. It is clear that the condition (2.1) equivalent to

$$\sum_{n=2}^{\infty} [n]_{p,q} |a_n| + \sum_{n=1}^{\infty} [n]_{p,q} |b_n| \leq 1. \tag{2.4}$$

Thus we have

$$\begin{aligned}
|D_{p,q}h(z)| &\geq 1 - \sum_{n=2}^{\infty} [n]_{p,q} |a_n| |z|^{n-1} \\
&> 1 - \sum_{n=2}^{\infty} \frac{\big([n]_{p,q} - 1 + |\tau|(1-\alpha)\big)\Phi_{p,q}(n,\lambda)}{|\tau|(1-\alpha)} |a_n| \\
&\geq \sum_{n=1}^{\infty} \frac{\big([n]_{p,q} + 1 - |\tau|(1-\alpha)\big)\Phi_{p,q}(n,\lambda)}{|\tau|(1-\alpha)} |b_n| \\
&> \sum_{n=1}^{\infty} [n]_{p,q} |b_n| |z|^{n-1} \geq |D_{p,q}g(z)|,
\end{aligned}$$

which proves $|D_{p,q}h(z)| > |D_{p,q}g(z)|$, that is, the function f is locally univalent and sense-preserving in $\mathbb{D}$. If $z_1, z_2 \in \mathbb{D}$ and for some p, q such that $pz_1 \neq qz_2$, then

$$\begin{aligned}
\left| \frac{(pz_1)^n - (qz_2)^n}{(pz_1) - (qz_2)} \right| &= \left| \sum_{l=1}^{n} (pz_1)^{l-1} (qz_2)^{n-l} \right| \\
&\leq \sum_{l=1}^{n} p^{l-1} q^{n-l} |z_1|^{l-1} |z_2|^{n-l} < [n]_{p,q}, \quad (n = 1,2,3,\dots).
\end{aligned}$$

Hence, using this relation with (2.1) and (2.4), we have

$$\begin{aligned}
\left|\frac{f(pz_1)-f(qz_2)}{h(pz_1)-h(qz_2)}\right| &\geq 1-\left|\frac{g(pz_1)-g(qz_2)}{h(pz_1)-h(qz_2)}\right| \\
&= 1-\left|\frac{\sum\limits_{n=1}^{\infty}\overline{b_n((pz_1)^n-(qz_2)^n)}}{(pz_1-qz_2)+\sum\limits_{n=2}^{\infty}a_n((pz_1)^n-(qz_2)^n)}\right| \\
&> 1-\frac{\sum\limits_{n=1}^{\infty}[n]_{p,q}|b_n|}{1-\sum\limits_{n=2}^{\infty}[n]_{p,q}|a_n|} \\
&\geq 1-\frac{\sum\limits_{n=1}^{\infty}\frac{\left([n]_{p,q}+1-|\tau|(1-\alpha)\right)\Phi_{p,q}(n,\lambda)}{|\tau|(1-\alpha)}|b_n|}{1-\sum\limits_{n=2}^{\infty}\frac{\left([n]_{p,q}-1+|\tau|(1-\alpha)\right)\Phi_{p,q}(n,\lambda)}{|\tau|(1-\alpha)}|a_n|} \geq 0.
\end{aligned}$$

This proves the univalence of f.

In view of (1.6) and using the fact that $\mathrm{Re}(w) > \alpha$ if and only if $|1-\alpha+w| > |1+\alpha-w|$, it suffices to show that

$$\begin{aligned}
&\left|(2\tau-\tau\alpha-1)\mathcal{H}_{p,q}^{\lambda}f(z)+zD_{p,q}(\mathcal{H}_{p,q}^{\lambda}f(z))\right| \\
&-\left|(\tau\alpha+1)\mathcal{H}_{p,q}^{\lambda}f(z)-zD_{p,q}(\mathcal{H}_{p,q}^{\lambda}f(z))\right| \geq 0.
\end{aligned} \tag{2.5}$$

Therefore, setting (1.4) and (1.7) into the left side of (2.5), we get

$$\begin{aligned}
&= \left|\tau(2-\alpha)z+\sum_{n=2}^{\infty}([n]_{p,q}-1+\tau(2-\alpha))\Phi_{p,q}(n,\lambda)a_nz^n\right. \\
&\quad \left.-\sum_{n=1}^{\infty}([n]_{p,q}+1-\tau(2-\alpha))\Phi_{p,q}(n,\lambda)\overline{b_nz^n}\right| \\
&\quad -\left|\tau\alpha z-\sum_{n=2}^{\infty}([n]_{p,q}-1-\tau\alpha)\Phi_{p,q}(n,\lambda)a_nz^n+\sum_{n=1}^{\infty}([n]_{p,q}+1+\tau\alpha))\Phi_{p,q}(n,\lambda)\overline{b_nz^n}\right| \\
&\geq 2|\tau|(1-\alpha)|z|-2\sum_{n=2}^{\infty}([n]_{p,q}-1+|\tau|(1-\alpha))\Phi_{p,q}(n,\lambda)|a_n||z|^n \\
&\quad -2\sum_{n=1}^{\infty}([n]_{p,q}+1-|\tau|(1-\alpha))\Phi_{p,q}(n,\lambda)|b_n||z|^n \\
&\geq |\tau|(1-\alpha)|z|\left(1-\sum_{n=2}^{\infty}\frac{\xi_n}{|\tau|(1-\alpha)}|a_n||z|^{n-1}-\sum_{n=1}^{\infty}\frac{\mu_n}{|\tau|(1-\alpha)}|b_n||z|^{n-1}\right) \geq 0,
\end{aligned}$$

by (2.1). This proves that $f \in \mathcal{SH}_{p,q}^{\lambda}(\tau,\alpha)$. The function

$$f(z) = z+\sum_{n=2}^{\infty}\frac{|\tau|(1-\alpha)}{\xi_n}x_nz^n+\sum_{n=1}^{\infty}\frac{|\tau|(1-\alpha)}{\mu_n}\overline{y_nz^n},$$

where $\sum\limits_{n=2}^{\infty}|x_n|+\sum\limits_{n=1}^{\infty}|y_n| = 1$ shows that the coefficient bound given by (2.1) is sharp. □

Theorem 7. *Let* $0 < q < p \leq 1$*,* $\tau \in \mathbb{C}\setminus\{0\}$*,* $0 \leq \alpha < 1$*,* $\lambda > -1$ *and* $z \in \mathbb{D}$*. Suppose* ξ_n *and*

μ_n given by (2.2) and (2.3) satisfy the conditions

$$\xi_n \geq \begin{cases} |\tau|(1-\alpha), & n=2,3,\ldots,l \\ \\ \xi_{l+1}, & n=l+1,l+2,\ldots, \end{cases} \tag{2.6}$$

and

$$\mu_n \geq |\tau|(1-\alpha),\ (n=2,3,\ldots). \tag{2.7}$$

If a function $f = h + \overline{g}$ of the form (1.3) with $b_1 = 0$ satisfies the condition (2.1), then

$$i)\ \operatorname{Re}\left(\frac{f(z)}{S_l(f)(z)}\right) \geq 1 - \frac{|\tau|(1-\alpha)}{\xi_{l+1}}, \tag{2.8}$$

$$ii)\ \operatorname{Re}\left(\frac{S_l(f)(z)}{f(z)}\right) \geq \frac{\xi_{l+1}}{\xi_{l+1} + |\tau|(1-\alpha)}. \tag{2.9}$$

These estimates are sharp for the function given by

$$f(z) = z + \frac{|\tau|(1-\alpha)}{\xi_{l+1}} z^{l+1}. \tag{2.10}$$

Proof. i) In order to prove (2.8), we may write

$$\psi_1(z) := \frac{\xi_{l+1}}{|\tau|(1-\alpha)}\left\{\frac{f(z)}{S_l(f)(z)} - \left(1 - \frac{|\tau|(1-\alpha)}{\xi_{l+1}}\right)\right\}$$

$$= 1 + \frac{\frac{\xi_{l+1}}{|\tau|(1-\alpha)} \sum\limits_{n=l+1}^{\infty} a_n z^n}{z + \sum\limits_{n=2}^{l} a_n z^n + \sum\limits_{n=2}^{\infty} \overline{b_n z^n}}.$$

It is sufficient to show that $\operatorname{Re}(\psi_1(z)) > 0$, or equivalently

$$\left|\frac{\psi_1(z) - 1}{\psi_1(z) + 1}\right| \leq 1.$$

Note that

$$\left|\frac{\psi_1(z) - 1}{\psi_1(z) + 1}\right| \leq \frac{\frac{\xi_{l+1}}{|\tau|(1-\alpha)} \sum\limits_{n=l+1}^{\infty} |a_n|}{2 - 2(\sum\limits_{n=2}^{l} |a_n| + \sum\limits_{n=2}^{\infty} |b_n|) - \frac{\xi_{l+1}}{|\tau|(1-\alpha)} \sum\limits_{n=l+1}^{\infty} |a_n|}.$$

This last expression is bounded above by 1 if and only if

$$\sum_{n=2}^{l} |a_n| + \sum_{n=2}^{\infty} |b_n| + \frac{\xi_{l+1}}{|\tau|(1-\alpha)} \sum_{n=l+1}^{\infty} |a_n| \leq 1. \tag{2.11}$$

In view of sufficient condition (2.1) with $b_1 = 0$, it is therefore sufficient to show that left side of (2.11) is bounded above by

$$\sum_{n=2}^{\infty} \frac{\xi_n}{|\tau|(1-\alpha)} |a_n| + \sum_{n=2}^{\infty} \frac{\mu_n}{|\tau|(1-\alpha)} |b_n|,$$

which is equivalent to

$$\sum_{n=2}^{l} \frac{\xi_n - |\tau|(1-\alpha)}{|\tau|(1-\alpha)} |a_n| + \sum_{n=2}^{\infty} \frac{\mu_n - |\tau|(1-\alpha)}{|\tau|(1-\alpha)} |b_n| + \sum_{n=l+1}^{\infty} \frac{\xi_n - \xi_{l+1}}{|\tau|(1-\alpha)} |a_n| \geq 0.$$

But the last inequality is true because of the given conditions (2.6) and (2.7). If we take

$f(z) = z + \frac{|\tau|(1-\alpha)}{\xi_{l+1}} z^{l+1}$ with $z = re^{i\pi/l}$ and r approaches to 1 from left, then we get

$$\frac{f(z)}{S_l(f)(z)} = 1 + \frac{|\tau|(1-\alpha)}{\xi_{l+1}} z^l \to 1 - \frac{|\tau|(1-\alpha)}{\xi_{l+1}},$$

which shows that the bound in (2.8) best possible. This completes the first part of the proof.

ii) Similarly, we have

$$\psi_2(z) := \frac{\xi_{l+1} + |\tau|(1-\alpha)}{|\tau|(1-\alpha)} \left\{ \frac{S_l(f)(z)}{f(z)} - \left(1 - \frac{|\tau|(1-\alpha)}{\xi_{l+1} + |\tau|(1-\alpha)}\right) \right\}$$

$$= 1 - \frac{\frac{\xi_{l+1}+|\tau|(1-\alpha)}{|\tau|(1-\alpha)} \sum\limits_{n=l+1}^{\infty} a_n z^n}{z + \sum\limits_{n=2}^{\infty} a_n z^n + \sum\limits_{n=2}^{\infty} \overline{b_n z^n}}.$$

Therefore,

$$\left|\frac{\psi_2(z) - 1}{\psi_2(z) + 1}\right| \le \frac{\frac{\xi_{l+1}+|\tau|(1-\alpha)}{|\tau|(1-\alpha)} \sum\limits_{n=l+1}^{\infty} |a_n|}{2 - 2(\sum\limits_{n=2}^{l} |a_n| + \sum\limits_{n=2}^{\infty} |b_n|) - \frac{\xi_{l+1}-|\tau|(1-\alpha)}{|\tau|(1-\alpha)} \sum\limits_{n=l+1}^{\infty} |a_n|} \le 1$$

if and only if

$$\sum_{n=2}^{l} |a_n| + \sum_{n=2}^{\infty} |b_n| + \frac{\xi_{l+1}}{|\tau|(1-\alpha)} \sum_{n=l+1}^{\infty} |a_n| \le 1. \tag{2.12}$$

Since the left side of (2.12) is bounded above by

$$\sum_{n=2}^{\infty} \frac{\xi_n}{|\tau|(1-\alpha)} |a_n| + \sum_{n=2}^{\infty} \frac{\mu_n}{|\tau|(1-\alpha)} |b_n|,$$

the proof is completed because of the arguments as used in the proof of part *(i)*. □

Theorem 8. *Let $0 < q < p \le 1$, $\tau \in \mathbb{C} \setminus \{0\}$, $0 \le \alpha < 1$, $\lambda > -1$ and $z \in \mathbb{D}$. Suppose ξ_n and μ_n given by (2.2) and (2.3) satisfy the conditions*

$$\mu_n \ge \begin{cases} |\tau|(1-\alpha), & n=2,3,\ldots,m \\ \mu_{m+1}, & n=m+1,m+2,\ldots, \end{cases} \tag{2.13}$$

and

$$\xi_n \ge |\tau|(1-\alpha), \ (n = 2, 3, \ldots). \tag{2.14}$$

If a function $f = h + \overline{g}$ of the form (1.3) with $b_1 = 0$ satisfies the condition (2.1), then

$$i) \ \operatorname{Re}\left(\frac{f(z)}{S_m(f)(z)}\right) \ge 1 - \frac{|\tau|(1-\alpha)}{\mu_{m+1}}, \tag{2.15}$$

$$ii) \ \operatorname{Re}\left(\frac{S_m(f)(z)}{f(z)}\right) \ge \frac{\mu_{m+1}}{\mu_{m+1} + |\tau|(1-\alpha)}. \tag{2.16}$$

These estimates are sharp for the function given by

$$f(z) = z + \frac{|\tau|(1-\alpha)}{\mu_{m+1}}\overline{z}^{m+1}. \tag{2.17}$$

Proof. *i)* In order to prove (2.15), we may write

$$\begin{aligned}\psi_3(z) &:= \frac{\mu_{m+1}}{|\tau|(1-\alpha)}\left\{\frac{f(z)}{S_m(f)(z)} - \left(1 - \frac{|\tau|(1-\alpha)}{\mu_{m+1}}\right)\right\} \\ &= 1 + \frac{\frac{\mu_{m+1}}{|\tau|(1-\alpha)}\sum\limits_{n=m+1}^{\infty}\overline{b_n z^n}}{z + \sum\limits_{n=2}^{\infty} a_n z^n + \sum\limits_{n=2}^{m}\overline{b_n z^n}}.\end{aligned}$$

It suffices to show that $\mathrm{Re}(\psi_3(z)) > 0$, or equivalently

$$\left|\frac{\psi_3(z)-1}{\psi_3(z)+1}\right| \le \frac{\frac{\mu_{m+1}}{|\tau|(1-\alpha)}\sum\limits_{n=m+1}^{\infty}|b_n|}{2 - 2(\sum\limits_{n=2}^{\infty}|a_n| + \sum\limits_{n=2}^{m}|b_n|) - \frac{\mu_{m+1}}{|\tau|(1-\alpha)}\sum\limits_{n=m+1}^{\infty}|b_n|} \le 1$$

if and only if

$$\sum_{n=2}^{\infty}|a_n| + \sum_{n=2}^{m}|b_n| + \frac{\mu_{m+1}}{|\tau|(1-\alpha)}\sum_{n=m+1}^{\infty}|b_n| \le 1. \tag{2.18}$$

It is now sufficient to show that left side of (2.18) is bounded above by

$$\sum_{n=2}^{\infty}\frac{\xi_n}{|\tau|(1-\alpha)}|a_n| + \sum_{n=2}^{\infty}\frac{\mu_n}{|\tau|(1-\alpha)}|b_n|,$$

which is equivalent to

$$\sum_{n=2}^{\infty}\frac{\xi_n - |\tau|(1-\alpha)}{|\tau|(1-\alpha)}|a_n| + \sum_{n=2}^{m}\frac{\mu_n - |\tau|(1-\alpha)}{|\tau|(1-\alpha)}|b_n| + \sum_{n=m+1}^{\infty}\frac{\mu_n - \mu_{m+1}}{|\tau|(1-\alpha)}|a_n| \ge 0.$$

Due to the given conditions (2.13) and (2.14), the above inequality is true. This completes the first part of the theorem.

To prove that $f(z) = z + \frac{|\tau|(1-\alpha)}{\mu_{m+1}}\overline{z}^{m+1}$ gives the sharp result, we observe that for $z = re^{i\pi/m+2}$ we have

$$\frac{f(z)}{S_m(f)(z)} = 1 + \frac{|\tau|(1-\alpha)}{\mu_{m+1}}r^m e^{-i(m+2)\frac{\pi}{m+2}} \to 1 - \frac{|\tau|(1-\alpha)}{\mu_{m+1}},$$

when $r \to 1^-$.

ii) Similarly, by using the method of proof of part *(i)*, we obtain the proof of (2.16). □

Theorem 9. *Let* $0 < q < p \le 1$, $\tau \in \mathbb{C}\setminus\{0\}$, $0 \le \alpha < 1$, $\lambda > -1$ *and* $z \in \mathbb{D}$. *Suppose* ξ_n *and* μ_n *given by (2.2) and (2.3) satisfy the conditions*

$$\xi_n \ge \begin{cases} |\tau|(1-\alpha), & n=2,3,\ldots,l \\ \xi_{l+1}, & n=l+1,l+2,\ldots, \end{cases} \tag{2.19}$$

$$\mu_n \ge \begin{cases} |\tau|(1-\alpha), & n=2,3,\ldots,l \\ \xi_{l+1}, & n=l+1,l+2,\ldots\ . \end{cases} \tag{2.20}$$

If a function $f = h + \overline{g}$ given by (1.3) with $b_1 = 0$ satisfies the condition (2.1), then

$$i)\ \operatorname{Re}\left(\frac{f(z)}{S_{l,m}(f)(z)}\right) \geq 1 - \frac{|\tau|(1-\alpha)}{\xi_{l+1}}, \tag{2.21}$$

$$ii)\ \operatorname{Re}\left(\frac{S_{l,m}(f)(z)}{f(z)}\right) \geq \frac{\xi_{l+1}}{\xi_{l+1} + |\tau|(1-\alpha)}. \tag{2.22}$$

These estimates are sharp for the function given by (2.10).

Proof. i) In order to prove (2.21), we may write

$$\begin{aligned}\psi_4(z) &:= \frac{\xi_{l+1}}{|\tau|(1-\alpha)}\left\{\frac{f(z)}{S_{l,m}(f)(z)} - \left(1 - \frac{|\tau|(1-\alpha)}{\xi_{l+1}}\right)\right\} \\ &= 1 + \frac{\frac{\xi_{l+1}}{|\tau|(1-\alpha)}\left(\sum\limits_{n=l+1}^{\infty} a_n z^n + \sum\limits_{n=m+1}^{\infty} \overline{b_n z^n}\right)}{z + \sum\limits_{n=2}^{l} a_n z^n + \sum\limits_{n=2}^{m} \overline{b_n z^n}}.\end{aligned}$$

It suffices to show that $\operatorname{Re}(\psi_4(z)) > 0$, or equivalently

$$\left|\frac{\psi_4(z)-1}{\psi_4(z)+1}\right| \leq \frac{\frac{\xi_{l+1}}{|\tau|(1-\alpha)}\left(\sum\limits_{n=l+1}^{\infty}|a_n| + \sum\limits_{n=m+1}^{\infty}|b_n|\right)}{2 - 2(\sum\limits_{n=2}^{l}|a_n| + \sum\limits_{n=2}^{m}|b_n|) - \frac{\xi_{l+1}}{|\tau|(1-\alpha)}\left(\sum\limits_{n=l+1}^{\infty}|a_n| + \sum\limits_{n=m+1}^{\infty}|b_n|\right)}.$$

This last expression is bounded above by 1 if and only if

$$\sum_{n=2}^{l}|a_n| + \sum_{n=2}^{m}|b_n| + \frac{\xi_{l+1}}{|\tau|(1-\alpha)}\left(\sum_{n=l+1}^{\infty}|a_n| + \sum_{n=m+1}^{\infty}|b_n|\right) \leq 1. \tag{2.23}$$

In view of (2.1) with $b_1 = 0$, it is now sufficient to show that the left side of (2.23) is bounded above by

$$\sum_{n=2}^{\infty}\frac{\xi_n}{|\tau|(1-\alpha)}|a_n| + \sum_{n=2}^{\infty}\frac{\mu_n}{|\tau|(1-\alpha)}|b_n|,$$

which is equivalent to

$$\begin{aligned}&\sum_{n=2}^{l}\frac{\xi_n - |\tau|(1-\alpha)}{|\tau|(1-\alpha)}|a_n| + \sum_{n=2}^{m}\frac{\mu_n - |\tau|(1-\alpha)}{|\tau|(1-\alpha)}|b_n| + \\ &\sum_{n=l+1}^{\infty}\frac{\xi_n - \xi_{l+1}}{|\tau|(1-\alpha)}|a_n| + \sum_{n=m+1}^{\infty}\frac{\mu_n - \xi_{l+1}}{|\tau|(1-\alpha)}|b_n| \geq 0.\end{aligned}$$

In view of (2.19) and (2.20), we conclude that the above inequality is true.

To see that $f(z) = z + \frac{|\tau|(1-\alpha)}{\xi_{l+1}} z^{l+1}$ gives the sharp result, we observe that for $z = re^{i\pi/l}$, we have

$$\frac{f(z)}{S_{l,m}(f)(z)} = 1 + \frac{|\tau|(1-\alpha)}{\xi_{l+1}} z^l \to 1 - \frac{|\tau|(1-\alpha)}{\xi_{l+1}},\ (r \to 1^-).$$

ii) Similarly, we prove (2.22). □

Theorem 10. *Let $0 < q < p \leq 1$, $\tau \in \mathbb{C} \setminus \{0\}$, $0 \leq \alpha < 1$, $\lambda > -1$ and $z \in \mathbb{D}$. Suppose ξ_n and*

μ_n given by (2.2) and (2.3) satisfy the conditions

$$\xi_n \geq \begin{cases} |\tau|(1-\alpha), & n=2,3,\dots,m \\ \\ \mu_{m+1}, & n=m+1,m+2,\dots, \end{cases} \tag{2.24}$$

$$\mu_n \geq \begin{cases} |\tau|(1-\alpha), & n=2,3,\dots,m \\ \\ \mu_{m+1}, & n=m+1,m+2,\dots . \end{cases} \tag{2.25}$$

If a function $f = h + \overline{g}$ given by (1.3) with $b_1 = 0$ satisfies the condition (2.1), then

$$i) \ \operatorname{Re}\left(\frac{f(z)}{S_{l,m}(f)(z)}\right) \geq \quad 1 - \frac{|\tau|(1-\alpha)}{\mu_{m+1}}, \tag{2.26}$$

$$ii) \ \operatorname{Re}\left(\frac{S_{l,m}(f)(z)}{f(z)}\right) \geq \frac{\mu_{m+1}}{\mu_{m+1} + |\tau|(1-\alpha)}. \tag{2.27}$$

These estimates are sharp for the function given by (2.17).

Proof. It is omitted because it is similar to the proof of Theorem 9. □

Declarations

Conflict of interest The authors declare no conflict of interest.

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