Appl. Math. J. Chinese Univ.
2024, 39(4): 571-583

The super-connectivity of graphs with two orbits

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Abstract. A graph G is said to be super-connected or simply super-κ, if each minimum vertex cut of G isolates a vertex. A graph G is said to be a k-vertex-orbit graph if there are k vertex orbits when $Aut(G)$ acts on $V(G)$. A graph G is said to be a k-edge-orbit graph if there are k edge orbits when $Aut(G)$ acts on edge set $E(G)$. In this paper, we give a necessary and sufficient condition for connected bipartite 2-vertex-orbit graphs to be super-κ. For 2-edge-orbit graphs, we give a sufficient condition for connected 2-edge-orbit graphs to be super-κ. In addition, we show that if G is a k-regular connected irreducible II-kind 2-edge-orbit graph with $k \leq 6$ and girth $g(G) \geq 6$, or G is a k-regular connected irreducible III-kind 2-edge-orbit graph with $k \leq 6$ and girth $g(G) \geq 8$, then G is super-connected.

§1 Introduction

With the rapid development of information networks, the network comes into focus. The topology of information network is very important to influence the network performance. When designing the underlying topology of a multiprocessor network, what we care about is the reliability of the network, that is, the ability of the network to function even when some vertices or edges fail. The underlying topology of a network is often modelled as a graph, hence, some classical notations of graph theory, such as the vertex connectivity and the edge connectivity, are utilized to measure the reliability of networks.

It is well known that the underlying topology of an interconnection network can be modelled by a graph G, and the connectivity $\kappa(G)$ of G is the minimum cardinality of a set $S \subseteq V(G)$ such that $G-S$ is either disconnected or trivial, which is an important measure for fault tolerance of the network. Whitney in [9] observed that $\kappa(G)$ never exceeds $\delta(G)$, the minimum degree of G. Graphs for which $\kappa(G) = \delta(G)$ are called *maximally connected*. However, this parameter has some intrinsic shortcomings. To overcome this shortcoming, Harary in [8] generalized the notion

Received: 2019-3-29. Revised: 2022-01-26.
MR Subject Classification: 05C10.
Keywords: super-connectivity, superatom, edge orbit.
Digital Object Identifier(DOI): https://doi.org/10.1007/s11766-024-3790-8.
Supported by the National Natural Science Foundation of Xinjiang(2020D04046), the National Natural Science Foundation of Shanxi(20210302123097), the National Natural Science Foundation of China(12371356, 11961067).
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of connectivity by introducing conditional connectivity. The *conditional connectivity* of a graph G with respect to some graph-theoretic property P is the size of the smallest set S of vertices (or edges), if such a set exists, such that $G-S$ is disconnected and every remaining component has property P. The conditional connectivity not only integrates all kinds of classical connectivity concepts, but also produces a large number of new concepts of connectivity in the network optimization design. On the basis of conditional connectivity, Boesch proposed the concept of super-connected graphs in [1]. A graph G is said to be *super-connected* or simply *super-*κ, if each minimum vertex cut of G isolated a vertex. In recent years, there are many studies on this subject, see [5,7,11,23] for example.

Let $G=(V,E)$ be a simple undirected connected graph with the vertex set $V=V(G)$ and the edge set $E=E(G)$. For each vertex $x\in V(G)$, the *neighborhood* $N(x)$ is defined as the set of vertices adjacent to x, and the *degree* $d(x)$ is the cardinality of $N(x)$, that is, $d(x)=|N(x)|$. The *minimum degree* of a graph G is $\delta(G)=\min\{d(x):x\in V(G)\}$, and the *maximum degree* of a graph G is $\Delta(G)=\max\{d(x):x\in V(G)\}$. If $\delta(G)=\Delta(G)=k$, then the graph is *regular* of degree k, where k is a positive integer. Denote by $\Gamma=Aut(G)$ the *automorphism group* of G. A graph G is said to be *edge transitive* if $Aut(G)$ acts transitively on $E(G)$, and G is said to be *vertex transitive* if $Aut(G)$ acts transitively on $V(G)$. For $v\in V(G)$, the set $\Gamma(v)=\{\varphi(v):\varphi\in\Gamma\}$ is called a Γ*-vertex-orbit*, and for $e\in E(G)$, the set $\Gamma(e)=\{\varphi(e):\varphi\in\Gamma\}$ is called a Γ*-edge-orbit*. Clearly, all Γ-vertex-orbits constitute a partition of $V(G)$, and all Γ-edge-orbits constitute a partition of $E(G)$. If there is only one Γ-vertex (edge)-orbit, then we say that G is Γ*-vertex (edge) transitive*. Clearly, vertex(edge) transitive is $Aut(G)$-vertex(edge) transitive. $Aut(G)$-vertex (edge)-orbits are simply called the vertex (edge) orbits of G. A graph G is said to be *reducible* if there exist two vertices u and v satisfying $N(u)=N(v)$, otherwise G is *irreducible*.

It is well known that the edge connectivity of vertex transitive graphs is regularity, and the connectivity of edge transitive graphs is equal to the minimum degree (to see [18]). If there are exactly two vertex orbits, then we say that G is a *2-vertex-orbit graph*. Similarly, we can define the $m(\geq 3)$*-vertex-orbit graph*, for a short *multiorbit graph*. Compared with the vertex transitive graph, the symmetry of multiorbit graphs becomes weaker. The study of the influence of the number of vertex orbits on connectivity becomes a natural extension of the study of vertex transitive graphs. In this paper, we mainly focus on the super-connectivity of the 2-vertex-orbit graph.

The *lexicographic product* of a graph G by a graph H, denoted by $G(H)$, is the graph with vertex set $V(G(H))=\{(u,v)|u\in V(G)$ and $v\in V(H)\}$, and edge set $E(G(H))=\{(u_1,v_1)(u_2,v_2)|$ either u_1 is adjacent to u_2 in G or $u_1=u_2$ and v_1 is adjacent to v_2 in $H\}$. Graphs with multiple edges are called *multiple* graphs. Graphs without edges are called *null* graphs. A null graph on m vertices is denoted by N_m. Denote by C_k the cycle of length k, when k is even, C_k is a bipartite graph with $V(C_k)=V_1\cup V_2$, for any $u\in V_1$ and $v\in V_2$, we replace u with N_m and v with N_n, respectively, if $uv\in E(G)$, then $G[N_m\cup N_n]$ is a complete bipartite graph with two parts of vertices number m and n, thus we get a new graph $G=C_k(N_m,N_n)$, where k is even. Denote by Q_3 the 3-cube, $\mathscr{L}(Q_3)$ the line graph of Q_3, and K_n the complete graph with n vertices, $K_{m,n}$ the complete bipartite graph with two parts of vertices number

m and n. For graph-theoretical terminology and notation not defined here we follow [2]. In a bipartite graph G, if for any two vertices u and v in a same part, there exists an automorphism $\varphi \in Aut(G)$ satisfying $\varphi(u) = v$, then G is said to be semi-transitive.

Up to now, there are not many researches on the super-connectivity of vertex (edge) transitive graphs. In [15], Meng described the maximal connectedness of vertex and edge transitive graphs, and proved when G is a connected vertex and edge transitive graph, G is not super-κ if and only if $G \cong C_n(N_m)(n \geq 6$ and $m \geq 1)$ or $G \cong \mathscr{L}(Q_3)(N_m)(m \geq 1)$. In [11], Liang et.al. characterized the super-connectivity of vertex transitive bipartite graphs. In [23], Zhang and Meng described the Super-connectivity of edge transitive graphs, and proved: If G is a connected edge transitive graph, and $\overline{G}$ is semi-transitive, then G is super-connected, where $\overline{G}$ is a quotient of G. If G is a connected irreducible edge transitive graph, then G is super-connected with the only exception when $G \cong C_n(n \geq 6)$ or $\mathscr{L}(Q_3)$.

So far, the study of multiple graphs is concentrated mainly on the 2-vertex-orbit graph. In [10], Liang and Meng characterized the maximal connectedness of bipartite 2-vertex-orbit graphs. In [12], Liu and Meng characterized the edge connectivity of regular 2-vertex-orbit graphs. In [20], Yang et.al. described the edge connectivity of 2-vertex-orbit graphs of the same size. In [14], Lin et.al. characterized the super restricted edge connectivity of regular 2-vertex-orbit graphs.

In this paper, we describe the super-connectivity of 2-vertex-orbit graphs. In section 2, we give some preliminaries. In section 3, we prove that if G is an irreducible connected bipartite 2-vertex-orbit graph, then G is super-connected. In section 4, we mainly research the Super-connectivity of 2-edge-orbit graphs.

§2 Preliminaries

Let $G = (V(G), E(G))$ be a connected graph and F be a non-empty subset of $V(G)$. Set $N(F) = \{x \in V(G) \setminus F :$ there exists $y \in F$ satisfying $xy \in E(G)\}$, $C(F) = F \cup N(F)$ and $R(F) = V(G) \setminus C(F)$. Clearly, the set $N(F)$ is a vertex cut set if $R(F) \neq \emptyset$. A vertex set $F \subset V(G)$ is said to be a *fragment* if $|N(F)| = \kappa(G)$ and $R(F) \neq \emptyset$. A fragment of minimum cardinality is called an *atom* of G. A fragment F with $2 \leq |F| \leq |V(G)| - \kappa(G) - 2$ is called a *strict fragment* of G. If there exists a strict fragment in G, then G is said to be *degenerate*. Clearly, if F is a strict fragment of G, so is $R(F)$. A strict fragment of G with minimum cardinality is called a *superatom* of G. The cardinality of a superatom of G is denoted by $\omega(G)$. If $A \subseteq V(G)$, then $G[A]$ denotes the subgraph of G induced by A.

An *imprimitive block* of G is a vertex set $F \subseteq V(G)$ such that, for any automorphism $\varphi \in Aut(G)$, either $\varphi(F) = F$ or $\varphi(F) \cap F = \emptyset$.

In [23], we can find the relationship between a superatom and a strict fragment of a graph G.

Theorem 2.1. *([23]) Let G be a connected degenerate graph with $\omega(G) \geq 3$ and $g(G) \geq 4$, where $g(G)$ is the girth of G. Let A be a superatom of G and B be a strict fragment of G. If $A \cap B \neq \emptyset$, then $A \subseteq B$.*

The following theorem plays an important role in discussing the vertex connectivity of graphs.

Theorem 2.2. *([18]) Let G be a connected graph, and A, F and C be its atom, fragment and minimum vertex cut of G, respectively. Then*

(1) $A \cap F = \emptyset$ or $A \subset F$;

(2) $A \cap C = \emptyset$ or $A \subset C$.

In the following, we will classify the connected graphs with two edge orbits. To this end, we introduce a well-known result.

Theorem 2.3. *([6]) Let G be a graph with $\delta(G) > 0$ and Γ be a subgroup of $Aut(G)$. If G is Γ-edge transitive, then either G is Γ-vertex transitive or G is a bipartite graph whose sets of bipartition are its Γ-vertex-orbits.*

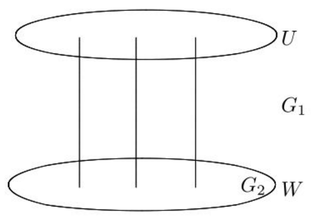
Let A be a vertex set (or an edge set) of G, and $G[A]$ be introduced by A. We have the following classification result for connected 2-edge-orbit graphs.

Theorem 2.4. *Let G be a connected 2-edge-orbit graph, E_1 and E_2 be its two edge orbits under $\Gamma = Aut(G)$, $G_1 = G[E_1]$ and $G_2 = G[E_2]$. Then one of the following cases occurs.*

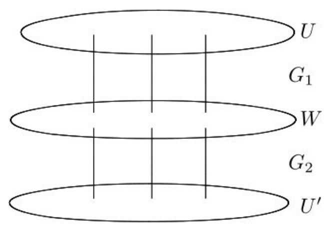
(i) $G = G_1 \cup G_2$, $V(G_1) = V(G_2) = V(G)$, thus G is the union of two edge disjoint Γ-vertex transitive graphs on $V(G)$.

(ii) $G = G_1 \cup G_2$, where G_1 is a bipartite graph with bipartition (U, W), U and W are its two Γ-vertex-orbits and $G_2 = G[W]$ is a Γ-vertex transitive graph (see Figure 1. (a)).

(iii) $G = G_1 \cup G_2$, where G_1 is a bipartite graph with bipartition (U, W), U and W are its two Γ-vertex-orbits, and G_2 is a bipartite graph with bipartition (U', W), U' and W are its two Γ-vertex-orbits, and $U \cap U' = \emptyset$ (see Figure 1. (b)).



(a) The II-kind 2-edge-orbit graph



(b) The III-kind 2-edge-orbit graph

Figure 1. The 2-edge-orbit graphs.

Proof. Since $\Gamma = Aut(G)$, we have $\Gamma \leq Aut(G_i)$ and G_i is Γ-edge transitive for $i = 1, 2$. Set $V_i = V(G_i)(i = 1, 2)$.

If $V_2 \neq V$, we take $v \in V \setminus V_2$. Since $\delta(G) > 0$, we have $v \in V_1$, and the edge set incident with v in G_1 is the edge set incident with v in G. By Theorem 2.3, either G_1 is Γ-vertex transitive or a bipartite graph. If G_1 is Γ-vertex transitive, then for any $u \in V(G_1)$, the edge set incident with

u in G is that in G_1. If $V_1 \neq V$, then G is not connected, a contradiction. Thus G_1 is a bipartite graph. Let U and W be its two Γ-vertex-orbits, then G_1 has bipartition (U, W). Without loss of generality, assume that $v \in U$, then $U \cap V_2 = \emptyset$. Since G is connected, $W \cap V_2 \neq \emptyset$. Since Γ is vertex transitive on W, we have $W \subseteq V_2$. If G_2 is Γ-vertex transitive, then, since vertices of G_2 have some neighbors in G, we have $W = V_2$ and therefore $G_2 = G[W]$(see *Figure 1.(a)*).

If G_2 is not Γ-vertex transitive, then G_2 has two Γ-vertex-orbits, say U' and W', and G_2 is a bipartite graph with bipartition (U', W'). Since G_2 is not Γ-vertex transitive, we have $U' \cap W = \emptyset$ or $W' \cap W = \emptyset$. Without loss of generality, assume that $U' \cap W = \emptyset$, then $W' \cap W \neq \emptyset$, and therefore $W' = W$(see *Figure 1.(b)*). □

Let the graph G of (i), (ii) and (iii) in Theorem 2.4 be denoted by I-kind, II-kind and III-kind 2-edge-orbit graph, respectively. By Theorem 2.2, we have the following result.

Theorem 2.5. *Let G be a connected 2-edge-orbit graph, A be a vertex atom with $|A| \geq 2$. Then $G[A]$ is a connected component of G_1 or G_2, where G_1 and G_2 are edge transitive parts of G.*

Proof. Let $E_i = E(G_i)$ for $i = 1, 2$. We show by contradiction that $G_i[A]$ is a connected component of G_i if $E(G[A]) \cap E_i \neq \emptyset$. If otherwise, then there exist two adjacent edges e_1 and e_2 in E_i satisfying $e_1 \in E(G[A])$ and $e_2 \notin E(G[A])$. Since E_i is an edge orbit, there exists some $\alpha \in Aut(G)$ satisfying $e_2 \in \alpha(e_1)$. But then, the two distinct atoms A and $\alpha(A)$ are not disjoint, contradicting Theorem 2.2. Thus if $E(G[A]) \cap E_i \neq \emptyset$ for $i = 1, 2$, then $G_i[A]$ is a connected component of G_i. But then G is not connected. Therefore, if $E(G[A]) \cap E_1 \neq \emptyset$, then $E(G[A]) \cap E_2 = \emptyset$, and $G_1[A] = G[A]$ is a connected component of G_1. □

§3 Super-connectivity of 2-vertex-orbit Graphs

In [10], Liang and Meng characterized the connected bipartite 2-vertex-orbit graphs which are maximally connected. In [11], Liang, Meng and Zhang gave a necessary and sufficient condition for the super-connectivity of vertex transitive bipartite graphs. In this section, we will characterize the super-connectivity of connected bipartite 2-vertex-orbit graphs.

The following result is about the connectivity of semi-transitive graphs.

Theorem 3.1. *([10]) If $G = (V, E)$ is a connected semi-transitive graph, then $\kappa(G) = \delta(G)$, where $\delta(G)$ is the minimum degree of G.*

Let $G = (V, E)$ be a connected semi-transitive graph. In this section, we use V_1 and V_2 to denote the two parts of G. Without loss of generality, we may assume that m and n are regularity of V_1 and V_2, respectively, and $m \leq n$, thus the minimum degree $\delta(G) = m$.

About the cardinality of a superatom of semi-transitive graphs, we have the following result.

Lemma 3.2. *Let G be a connected degenerate semi-transitive graph. Then $\omega(G) = 2$.*

Proof. Clearly, $\omega(G) \geq 2$. Assume $\omega(G) \geq 3$. Let A be a superatom of G. Since G is a semi-transitive graph, we have $A_i = A \cap V_i \neq \emptyset$ for $i = 1, 2$, and each vertex of G lies in a superatom. We have $g(G) \geq 4$ since G is a bipartite graph. By Theorem 2.1, distinct superatoms are disjoint, thus $V(G)$ is a disjoint union of distinct superatoms.

Set $C(A) = A \cup N(A)$ and $R(A) = V(G) \setminus C(A)$. Since G is a degenerate graph, $R(A)$ is a strict fragment of G. By Theorem 2.1, $R(A)$ is a disjoint union of distinct superatoms of G, and so $N(A)$ is also a disjoint union of distinct superatoms of G, thus $\omega(G) \leq |N(A)| = \kappa(G) = \delta(G) = m$. If $\omega(G) = m$, then $N(A)$ is a superatom of G, and $|A \cup N(A)| = 2m$. Since $\delta(G) = m$, we have $m = n$ and $G[A \cup N(A)] \cong K_{m,m}$ is a connected component of G, which contradicts that G is connected. If $\omega(G) < m$, then $|A \cup N(A)| < 2m$. $N(x) \subseteq A \cup N(A)$ for any $x \in A$, and $G[A \cup N(A)]$ is a bipartite subgraph of G, thus there exists a vertex $x \in A$ satisfying $|N(x)| < m = \delta(G)$, a contradiction. □

Let A be a superatom of G, and $G[A]$ be introduced by A. If $G[A]$ is connected, then we have the following result.

Lemma 3.3. *Let G be a connected degenerate semi-transitive graph, and A be a superatom of G. If $G[A] \cong K_2$, then $G \cong C_k$ for some $k \geq 6$.*

Proof. If $m \neq n$, that is, $m < n$, then since $G[A] \cong K_2$, we have $m + n - 2 = |N(A)| = \kappa(G) = m$, and so $n = 2$ and $m = 1$, and $G[A \cup N(A)] \cong K_{1,2}$ is a connected component of G, a contradiction. Thus $m = n$. Since $2(m-1) = |N(A)| = \kappa(G) = m$, we have $m = 2$ and $G \cong C_k$, where $k \geq 6$ because C_k is not degenerate for $k \leq 5$. □

Since C_k is vertex transitive, in the following, we assume that G is a connected degenerate bipartite 2-vertex-orbit graph. Now define an equivalence relation R on $V(G)$: for $v_1, v_2 \in V(G)$, $v_1 R v_2$ if and only if $N(v_1) = N(v_2)$. According to this equivalence and $V(G) = V_1 \cup V_2$, V_1 is partitioned into some non-empty sets $F_{11}, \cdots, F_{1p} (p \geq 1)$, and V_2 is partitioned into some non-empty sets $F_{21}, \cdots, F_{2q} (q \geq 1)$. Thus F_{1i} and F_{2j} are imprimitive blocks of V_1 and V_2, respectively, and $G[F_{1i}]$ and $G[F_{2j}]$ are independent sets for any $1 \leq i \leq p$ and $1 \leq j \leq q$, where $G[F_{1i}]$ and $G[F_{2j}]$ are the subgraph of G induced by F_{1i} and F_{2j}, respectively.

Define a quotient graph $\overline{G} = G/R$ of G as follows: the vertices of $\overline{G}$ are $f_{1i} (1 \leq i \leq p)$ and $f_{2j} (1 \leq j \leq q)$, f_{1i} and f_{2j} are adjacent in $\overline{G}$ if and only if there is one vertex in F_{1i} which is adjacent to some vertex of F_{2j} in G. Clearly, the quotient graph $\overline{G}$ is irreducible for any graph G.

We have the following results about the quotient graph $\overline{G}$ of a graph G, where G is a connected degenerate bipartite 2-vertex-orbit graph.

Lemma 3.4. *$\overline{G}$ is an irreducible connected semi-transitive graph.*

Proof. Clearly, $\overline{G}$ is connected. Since G is a bipartite 2-vertex-orbit graph and $F_{1i} (1 \leq i \leq p)$ and $F_{2j} (1 \leq j \leq q)$ are imprimitive blocks of G, for any $u \in F_{1l}$ and $v \in F_{1k}$ $(1 \leq l \neq k \leq p)$, $uv \notin E(G)$, we have $f_{1l} f_{1k} \notin E(\overline{G})$, we can see that $\overline{G}$ is a bipartite graph. For any $l \neq k$, there exists $\varphi_1, \varphi_2 \in Aut(G)$ satisfying $\varphi_1(F_{1l}) = F_{1k}$ and $\varphi_2(F_{2s}) = F_{2t}$, thus the restriction of φ_1 and φ_2 on $V(\overline{G})$ is an automorphism of $\overline{G}$ mapping f_{1l} to f_{1k} and f_{2s} to f_{2k}, thus the number of vertex orbits of $\overline{G}$ is at most two. □

Lemma 3.5. *$\overline{G}$ is not super-κ if and only if $\overline{G} \cong C_k (k \geq 6)$, where k is even.*

Proof. The sufficiency is obvious.

Now we prove the necessity. Assume that $\overline{G}$ is not super-κ, then $\overline{G}$ is degenerate. Let A be a superatom of $\overline{G}$. By Lemma 3.2, we have $|A| = 2$. $\overline{G}[A]$ cannot be an independent set because $\overline{G}$ is irreducible, thus $\overline{G}[A] \cong K_2$. By Lemma 3.3, $\overline{G} \cong C_k$ for some $k \geq 6$. □

Since C_k is vertex transitive, by Lemmas 3.4 and 3.5, we have the following theorems about the super-connectivity of connected bipartite 2-vertex-orbit graphs.

Theorem 3.6. *Let G be an irreducible connected bipartite 2-vertex-orbit graph. Then G is super-κ.*

Theorem 3.7. *Let G be a connected bipartite 2-vertex-orbit graph. Then G is not super-κ if and only if $G \cong C_k(N_m, N_n)$, where $k \geq 6$ is even and $m \neq n$.*

§4 Super-connectivity of 2-edge-orbit Graphs

In the former part of this section, we will characterize the super-connected graphs with two edge orbits. Let $G = (V, E)$ be a connected 2-edge-orbit graph, $\Gamma = Aut(G)$, E_1 and E_2 be its two edge orbits. Then $E_1 \cup E_2 = E$. Let $G_i = G[E_i]$ be the subgraph of G induced by E_i, $V_i = V(G_i)$. Then $\Gamma \leq Aut(G_i)$ and G_i is Γ-edge transitive for $i = 1, 2$.

In the following, if G is a II-kind 2-edge-orbit graph, assume $G = G_1 \cup G_2$, where G_1 is a bipartite graph with bipartition (U, W), U and W are its two Γ-vertex-orbits and $G_2 = G[W]$ is a Γ-vertex transitive graph. In the subgraph G_1, the regularity of the vertex in U is k_1 and the regularity of the vertex in W is k_1'. In the subgraph G_2, the regularity of the vertex is k_2.

If G is a III-kind 2-edge-orbit graph, assume $G = G_1 \cup G_2$, where G_1 is a bipartite graph with bipartition (U, W), U and W are its two Γ-vertex-orbits, and G_2 is a bipartite graph with bipartition (U', W), U' and W are its two Γ-vertex-orbits, and $U \cap U' = \emptyset$. In the subgraph G_1, the regularity of the vertex in U is k_1 and the regularity of the vertex in W is k_1'. In the subgraph G_2, the regularity of the vertex in U' is k_2 and the regularity of the vertex in W is k_2'.

In the next discussion, we need the following results.

Theorem 4.1. *Let G be a 2-edge-orbit graph. Let G_1 and G_2 be its edge transitive parts at least one of which is not connected. Let H_i be a connected component (if any) of G_i for $i = 1, 2$. If $\kappa(G) < \delta(G)$, we have*

(1) If $V(H_i) \subsetneq V(G)$ for $i = 1, 2$, then $\kappa(G)$=$min\{|N_{G_1}(V(H_2))|, |N_{G_2}(V(H_1))|\}$;

(2) If $V(H_1) \subsetneq V(G)$ and $V(H_2) = V(G)$, then $\kappa(G) = |N_{G_2}(V(H_1))|$;

(3) If $V(H_1) = V(G)$ and $V(H_2) \subsetneq V(G)$, then $\kappa(G) = |N_{G_1}(V(H_2))|$.

Proof. Let A be a vertex atom of G. Since $\kappa(G) < \delta(G)$, we have $|A| \geq 2$, and by Theorem 2.5 we see that $G[A]$ is a connected component of G_1 or G_2. If $G[A]$ is a connected component G_1, then $\kappa(G) = |N_G(A)| = |N_{G_2}(V(H_1))|$. If $G[A]$ is a connected component of G_2, then $\kappa(G) = |N_G(A)| = |N_{G_1}(V(H_2))|$. □

Corollary 4.2. *Let G be a connected II-kind 2-edge-orbit graph. Let G_1 and G_2 be its edge transitive parts such that G_1 is not connected and G_2 is connected. Then $\kappa(G) = \delta(G)$.*

Proof. By contradiction. Let A be a vertex atom of G. If $\kappa(G) < \delta(G)$, then $G[A]$ is a connected component of G_1. Since $G[A]$ is connected, we have $A \cap U \neq \emptyset$ and $A \cap W \neq \emptyset$. Let $A_1 = A \cap U$ and $A_2 = A \cap W$. Since U and W are Γ-vertex-orbits, each vertex must be in a vertex atom of G, and each vertex atom has non-empty intersection both with U and W. On the other hand, $N_G(A) = N_{G_2}(A_2) \subset W$, which contradicts that A is a vertex atom. □

In [9], the authors characterized the maximal connectedness of II-kind and III-kind 2-edge-orbit graphs. The results follow:

Theorem 4.3. *([9]) Let G be a connected II-kind 2-edge-orbit graph. If $|U| \geq |W|$, then $\kappa(G) = \delta(G)$.*

Theorem 4.4. *([9]) Let G be a connected III-kind 2-edge-orbit graph.*
(1) If $|W| \leq min\{|U|, |U'|\}$, then $\kappa(G) = \delta(G)$;
(2) If H_i is one component of G_i for $i = 1, 2$ satisfying $|N_{G_1}(V(H_2))| = |N_{G_2}(V(H_1))|$, then $\kappa(G) = \delta(G)$.

Theorem 4.5. *([9]) Let G be a k-regular connected 2-edge-orbit graph with $k \leq 6$.*
(1) If G is a II-kind 2-edge-orbit graph with $g(G) \geq 6$, then $\kappa(G) = k$;
(2) If G is a III-kind 2-edge-orbit graph with $g(G) \geq 8$, then $\kappa(G) = k$.

We have the following result about the cardinality of a superatom for connected II-kind 2-edge-orbit graphs.

Lemma 4.6. *Let G be a connected degenerate II-kind 2-edge-orbit graph with $g(G) \geq 4$. If one of the following conditions occurs, then $\omega(G) = 2$.*
(1) G_1 is not connected and G_2 is connected;
(2) $|U| \geq |W|$ and $G_2 \ncong K_2 \cup \cdots \cup K_2$.

Proof. Clearly, $\omega(G) \geq 2$. Suppose $\omega(G) \geq 3$. Since $g(G) \geq 4$, the intersection of any distinct superatoms of G is empty by Theorem 2.1. Let A be a superatom of G. By Theorem 2.5, $G[A]$ is a connected component of G_1 or G_2.

(1) Since G_2 is connected, if $G[A]$ is a connected component of G_2, then $N(A) = U$ and $R(A) = \emptyset$, a contradiction. Thus $G[A]$ is a connected component of G_1. Set $A_1 = A \cap U$ and $A_2 = A \cap W$. Since U and W are Γ-vertex-orbits of G, each superatom has a non-empty intersection both with U and W, and each vertex of G_1 lies in a superatom. By Theorem 2.1, $G - G[A \cup N(A)]$ is a disjoint union of distinct superatoms, and so there is a superatom A' satisfying $v \in A'$ for some $v \in N(A)$, and $A' \subseteq N(A)$. Hence, we obtain a contradiction by $N(A) \subseteq W$.

(2) Since $|U| \geq |W|$, we have $k_1 \leq k_1'$ and $\kappa(G) = k_1 = \min\{k_1, k_1' + k_2\}$. If $G[A]$ is a connected component of G_1, then $A_1 = A \cap U \neq \emptyset$. By a similar argument as (1), we also have a contradiction. If $G[A]$ is a connected component of G_2, then $A \subseteq W$ and $k_1 = |N(A)| \geq k_1'$. Since $k_1 \leq k_1'$, we have $|A| = 1$, a contradiction. □

We have the following result about the cardinality of a superatom for k-regular connected II-kind 2-edge-orbit graphs.

Lemma 4.7. *Let G be a k-regular connected degenerate II-kind 2-edge-orbit graph with $k \le 6$ and $g(G) \ge 6$. Then $\omega(G) = 2$.*

Proof. Clearly, $\omega(G) \ge 2$. Suppose $\omega(G) \ge 3$. Since $g(G) \ge 6 > 4$, the intersection of any distinct superatoms of G is empty by Theorem 2.1. Let A be a superatom of G. By Theorem 2.5, $G[A]$ is a connected component of G_1 or G_2.

If $G[A]$ is a connected component of G_1, then $A_1 = A \cap U \neq \emptyset$, $A_2 = A \cap W \neq \emptyset$. Since U and W are Γ-vertex-orbits of G, each superatom has a non-empty intersection both with U and W, and each vertex of G_1 lies in a superatom. By Theorem 2.1, $G - G[A \cup N(A)]$ is a disjoint union of distinct superatoms, and so there is a superatom A' satisfying $v \in A'$ for some $v \in N(A)$, and $A' \subseteq N(A)$. Hence, we obtain a contradiction by $N(A) \subseteq W$.

If $G[A]$ is a connected component of G_2, then $A \subseteq W$. Since G_2 is a vertex and edge transitive graph, if $k_2 = 1$, then $G_2[A] \cong K_2$, a contradiction. Hence, $k_2 \ge 2$.

Set $N_{G_2}(v) = \{v_1, v_2, ..., v_{k_2}\}$ for some $v \in A$, and $N_{G_2}(v_1) = \{u_1, u_2, ..., u_{k_2-1}, v\}$. Clearly, $N_{G_2}(v) \cap N_{G_2}(v_1) = \emptyset$. Since $g(G) \ge 6$, we have $N_{G_1}(w_i) \cap N_{G_1}(w_j) = \emptyset$ for any $w_i, w_j \in N_{G_2}(v) \cup N_{G_2}(v_1)$. Thus, $|N(A)| \ge k_1' + k_1'k_2 + k_1(k_2 - 1) = 2k_1'k_2$. Since $2k_1'k_2 - k_1' - k_2 = k_1'(k_2 - 1) + k_2(k_1 - 1) > 0$, we obtain a contradiction by $|N(A)| = k = k_1' + k_2$. □

When G is a connected II-kind 2-edge-orbit graph, we have the following theorem about the super-connectivity of G.

Theorem 4.8. *Let G be a connected irreducible II-kind 2-edge-orbit graph with $g(G) \ge 4$. If one of the following conditions occurs, then G is super-connected.*

(1) G_1 is not connected and G_2 is connected;

(2) $|U| \ge |W|$ and $G_2 \ncong K_2 \cup \cdots \cup K_2$.

Proof. Suppose G is not super-connected. Let A be a superatom of G. Since G is irreducible, by Lemma 4.6, we have $G[A] \cong K_2$. By Theorem 2.5, $G[A]$ is a connected component of G_1 or G_2.

(1) Since G_2 is connected, $G[A]$ is a connected component of G_1. Set $A_1 = A \cap U$ and $A_2 = A \cap W$. Since $G[A] \cong K_2$, we have $|A_1| = |A_2| = 1$, $k_1 = k_1' = 1$, and $|N(A)| = k_2 \cdot |A_2| = k_2 = \min\{k_1, k_1' + k_2\} = 1$, and so $G_2 \cong K_2$, a contradiction.

(2) If $G[A]$ is a connected component of G_1. Set $A_1 = A \cap U$ and $A_2 = A \cap W$, then $|A_1| = |A_2| = 1$, $k_1 = k_1' = 1$, and $|U| = |W|$. Since $|N(A)| = k_2 = 1$, we have $G_2 \cong K_2 \cup \cdots \cup K_2$, a contradiction.

If $G[A]$ is a component of G_2, then since $G[A] \cong K_2$ and G_2 is an edge transitive graph, we have $G_2 \cong K_2 \cup \cdots \cup K_2$, a contradiction. □

In Theorem 4.8, we demand $g(G) \ge 4$. If $g(G) = 3$, then G is not super-connected and we give a remark in the following.

Remark 1. *Let G (see Figure 2.) be a II-kind 2-edge-orbit graph with $g(G) = 3$, where $G_1 \cong K_{1,4} \cup K_{1,4} \cup K_{1,4}$, and $G_2 \cong \mathscr{L}(Q_3)$, then $\kappa(G) = \delta(G) = 4$, $\{v_1, v_2, v_3, v_4\}$ is a minimum vertex cut of G, and there is not a singleton in $G - \{v_1, v_2, v_3, v_4\}$, thus G is not super-connected.*

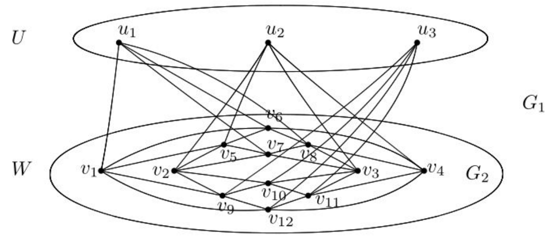


Figure 2. A II-kind 2-edge-orbit graph with g(G)=3.

If $G_2 \cong K_2 \cup \cdots \cup K_2$, then G is also not super-connected, and we give a remark in the following.

Remark 2. *Let G (see Figure 3.) be a II-kind 2-edge-orbit graph with $|U| \geq |W|$ and $G_2 \cong K_2 \cup \cdots \cup K_2$, where $G_1 \cong K_{2,3} \cup K_{2,3} \cup K_{2,3} \cup K_{2,3}$ and $G_2 \cong K_2 \cup K_2 \cup K_2 \cup K_2$, then $\kappa(G) = \delta(G) = 2$, $\{v_3, v_5\}$ is a minimum vertex cut of G, and there is not a singleton in $G \backslash \{v_3, v_5\}$, thus G is not super-connected.*

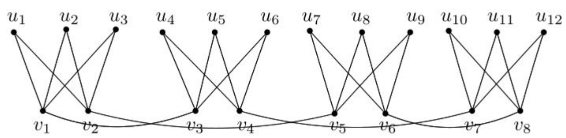


Figure 3. A II-kind 2-edge-orbit graph with $|U| \geq |W|$ and $G_2 \cong K_2 \cup \cdots \cup K_2$.

When G is a k-regular connected II-kind 2-edge-orbit graph, we have the following theorem about the super-connectivity of G.

Theorem 4.9. *Let G be a k-regular connected irreducible II-kind 2-edge-orbit graph with $k \leq 6$ and $g(G) \geq 6$. Then G is super-connected.*

Proof. Suppose G is not super-connected. Let A be a superatom of G. Since G is irreducible, by Lemma 4.7, we have $G[A] \cong K_2$. By Theorem 2.5, $G[A]$ is a connected component of G_1 or G_2. If $G[A]$ is a component of G_1, then $k_1 = k'_1 = 1$ and $G \cong G[A]$, a contradiction. If $G[A]$ is a component of G_2, then $|N(A)| = 2k'_1$, $k_2 = 1$, and $|N(A)| = \kappa(G) = k = k'_1 + k_2$. Thus $k'_1 = k_2 = 1$ and $k_1 = 2$, and so G is a cycle, a contradiction. □

Next, we will consider the super-connectivity for the connected III-kind 2-edge-orbit graph. In the first, we have the following result about the cardinality of a superatom for connected III-kind 2-edge-orbit graphs.

Lemma 4.10. *Let G be a connected degenerate III-kind 2-edge-orbit graph. If one of the following conditions occurs, then $\omega(G) = 2$.*

(1) $|W| \leq min\{|U|, |U'|\}$;

(2) H_i is one component of G_i for $i = 1, 2$ satisfying $|N_{G_1}(V(H_2))| = |N_{G_2}(V(H_1))|$.

Proof. Clearly, $\omega(G) \geq 2$. Suppose $\omega(G) \geq 3$. Since G is a III-kind 2-edge-orbit graph, we have $g(G) \geq 4$, and the intersection of any distinct superatoms of G is empty by Theorem 2.1. Let A be a superatom of G. By Theorem 2.5, $G[A]$ is a connected component of G_1 or G_2.

(1) Since $|W| \leq \min\{|U|, |U'|\}$, we have $k_1 \leq k_1'$ and $k_2 \leq k_2'$. Without loss of generality, assume that $G[A]$ is a connected component of G_1, then $A_1 = A \cap U \neq \emptyset$ and $A_2 = A \cap W \neq \emptyset$.

When $k_1 \leq k_2$. $k_1 = |N(A)| = |N_{G_2}(A_2)| \geq k_2'$. Hence $k_1 = k_2 = k_1' = k_2'$. If $|A_2| \geq 2$, then $N_{G_2}(v_1) = N_{G_2}(v_2)$ for any $v_1, v_2 \in A_2$, and so $G[A \cup N(A)]$ is a component of G, a contradiction. If $|A_2| = 1$, then $N(v) = N_{G_1}(v) = A_2$ for any $v \in A_1$, that is $k_1 = 1$. Hence, $G[A \cup N(A)]$ is a component of G, a contradiction.

When $k_1 > k_2$. By a similar argument as above, we observe a contradiction.

(2) Without loss of generality, assume that $G[A]$ is a connected component of G_1. Then $G[A] \cong H_1$, $A_1 = A \cap U \neq \emptyset$, and $A_2 = A \cap W \neq \emptyset$.

Since U and W are Γ-vertex-orbits of G, each vertex of G_1 lies in a superatom, and each superatom has a non-empty intersection both with U and W. Since H_2 is a component of G_2, we have $V(H_2) \cap W \neq \emptyset$, and there is a superatom A' satisfying $v \in A'$ for some $v \in V(H_2) \cap W$. $V(H_2)$ is a strict fragment of G by $|N_{G_1}(V(H_2))| = |N_{G_2}(V(H_1))|$. By Theorem 2.1, $A' \subseteq V(H_2)$, and $A' \cap U = \emptyset$, we obtain a contradiction in which each superatom is disjoint with U and W. □

We have the following result about the cardinality of a superatom for k-regular connected III-kind 2-edge-orbit graphs.

Lemma 4.11. *Let G be a k-regular connected degenerate III-kind 2-edge-orbit graph with $k \leq 6$ and $g(G) \geq 8$. Then $\omega(G) = 2$.*

Proof. Clearly, $\omega(G) \geq 2$. Suppose $\omega(G) \geq 3$. Since $g(G) \geq 8 > 4$, the intersection of any distinct superatoms of G is empty by Theorem 2.1. Let A be a superatom of G. By Theorem 2.5, $G[A]$ is a connected component of G_1 or G_2.

Without loss of generality, assume that $G[A]$ is a connected component of G_1. Set $A_1 = A \cap U$ and $A_2 = A \cap W$, and $k = k_1 = k_2 = k_1' + k_2'$.

When $k_1' = 1$. Since $G[A]$ is a connected component of G_1, we have $|A_1| = 1$ and $G[A] \cong K_{1,k}$. By $g(G) \geq 8$, we have $|N(A)| = k_2'|A_2| = k = 1 + k_2'$. Hence, $k_2' = 1$, $|A_2| = 2$, and $k = 2$. Thus, G is a cycle, a contradiction.

When $k_1' \geq 2$. Since G_1 is a 2-vertex-orbit graph, we have $|A_1| \geq 2$. Assume $v_1, v_2 \in A_1$. Set $N(v_1) = \{u_{11}, u_{12}, ..., u_{1k}\}$, $N(v_2) = \{u_{21}, u_{22}, ..., u_{2k}\}$. Clearly, $|N(v_1) \cap N(v_2)| \leq 1$ by $g(G) \geq 8$. Hence, $|N(A)| \geq |N_{G_2}(N(v_1) \cup N(v_2))| \geq k_2'(2k-1)$. Since $k_2'(2k-1) - k = k(k_2' - 1) + k_2'k - 1 > 0$, we have a contradiction by $|N(A)| = k$. □

When G is a connected III-kind 2-edge-orbit graph, we have the following theorem about the super-connectivity of G.

Theorem 4.12. *Let G be a connected irreducible III-kind 2-edge-orbit graph. If one of the following conditions occurs, then G is super-connected.*

(1) $|W| \leq min\{|U|, |U'|\}$;

(2) H_i is a connected component of G_i for $i = 1, 2$ satisfying $|N_{G_1}(V(H_2))| = |N_{G_2}(V(H_1))|$.

Proof. Suppose G is not super-connected. Let A be a superatom of G. By Lemma 4.10, $G[A] \cong K_2$. By Theorem 2.5, $G[A]$ is a connected component of G_1 or G_2. Without loss of generality, we assume that $G[A]$ is a connected component of G_1. Set $A_1 = A \cap U$ and $A_2 = A \cap W$. Since $G[A] \cong K_2$, we have $|A_1| = |A_2| = 1$ and $k_1 = k'_1 = 1$, and so $|N(A)| = k_2 = 1$, then $G[A \cup N(A)]$ is a component of G, a contradiction. □

When G is a k-regular III-kind 2-edge-orbit graph, we have the following theorem about the super-connectivity of G.

Theorem 4.13. *Let G be a k-regular connected irreducible III-kind 2-edge-orbit graph with $k \leq 6$ and $g(G) \geq 8$. Then G is super-connected.*

Proof. Suppose G is not super-connected. Let A be a superatom of G. By Lemma 4.11, $G[A] \cong K_2$. By Theorem 2.5, $G[A]$ is a connected component of G_1 or G_2. Without loss of generality, we assume that $G[A]$ is a connected component of G_1. Set $A_1 = A \cap U$ and $A_2 = A \cap W$. Since $G[A] \cong K_2$, we have $|A_1| = |A_2| = 1$ and $k_1 = k'_1 = 1$. Since G is a k-regular graph, we have $k = k_1 = k_2 = k'_1 + k'_2 = 1$, a contradiction. □

Declarations

Conflict of interest The authors declare no conflict of interest.

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