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Weighted multilinear p-adic Hardy operators and commutators on p-adic Herz-type spaces

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Abstract. In this paper, we introduce the weighted multilinear p-adic Hardy operator and weighted multilinear p-adic Cesàro operator, we also obtain the boundedness of these two operators on the product of p-adic Herz spaces and p-adic Morrey-Herz spaces, the corresponding operator norms are also established in each case. Moreover, the boundedness of commutators of these two operators with symbols in central bounded mean oscillation spaces and Lipschitz spaces on p-adic Morrey-Herz spaces are also given.

§1 Introduction

Hensel first introduced p-adic numbers in 1908, generalized the concepts of absolute values of real numbers and complex numbers, and developed the theory of assignment. It is not only a simple and powerful tool for studying algebraic number theory, but also has important applications in univariate algebraic function theory, closed loops and algebraic geometry $[1, 2, 3, 4]$.

In recent years, p-adic analysis has received extensive attention for its applications in mathematical physics. Especially, p-adic harmonic analysis, p-adic partial differential equations and p-adic wavelets have been studied more. p-adic harmonic analysis has attracted more and more attention (see $[5, 6, 7]$). Recently, the boundedness of many operators in harmonic analysis has been studied with the p-adic function space as the base space. The p-adic function space class has become another emerging field $[8, 9, 10]$.

For a prime number p, let $\mathbb{Q}_p$ be the field of p-adic numbers. This field is the completion of the field of rational numbers $\mathbb{Q}$ with respect to the non-Archimedean p-adic norm $|\cdot|_p$. This norm is defined as follows: if $x=0$, $|0|_p=0$; if $x\neq 0$ is an arbitrary rational number with the unique representation $x=p^{\gamma}\frac{m}{n}$, where m, n are not divisible by p, $\gamma=\gamma(x)\in\mathbb{Z}$, then $|x|_p=p^{-\gamma}$. This norm satisfies the following properties:

(i)$|x|_p\geq 0$, $\forall x\in\mathbb{Q}_p$, $|x|_p=0\Leftrightarrow x=0$;

(ii)$|xy|_p=|x|_p|y|_p$, $\forall x,y\in\mathbb{Q}_p$;

(iii)$|x+y|_p\leq\max(|x|_p,|y|_p)$, $\forall x,y\in\mathbb{Q}_p$, and when $|x|_p\neq|y|_p$, we have $|x+y|_p=\max(|x|_p,|y|_p)$.

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For the p-adic analysis, a non-zero element x of $\mathbb{Q}_p$ is uniquely represented as a canonical form

$$x = p^{\gamma}(x_0 + x_1 p + x_2 p^2 + \cdots),$$

where $\gamma = \gamma(x) \in \mathbb{Z}$, $x_k = 0, 1, \cdots, p-1$, $x_0 \neq 0$, $k = 0, 1, \cdots$.

$\mathbb{Q}_p^n = \mathbb{Q}_p \times \mathbb{Q}_p \cdots \times \mathbb{Q}_p$ contains all n-tuples of $\mathbb{Q}_p$, the norm on $\mathbb{Q}_p^n$ is

$$|x|_p = \max_{1 \le k \le n} |x_k|_p, \qquad x = (x_1, x_2, \cdots, x_n) \in \mathbb{Q}_p^n.$$

Denoted by

$$B_\gamma(a) = \{x \in \mathbb{Q}_p^n : |x - a|_p \le p^\gamma\}$$

be a ball of radius p^γ with center at $a \in \mathbb{Q}_p^n$. Similarly, denote by

$$S_\gamma(a) = \{x \in \mathbb{Q}_p^n : |x - a|_p = p^\gamma\}$$

the sphere with center at $a \in \mathbb{Q}_p^n$ and radius p^γ, $\gamma \in \mathbb{Z}$. Clearly $S_\gamma(a) = B_\gamma(a) \setminus B_{\gamma-1}(a)$ and

$$B_\gamma(a) = \bigcup_{k \le \gamma} S_k(a).$$

We set $B_\gamma(0) = B_\gamma$ and $S_\gamma(0) = S_\gamma$. Let $\mathbb{Z}_p = \{x \in \mathbb{Q}_p : |x|_p \le 1\}$ be the class of all p-adic integers in $\mathbb{Q}_p$, and let $\mathbb{Z}_p^* = \mathbb{Z}_p \setminus \{0\}$.

Since $\mathbb{Q}_p^n$ is a locally compact commutative group under addition, it follows from the standard theory that there exists a Haar measure dx on $\mathbb{Q}_p^n$, which is unique up to positive constant multiple and is translation invariant. This measure is unique by normalizing dx such that

$$\int_{B_0} dx = |B_0| = 1,$$

where $|B|$ denotes the Haar measure of a measurable subset B of $\mathbb{Q}_p^n$. By simple calculation, we can obtain that

$$|B_\gamma(a)| = p^{\gamma n}, \qquad |S_\gamma(a)| = p^{\gamma n}(1 - p^{-n})$$

for any $a \in \mathbb{Q}_p^n$. For more information about p-adic, one can refer to [11, 12, 13].

In 1920, Hardy [14] introduced the classical Hardy operators, which are defined by

$$\mathcal{H}(f)(x) = \frac{1}{x} \int_0^x f(y)dy,$$

where the function f is a nonnegative integrable function on $\mathbb{R}^+$. The celebrated Hardy's integral inequality was proved If $1 < p < \infty$, then

$$\|\mathcal{H}(f)\|_{L^p(\mathbb{R}^+)} \le \frac{p}{p-1} \|f\|_{L^p(\mathbb{R}^+)},$$

where $\frac{p}{p-1}$ is the best constant.

In 1995, Christ and Grafakos [15] gave the definition of the n-dimensional Hardy operator

$$H(f)(x) = \frac{1}{|B(0, |x|)|} \int_{B(0,|x|)} f(y)dy,$$

where f is a non-negative measurable function on $\mathbb{R}^n \backslash \{0\}$. The operator norm of operator H from $L^p(\mathbb{R}^n)$ to $L^p(\mathbb{R}^n)$ is equivalent to the operator norm of $\mathcal{H}$. $\|H\|_{L^p(\mathbb{R}^n) \to L^p(\mathbb{R}^n)} = \frac{p}{p-1}$.

In 2012, Fu et al. [16] defined the n-dimensional p-adic Hardy operator as follows

$$H_p(f)(x) = \frac{1}{|B(0, |x|_p)|} \int_{B(0,|x|_p)} f(y)dy, \quad x \in \mathbb{Q}_p^n \setminus \{0\},$$

where f is a nonnegative measurable function on $\mathbb{Q}_p^n$, $B(0, |x|_p)$ is a ball in $\mathbb{Q}_p^n$ with a center at $0 \in \mathbb{Q}_p^n$ and radius $|x|_p$. They obtained the sharp estimates of p-adic Hardy operators on p-adic weighted Lebesgue spaces.

In 1984, Carton-Lebrun and Fosset [17] defined the weighted Hardy operator $\mathcal{H}_\varphi$ as

$$\mathcal{H}_\varphi(f)(x)=\int_0^1 f(tx)\varphi(t)dt,\quad x\in\mathbb{R}^n,$$

where φ: $[0,1]\to[0,\infty)$ is a measurable function, the authors established the boundedness of operator $\mathcal{H}_\varphi$ on the Lebesgue spaces and BMO($\mathbb{R}^n$).

In 2001, J Xiao [18] proved that $\mathcal{H}_\varphi$ is bounded on $L^p(\mathbb{R}^n)$ if and only if

$$\int_0^1 t^{-\frac{n}{p}}\varphi(t)dt<\infty,$$

and the corresponding operator norm was worked out.

In 2006, K S Rim, J Lee [19] considered the p-adic form of the weighted Hardy operator $\mathcal{H}^p_\varphi$ as follows

$$U^p_\varphi(f)(x)=\int_{\mathbb{Z}_p^*} f(tx)\varphi(t)dt,\quad x\in\mathbb{Q}_p^n,$$

moreover, the authors also introduced the weighted Cesàro operator to the p-adic field as follows

$$V^p_\varphi(f)(x)=\int_{\mathbb{Z}_p^*} f(\frac{x}{t})|t|_p^{-n}\varphi(t)dt,\quad x\in\mathbb{Q}_p^n,$$

observe that the weighted Hardy operator U_φ and the weighted Cesaro operator V_φ are mutually adjoint.

In this note, we establish the sharp bounds of weighted multilinear p-adic Hardy operators $U^m_{\varphi,p}$ and $V^m_{\varphi,p}$ on the product of p-adic Herz spaces and p-adic Morrey-Herz spaces, we also consider the boundedness of commutators of $U^m_{\varphi,p}$ and $V^m_{\varphi,p}$ on these spaces. Firstly, we introduce the weighted multilinear p-adic Hardy operators as follows.

Definition 1.1. Let φ: $\mathbb{Z}_p^*\times\mathbb{Z}_p^*\times\cdots\times\mathbb{Z}_p^*\to[0,+\infty)$, $m\in\mathbb{N}$, $x\in\mathbb{Q}_p^n$, $\vec{f}=(f_1,f_2,\cdots,f_m)$, $t=(t_1,t_2,\cdots,t_m)$, and $f_i(i=1,2,\cdots,m)$ are measurable functions on $\mathbb{Q}_p^n$, the weighted multilinear p-adic Hardy operator $U^m_{\varphi,p}$ is defined as

$$U^m_{\varphi,p}(\vec{f})(x)=\int_{(\mathbb{Z}_p^*)^m}\prod_{i=1}^m f_i(t_ix)\varphi(t)dt.$$

Definition 1.2. Let φ: $\mathbb{Z}_p^*\times\mathbb{Z}_p^*\times\cdots\times\mathbb{Z}_p^*\to[0,+\infty)$, $m\in\mathbb{N}$, $x\in\mathbb{Q}_p^n$, $\vec{f}=(f_1,f_2,\cdots,f_m)$, $t=(t_1,t_2,\cdots,t_m)$, and $f_i(i=1,2,\cdots,m)$ are measurable functions on $\mathbb{Q}_p^n$, the weighted multilinear p-adic Hardy-cesàro operator $V^m_{\varphi,p}$ is defined as

$$V^m_{\varphi,p}(\vec{f})(x)=\int_{(\mathbb{Z}_p^*)^m}\prod_{i=1}^m f_i(\frac{x}{t_i})|t_i|_p^{-n}\varphi(t)dt.$$

In 2017, N M Chuong, H D Hung, N T Hong [25] obtained the boundedness of a weighted multilinear Hardy operator and commutators on the product of Herz spaces and Morrey-Herz spaces in the Euclidean case. In this paper, we extend it to the p-adic case.

The outline of this paper is as follows. In Section 2, we give the notations and definitions that we shall use in this paper. We defined the p-adic Herz spaces and p-adic Morrey-Herz spaces. In Section 3, we state the main results on the boundedness of the weighted multilinear p-adic Hardy operator on these spaces. We also work out the norms of operator on such spaces. On the other hand, we obtain the boundedness of commutator operators of $U^m_{\varphi,p}$ and $V^m_{\varphi,p}$ with symbols in the p-adic central BMO spaces and p-adic Lipschitz spaces on the p-adic Morrey-Herz spaces. In Section 4, we give proofs of related theorems.

§2 Some Preliminaries

Definition 2.1. Let $\alpha \in \mathbb{R}$, $0 < q, l < \infty$, the p-adic Herz space $K_q^{\alpha,l}(\mathbb{Q}_p^n)$ is defined by

$$K_q^{\alpha,l}(\mathbb{Q}_p^n) = \left\{ f \in L_{loc}^q(\mathbb{Q}_p^n \setminus \{0\}) : \|f\|_{K_q^{\alpha,l}(\mathbb{Q}_p^n)} < \infty \right\},$$

and

$$\|f\|_{K_q^{\alpha,l}(\mathbb{Q}_p^n)} = \left(\sum_{k=-\infty}^{\infty} p^{k\alpha l} \|f\chi_k\|_{L^q(\mathbb{Q}_p^n)}^l \right)^{\frac{1}{l}}.$$

We denote by χ_k the characteristic function of the sphere S_k. It is obvious that $K_q^{0,q}(\mathbb{Q}_p^n) = L^q(\mathbb{Q}_p^n)$ and $K_q^{\frac{\alpha}{q},q}(\mathbb{Q}_p^n) = L^q(\mathbb{Q}_p^n, |x|_p^\alpha)$.

Definition 2.2. Let $1 \le q < \infty$ and $-\frac{1}{q} \le \lambda < 0$, the p-adic Morrey spaces $L^{q,\lambda}(\mathbb{Q}_p^n)$ is defined by

$$L^{q,\lambda}(\mathbb{Q}_p^n) = \left\{ f \in L_{loc}^q(\mathbb{Q}_p^n) : \|f\|_{L^{q,\lambda}(\mathbb{Q}_p^n)} < \infty \right\},$$

and

$$\|f\|_{L^{q,\lambda}(\mathbb{Q}_p^n)} = \sup_{a \in \mathbb{Q}_p^n, \gamma \in \mathbb{Z}} \left(\frac{1}{|B_\gamma(a)|_p^{1+\lambda q}} \int_{B_\gamma(a)} |f(x)|^q dx \right)^{\frac{1}{q}}.$$

It is clear that $L^{q,-\frac{1}{q}}(\mathbb{Q}_p^n) = L^q(\mathbb{Q}_p^n)$ and $L^{q,0}(\mathbb{Q}_p^n) = L^\infty(\mathbb{Q}_p^n)$.

Definition 2.3. Let $\alpha \in \mathbb{R}$, $0 < q, l < \infty$, and λ be a non-negative real number. The p-adic Morrey-Herz space $MK_{l,q}^{\alpha,\lambda}(\mathbb{Q}_p^n)$ is defined by

$$MK_{l,q}^{\alpha,\lambda}(\mathbb{Q}_p^n) = \left\{ f \in L_{loc}^q(\mathbb{Q}_p^n \setminus \{0\}) : \|f\|_{MK_{l,q}^{\alpha,\lambda}(\mathbb{Q}_p^n)} < \infty \right\},$$

and

$$\|f\|_{MK_{l,q}^{\alpha,\lambda}(\mathbb{Q}_p^n)} = \sup_{k_0 \in \mathbb{Z}} p^{-k_0\lambda} \left(\sum_{k=-\infty}^{k_0} p^{k\alpha l} \|f\chi_k\|_{L^q(\mathbb{Q}_p^n)}^l \right)^{\frac{1}{l}}.$$

It is easy to see that $MK_{l,q}^{\alpha,0}(\mathbb{Q}_p^n) = K_q^{\alpha,l}(\mathbb{Q}_p^n)$ and $L^{q,\lambda}(\mathbb{Q}_p^n) \subseteq MK_{q,q}^{0,\lambda}(\mathbb{Q}_p^n)$.

Definition 2.4. Let $1 \le q < \infty$. A function $f \in L_{loc}^q(\mathbb{Q}_p^n)$ is said to be $CMO^q(\mathbb{Q}_p^n)$, if

$$\|f\|_{CMO^q(\mathbb{Q}_p^n)} = \sup_{\gamma \in \mathbb{Z}} \left(\frac{1}{|B_\gamma(0)|} \int_{B_\gamma(0)} |f(x) - f_{B_\gamma(0)}|^q dx \right)^{\frac{1}{q}} < \infty,$$

where

$$f_{B_\gamma(0)} = \frac{1}{|B_\gamma(0)|} \int_{B_\gamma(0)} f(x) dx.$$

Definition 2.5. Let β be a positive real number, the Lipschitz space $Lip^\beta(\mathbb{Q}_p^n)$ is defined as the space of all measurable functions f on $\mathbb{Q}_p^n$ such that

$$\|f\|_{Lip^\beta(\mathbb{Q}_p^n)} = \sup_{x, h \in \mathbb{Q}_p^n, h \ne 0} \frac{|f(x+h) - f(x)|}{|h|_p^\beta} < \infty.$$

Let b be a measurable, locally integrable function, and T be a linear operator, the commutator $[b, T]$ is defined by

$$[b, T]f = bTf - T(bf).$$

In [20], R Coifman, R Rochberg, G Weiss proved that the commutator $[b, T]$, where T is a Calderón-Zygmund singular integral operator, is bounded on $L^p(\mathbb{R}^n)$, $1 < p < \infty$, if and only if $b \in BMO(\mathbb{R}^n)$. In this paper, we introduce the definition for the multilinear version of the commutator of the weighted p-adic Hardy operators.

Definition 2.6. Let φ: $\mathbb{Z}_p^* \times \mathbb{Z}_p^* \times \cdots \times \mathbb{Z}_p^* \to [0, +\infty)$, $m \in \mathbb{N}$, $x \in \mathbb{Q}_p^n$, $\vec{f} = (f_1, f_2, \cdots, f_m)$, $t = (t_1, t_2, \cdots, t_m)$, and $f_i (i = 1, 2, \cdots, m)$ are measurable functions on $\mathbb{Q}_p^n$, $b_i (i = 1, 2, \cdots, m)$ are locally integral functions on $\mathbb{Q}_p^n$, $U_{\varphi,p}^{m,\vec{b}}$ and $V_{\varphi,p}^{m,\vec{b}}$ are defined by

$$U_{\varphi,p}^{m,\vec{b}}(\vec{f})(x) = \int_{(\mathbb{Z}_p^*)^m} \left(\prod_{i=1}^{m} f_i(t_i x) \right) \left(\prod_{i=1}^{m} \big(b_i(x) - b_i(t_i x)\big) \right) \varphi(t) dt, \qquad x \in \mathbb{Q}_p^n.$$

$$V_{\varphi,p}^{m,\vec{b}}(\vec{f})(x) = \int_{(\mathbb{Z}_p^*)^m} \left(\prod_{i=1}^{m} f_i(\frac{x}{t_i}) |t_i|_p^{-n} \right) \left(\prod_{i=1}^{m} \big(b_i(x) - b_i(\frac{x}{t_i})\big) \right) \varphi(t) dt, \qquad x \in \mathbb{Q}_p^n.$$

The main theorem is given as follows.

§3 Main Results

Theorem 3.1. *Let $\alpha, \alpha_i \in \mathbb{R}$, $1 \le l, l_i, q, q_i < \infty$ and $\alpha_1 + \alpha_2 + \cdots + \alpha_m = \alpha$, $\frac{1}{q_1} + \frac{1}{q_2} + \cdots + \frac{1}{q_m} = \frac{1}{q}$, $\frac{1}{l_1} + \frac{1}{l_2} + \cdots + \frac{1}{l_m} = \frac{1}{l}$, $i = 1, 2, \cdots, m$. Then $U_{\varphi,p}^m$ is bounded from $K_{q_1}^{\alpha_1, l_1}(\mathbb{Q}_p^n) \times K_{q_2}^{\alpha_2, l_2}(\mathbb{Q}_p^n) \times \cdots \times K_{q_m}^{\alpha_m, l_m}(\mathbb{Q}_p^n)$ to $K_q^{\alpha, l}(\mathbb{Q}_p^n)$ If*

$$\mathcal{A}_m = \int_{(\mathbb{Z}_p^*)^m} \prod_{i=1}^{m} |t_i|_p^{-\frac{n}{q_i} - \alpha_i} \varphi(t) dt < \infty.$$

Conversely, if $l_1 = l_2 = \cdots = l_m = ml$, $U_{\varphi,p}^m$ is bounded from $K_{q_1}^{\alpha_1, l_1}(\mathbb{Q}_p^n) \times K_{q_2}^{\alpha_2, l_2}(\mathbb{Q}_p^n) \times \cdots \times K_{q_m}^{\alpha_m, l_m}(\mathbb{Q}_p^n)$ to $K_q^{\alpha, l}(\mathbb{Q}_p^n)$, Then $\mathcal{A}_m < \infty$. Moreover, one has

$$\|U_{\varphi,p}^m\|_{K_{q_1}^{\alpha_1, l_1}(\mathbb{Q}_p^n) \times K_{q_2}^{\alpha_2, l_2}(\mathbb{Q}_p^n) \times \cdots \times K_{q_m}^{\alpha_m, l_m}(\mathbb{Q}_p^n) \to K_q^{\alpha, l}(\mathbb{Q}_p^n)} = \mathcal{A}_m.$$

Then $V_{\varphi,p}^m$ is bounded from $K_{q_1}^{\alpha_1, l_1}(\mathbb{Q}_p^n) \times K_{q_2}^{\alpha_2, l_2}(\mathbb{Q}_p^n) \times \cdots \times K_{q_m}^{\alpha_m, l_m}(\mathbb{Q}_p^n)$ to $K_q^{\alpha, l}(\mathbb{Q}_p^n)$ If

$$\widetilde{\mathcal{A}}_m = \int_{(\mathbb{Z}_p^*)^m} \prod_{i=1}^{m} |t_i|_p^{n(\frac{1}{q_i} - 1) + \alpha_i} \varphi(t) dt < \infty.$$

Conversely, if $l_1 = l_2 = \cdots = l_m = ml$, $V_{\varphi,p}^m$ is bounded from $K_{q_1}^{\alpha_1, l_1}(\mathbb{Q}_p^n) \times K_{q_2}^{\alpha_2, l_2}(\mathbb{Q}_p^n) \times \cdots \times K_{q_m}^{\alpha_m, l_m}(\mathbb{Q}_p^n)$ to $K_q^{\alpha, l}(\mathbb{Q}_p^n)$, Then $\widetilde{\mathcal{A}}_m < \infty$. Moreover, one has

$$\|V_{\varphi,p}^m\|_{K_{q_1}^{\alpha_1, l_1}(\mathbb{Q}_p^n) \times K_{q_2}^{\alpha_2, l_2}(\mathbb{Q}_p^n) \times \cdots \times K_{q_m}^{\alpha_m, l_m}(\mathbb{Q}_p^n) \to K_q^{\alpha, l}(\mathbb{Q}_p^n)} = \widetilde{\mathcal{A}}_m.$$

Theorem 3.2. *Let $\alpha, \alpha_i \in \mathbb{R}$, $\lambda, \lambda_i \ge 0$, $1 \le l, l_i, q, q_i < \infty$ and $\alpha_1 + \alpha_2 + \cdots + \alpha_m = \alpha$, $\lambda = \lambda_1 + \lambda_2 + \cdots + \lambda_m$, $\frac{1}{q_1} + \frac{1}{q_2} + \cdots + \frac{1}{q_m} = \frac{1}{q}$, $\frac{1}{l_1} + \frac{1}{l_2} + \cdots + \frac{1}{l_m} = \frac{1}{l}$, $i = 1, 2, \cdots, m$. Then $U_{\varphi,p}^m$ is bounded from $MK_{l_1,q_1}^{\alpha_1,\lambda_1}(\mathbb{Q}_p^n) \times MK_{l_2,q_2}^{\alpha_2,\lambda_2}(\mathbb{Q}_p^n) \times \cdots \times MK_{l_m,q_m}^{\alpha_m,\lambda_m}(\mathbb{Q}_p^n)$ to $MK_{l,q}^{\alpha,\lambda}(\mathbb{Q}_p^n)$If*

$$\mathcal{B}_m = \int_{(\mathbb{Z}_p^*)^m} \prod_{i=1}^{m} |t_i|_p^{\lambda_i - \frac{n}{q_i} - \alpha_i} \varphi(t) dt < \infty.$$

Conversely, if $l_1 = l_2 = \cdots = l_m = ml$ and $\lambda_1 = \lambda_2 = \cdots = \lambda_m = \frac{1}{m}\lambda$, $U_{\varphi,p}^m$ is bounded from $MK_{l_1,q_1}^{\alpha_1,\lambda_1}(\mathbb{Q}_p^n) \times MK_{l_2,q_2}^{\alpha_2,\lambda_2}(\mathbb{Q}_p^n) \times \cdots \times MK_{l_m,q_m}^{\alpha_m,\lambda_m}(\mathbb{Q}_p^n)$ to $MK_{l,q}^{\alpha,\lambda}(\mathbb{Q}_p^n)$, Then $\mathcal{B}_m < \infty$. M oreover, one has

$$\|U_{\varphi,p}^m\|_{MK_{l_1,q_1}^{\alpha_1,\lambda_1}(\mathbb{Q}_p^n) \times MK_{l_2,q_2}^{\alpha_2,\lambda_2}(\mathbb{Q}_p^n) \times \cdots \times MK_{l_m,q_m}^{\alpha_m,\lambda_m}(\mathbb{Q}_p^n) \to MK_{l,q}^{\alpha,\lambda}(\mathbb{Q}_p^n)} = \mathcal{B}_m.$$

Then $V_{\varphi,p}^m$ is bounded from $MK_{l_1,q_1}^{\alpha_1,\lambda_1}(\mathbb{Q}_p^n) \times MK_{l_2,q_2}^{\alpha_2,\lambda_2}(\mathbb{Q}_p^n) \times \cdots \times MK_{l_m,q_m}^{\alpha_m,\lambda_m}(\mathbb{Q}_p^n)$ to $MK_{l,q}^{\alpha,\lambda}(\mathbb{Q}_p^n)$

If

$$\widetilde{\mathcal{B}}_m=\int_{(\mathbb{Z}_p^*)^m}\prod_{i=1}^m|t_i|_p^{\alpha_i-\lambda_i+(\frac{1}{q_i}-1)n}\varphi(t)dt<\infty.$$

Conversely, if $l_1=l_2=\cdots=l_m=ml$ *and* $\lambda_1=\lambda_2=\cdots=\lambda_m=\frac{1}{m}\lambda$, $V_{\varphi,p}^m$ *is bounded from* $MK_{l_1,q_1}^{\alpha_1,\lambda_1}(\mathbb{Q}_p^n)\times MK_{l_2,q_2}^{\alpha_2,\lambda_2}(\mathbb{Q}_p^n)\times\cdots\times MK_{l_m,q_m}^{\alpha_m,\lambda_m}(\mathbb{Q}_p^n)$ *to* $MK_{l,q}^{\alpha,\lambda}(\mathbb{Q}_p^n)$, *Then* $\widetilde{\mathcal{B}}_m<\infty$. *Moreover, one has*

$$\|V_{\varphi,p}^m\|_{MK_{l_1,q_1}^{\alpha_1,\lambda_1}(\mathbb{Q}_p^n)\times MK_{l_2,q_2}^{\alpha_2,\lambda_2}(\mathbb{Q}_p^n)\times\cdots\times MK_{l_m,q_m}^{\alpha_m,\lambda_m}(\mathbb{Q}_p^n)\to MK_{l,q}^{\alpha,\lambda}(\mathbb{Q}_p^n)}=\widetilde{\mathcal{B}}_m.$$

Theorem 3.3. *Let* $\alpha,\alpha_i\in\mathbb{R}$, $\lambda,\lambda_i\geq 0$, $0<l<\infty$, $1\leq l_i,q,p_i,q_i<\infty$ *and* $\alpha_1+\alpha_2+\cdots+\alpha_m=\alpha$, $\lambda=\lambda_1+\lambda_2+\cdots+\lambda_m$, $\frac{1}{q_1}+\frac{1}{q_2}+\cdots+\frac{1}{q_m}+\frac{1}{p_1}+\frac{1}{p_2}+\cdots+\frac{1}{p_m}=\frac{1}{q}$, $\frac{1}{l_1}+\frac{1}{l_2}+\cdots+\frac{1}{l_m}=\frac{1}{l}$, $i=1,2,\cdots,m$. $b=(b_1,b_2,\cdots,b_m)\in CMO^{p_1}(\mathbb{Q}_p^n)\times CMO^{p_2}(\mathbb{Q}_p^n)\times\cdots\times CMO^{p_m}(\mathbb{Q}_p^n)$. *Then* $U_{\varphi,p}^{m,\vec{b}}$ *is bounded from* $MK_{l_1,q_1}^{\alpha_1+\frac{n}{p_1},\lambda_1}(\mathbb{Q}_p^n)\times MK_{l_2,q_2}^{\alpha_2+\frac{n}{p_2},\lambda_2}(\mathbb{Q}_p^n)\times\cdots\times MK_{l_m,q_m}^{\alpha_m+\frac{n}{p_m},\lambda_m}(\mathbb{Q}_p^n)$ *to* $MK_{l,q}^{\alpha,\lambda}(\mathbb{Q}_p^n)$, *when* $1\leq l<\infty$ *and* $\lambda\geq 0$ *or when* $\lambda>0$ *and* $0<l<1$ *if*

$$\mathcal{C}_m=\int_{(\mathbb{Z}_p^*)^m}\prod_{i=1}^m|t_i|_p^{\lambda_i-\alpha_i-\frac{n}{p_i}-\frac{n}{q_i}}\log_p\frac{p}{|t_i|_p}\varphi(t)dt<\infty.$$

Then $V_{\varphi,p}^{m,\vec{b}}$ *is bounded from* $MK_{l_1,q_1}^{\alpha_1+\frac{n}{p_1},\lambda_1}(\mathbb{Q}_p^n)\times MK_{l_2,q_2}^{\alpha_2+\frac{n}{p_2},\lambda_2}(\mathbb{Q}_p^n)\times\cdots\times MK_{l_m,q_m}^{\alpha_m+\frac{n}{p_m},\lambda_m}(\mathbb{Q}_p^n)$ *to* $MK_{l,q}^{\alpha,\lambda}(\mathbb{Q}_p^n)$, *when* $1\leq l<\infty$ *and* $\lambda\geq 0$ *or when* $\lambda>0$ *and* $0<l<1$ *if*

$$\widetilde{\mathcal{C}}_m=\int_{(\mathbb{Z}_p^*)^m}\prod_{i=1}^m|t_i|_p^{\alpha_i-\lambda_i+\frac{n}{p_i}+(\frac{1}{q_i}-1)n}\log_p\frac{p}{|t_i|_p}\varphi(t)dt<\infty.$$

Theorem 3.4. *Let* $\alpha,\alpha_i\in\mathbb{R}$, $\lambda,\lambda_i\geq 0$, $0<l<\infty$, $1\leq l_i,q,p_i,q_i<\infty$ *and* $\alpha_1+\alpha_2+\cdots+\alpha_m=\alpha$, $\lambda=\lambda_1+\lambda_2+\cdots+\lambda_m$, $\frac{1}{q_1}+\frac{1}{q_2}+\cdots+\frac{1}{q_m}+\frac{1}{p_1}+\frac{1}{p_2}+\cdots+\frac{1}{p_m}=\frac{1}{q}$, $\frac{1}{l_1}+\frac{1}{l_2}+\cdots+\frac{1}{l_m}=\frac{1}{l}$, $i=1,2,\cdots,m$. $b=(b_1,b_2,\cdots,b_m)\in Lip^{\beta_1}(\mathbb{Q}_p^n)\times Lip^{\beta_2}(\mathbb{Q}_p^n)\times\cdots\times Lip^{\beta_m}(\mathbb{Q}_p^n)$, $0<\beta_i<1$. *Then* $U_{\varphi,p}^{m,\vec{b}}$ *is bounded from* $MK_{l_1,q_1}^{\alpha_1+\beta_1+\frac{n}{p_1},\lambda_1}(\mathbb{Q}_p^n)\times MK_{l_2,q_2}^{\alpha_2+\beta_2+\frac{n}{p_2},\lambda_2}(\mathbb{Q}_p^n)\times\cdots\times MK_{l_m,q_m}^{\alpha_m+\beta_m+\frac{n}{p_m},\lambda_m}(\mathbb{Q}_p^n)$ *to* $MK_{l,q}^{\alpha,\lambda}(\mathbb{Q}_p^n)$, *when* $1\leq l<\infty$ *and* $\lambda\geq 0$ *or when* $\lambda>0$ *and* $0<l<1$ *if*

$$\mathcal{D}_m=\int_{(\mathbb{Z}_p^*)^m}\prod_{i=1}^m|t_i|_p^{\lambda_i-\alpha_i-\beta_i-\frac{n}{p_i}-\frac{n}{q_i}}|1-t_i|_p^{\beta_i}\varphi(t)dt<\infty.$$

Then $V_{\varphi,p}^{m,\vec{b}}$ *is bounded from* $MK_{l_1,q_1}^{\alpha_1+\beta_1+\frac{n}{p_1},\lambda_1}(\mathbb{Q}_p^n)\times MK_{l_2,q_2}^{\alpha_2+\beta_2+\frac{n}{p_2},\lambda_2}(\mathbb{Q}_p^n)\cdots\times MK_{l_m,q_m}^{\alpha_m+\beta_m+\frac{n}{p_m},\lambda_m}$ *to* $MK_{l,q}^{\alpha,\lambda}(\mathbb{Q}_p^n)$, *when* $1\leq l<\infty$ *and* $\lambda\geq 0$ *or when* $\lambda>0$ *and* $0<l<1$ *if*

$$\widetilde{\mathcal{D}}_m=\int_{(\mathbb{Z}_p^*)^m}\prod_{i=1}^m|t_i|_p^{\alpha_i-\lambda_i+\beta_i+\frac{n}{p_i}+(\frac{1}{q_i}-1)n}|1-\frac{1}{t_i}|_p^{\beta_i}\varphi(t)dt<\infty.$$

§4 Proof of The Theorem

4.1 Proof of Theorem 3.1

We only consider the case of $m=2$, the case of $m\geq 3$ is similarly available. By Minkowski's inequality and Hölder's inequality

$$\|U_{\varphi,p}^2(f_1,f_2)\chi_k\|_{L^q(\mathbb{Q}_p^n)}$$

$$= \left(\int_{S_k} \left| \int_{(\mathbb{Z}_p^*)^2} f_1(t_1x) f_2(t_2x) \varphi(t) dt \right|^q dx \right)^{\frac{1}{q}}$$
$$\leq \int_{(\mathbb{Z}_p^*)^2} \left(\int_{S_k} |f_1(t_1x) f_2(t_2x)|^q dx \right)^{\frac{1}{q}} \varphi(t) dt$$
$$\leq \int_{(\mathbb{Z}_p^*)^2} \left(\int_{S_k} |f_1(t_1x)|^{q_1} dx \right)^{\frac{1}{q_1}} \left(\int_{S_k} |f_2(t_2x)|^{q_2} dx \right)^{\frac{1}{q_2}} \varphi(t) dt$$
$$= \int_{(\mathbb{Z}_p^*)^2} \left(\int_{t_1S_k} |f_1(y_1)|^{q_1} dy_1 \right)^{\frac{1}{q_1}} \left(\int_{t_2S_k} |f_2(y_2)|^{q_2} dy_2 \right)^{\frac{1}{q_2}} |t_1|_p^{-\frac{n}{q_1}} |t_2|_p^{-\frac{n}{q_2}} \varphi(t) dt.$$

There exist n_1, n_2 such that $|t_1|_p = p^{n_1}$, $|t_2|_p = p^{n_2}$,

$$\|U_{\varphi,p}^2(f_1,f_2)\chi_k\|_{L^q(\mathbb{Q}_p^n)} = \int_{(\mathbb{Z}_p^*)^2} \|f_1\chi_{k+n_1}\|_{L^{q_1}(\mathbb{Q}_p^n)} \|f_2\chi_{k+n_2}\|_{L^{q_2}(\mathbb{Q}_p^n)} |t_1|_p^{-\frac{n}{q_1}} |t_2|_p^{-\frac{n}{q_2}} \varphi(t) dt.$$

$$\|U_{\varphi,p}^2(f_1,f_2)\|_{K_q^{\alpha,l}(\mathbb{Q}_p^n)}$$
$$= \left[\sum_{k=-\infty}^{\infty} p^{k\alpha l} \left(\int_{(\mathbb{Z}_p^*)^2} \|f_1\chi_{k+n_1}\|_{L^{q_1}(\mathbb{Q}_p^n)} \|f_2\chi_{k+n_2}\|_{L^{q_2}(\mathbb{Q}_p^n)} |t_1|_p^{-\frac{n}{q_1}} |t_2|_p^{-\frac{n}{q_2}} \varphi(t) dt \right)^l \right]^{\frac{1}{l}}$$
$$\leq \int_{(\mathbb{Z}_p^*)^2} \left(\sum_{k=-\infty}^{\infty} p^{k\alpha l} \|f_1\chi_{k+n_1}\|_{L^{q_1}(\mathbb{Q}_p^n)}^l \|f_2\chi_{k+n_2}\|_{L^{q_2}(\mathbb{Q}_p^n)}^l \right)^{\frac{1}{l}} |t_1|_p^{-\frac{n}{q_1}} |t_2|_p^{-\frac{n}{q_2}} \varphi(t) dt$$
$$\leq \int_{(\mathbb{Z}_p^*)^2} \left(\sum_{k=-\infty}^{\infty} p^{k\alpha_1 l_1} \|f_1\chi_{k+n_1}\|_{L^{q_1}(\mathbb{Q}_p^n)}^{l_1} \right)^{\frac{1}{l_1}} \left(\sum_{k=-\infty}^{\infty} p^{k\alpha_2 l_2} \|f_2\chi_{k+n_2}\|_{L^{q_2}(\mathbb{Q}_p^n)}^{l_2} \right)^{\frac{1}{l_2}}$$
$$\times |t_1|_p^{-\frac{n}{q_1}} |t_2|_p^{-\frac{n}{q_2}} \varphi(t) dt$$
$$= \|f_1\|_{K_{q_1}^{\alpha_1,l_1}(\mathbb{Q}_p^n)} \|f_2\|_{K_{q_2}^{\alpha_2,l_2}(\mathbb{Q}_p^n)} \int_{(\mathbb{Z}_p^*)^2} |t_1|_p^{-\frac{n}{q_1}-\alpha_1} |t_2|_p^{-\frac{n}{q_2}-\alpha_2} \varphi(t) dt,$$
$$\|U_{\varphi,p}^2\|_{K_{q_1}^{\alpha_1,l_1}(\mathbb{Q}_p^n)\times K_{q_2}^{\alpha_2,l_2}(\mathbb{Q}_p^n)\to K_q^{\alpha,l}(\mathbb{Q}_p^n)} \leq \int_{(\mathbb{Z}_p^*)^2} |t_1|_p^{-\frac{n}{q_1}-\alpha_1} |t_2|_p^{-\frac{n}{q_2}-\alpha_2} \varphi(t) dt.$$

Next, the opposite of the proof is given and we shall consider the following function,

$$f_1(x) = \begin{cases} 0, & |x|_p < 1, \\ |x|_p^{-\frac{n}{q_1}-\alpha_1-\frac{1}{p^m}}, & |x|_p \geq 1. \end{cases}$$
$$f_2(x) = \begin{cases} 0, & |x|_p < 1, \\ |x|_p^{-\frac{n}{q_2}-\alpha_2-\frac{1}{p^m}}, & |x|_p \geq 1. \end{cases}$$

It is easy to see that when $k < 0$, then $f_1\chi_k = 0$, $f_2\chi_k = 0$. For $k \geq 0$, we have

$$\|f_1\chi_k\|_{L^{q_1}(\mathbb{Q}_p^n)}^{q_1} = \int_{S_k} |x|_p^{-n-\alpha_1 q_1-\frac{q_1}{p^m}} dx = (1-p^{-n}) p^{k(-\alpha_1 q_1-\frac{q_1}{p^m})},$$

$$\|f_1\|_{K_{q_1}^{\alpha_1,l_1}(\mathbb{Q}_p^n)} = (1-p^{-n})^{\frac{1}{q_1}} \frac{p^{\frac{1}{p^m}}}{(p^{\frac{l_1}{p^m}}-1)^{\frac{1}{l_1}}}.$$

Similarly,

$$\|f_2\|_{K_{q_2}^{\alpha_2,l_2}(\mathbb{Q}_p^n)} = (1-p^{-n})^{\frac{1}{q_2}} \frac{p^{\frac{1}{p^m}}}{(p^{\frac{l_2}{p^m}}-1)^{\frac{1}{l_2}}},$$

$$U^2_{\varphi,p}(f_1,f_2)(x)=\begin{cases}0, & |x|_p<1,\\ |x|_p^{-\frac{n}{q}-\alpha-\frac{2}{p^m}}\int_{\frac{1}{|x|_p}\le|t_1|_p\le 1}\int_{\frac{1}{|x|_p}\le|t_2|_p\le 1}\prod_{i=1}^{2}|t_i|_p^{-\frac{n}{q_i}-\alpha_i-\frac{1}{p^m}}\varphi(t)dt, & |x|_p\ge 1.\end{cases}$$

For any $m\le k$, $p^{-m}\ge p^{-k}$,

$$\begin{aligned}&\|U^2_{\varphi,p}(f_1,f_2)\chi_k\|^q_{L^q(\mathbb{Q}_p^n)}\\ &=\int_{S_k}|x|_p^{-n-\alpha q-\frac{2q}{p^m}}dx\Big(\int_{p^{-k}\le|t_1|_p\le 1}\int_{p^{-k}\le|t_2|_p\le 1}\prod_{i=1}^{2}|t_i|_p^{-\frac{n}{q_i}-\alpha_i-\frac{1}{p^m}}\varphi(t)dt\Big)^q\\ &\ge p^{k(-\alpha q-\frac{2q}{p^m})}(1-p^{-n})\Big(\int_{p^{-m}\le|t_1|_p\le 1}\int_{p^{-m}\le|t_2|_p\le 1}\prod_{i=1}^{2}|t_i|_p^{-\frac{n}{q_i}-\alpha_i-\frac{1}{p^m}}\varphi(t)dt\Big)^q.\end{aligned}$$

Then

$$\begin{aligned}&\|U^2_{\varphi,p}(f_1,f_2)\|_{K_q^{\alpha,l}(\mathbb{Q}_p^n)}\\ &\ge\int_{p^{-m}\le|t_1|_p\le 1}\int_{p^{-m}\le|t_2|_p\le 1}\prod_{i=1}^{2}|t_i|_p^{-\frac{n}{q_i}-\alpha_i-\frac{1}{p^m}}\varphi(t)dt\\ &\times\Big(\sum_{k=m}^{\infty}p^{k\alpha l}p^{k(-\alpha l-\frac{2l}{p^m})}\Big)^{\frac{1}{l}}(1-p^{-n})^{\frac{1}{q}}\\ &=p^{\frac{-2m}{p^m}}\int_{p^{-m}\le|t_1|_p\le 1}\int_{p^{-m}\le|t_2|_p\le 1}\prod_{i=1}^{2}|t_i|_p^{-\frac{n}{q_i}-\alpha_i-\frac{1}{p^m}}\varphi(t)dt\\ &\times(1-p^{-n})^{\frac{1}{q_1}}(1-p^{-n})^{\frac{1}{q_2}}\frac{p^{\frac{1}{p^m}}}{(p^{\frac{2l}{p^m}}-1)^{\frac{1}{l_1}}}\frac{p^{\frac{1}{p^m}}}{(p^{\frac{2l}{p^m}}-1)^{\frac{1}{l_2}}}.\end{aligned}$$

When $l_1=l_2=2l$,

$$\begin{aligned}&\|U^2_{\varphi,p}(f_1,f_2)\|_{K_q^{\alpha,l}(\mathbb{Q}_p^n)}\\ &\ge p^{\frac{-2m}{p^m}}\|f_1\|_{K_{q_1}^{\alpha_1,l_1}(\mathbb{Q}_p^n)}\|f_2\|_{K_{q_2}^{\alpha_2,l_2}(\mathbb{Q}_p^n)}\int_{p^{-m}\le|t_1|_p\le 1}\int_{p^{-m}\le|t_2|_p\le 1}\prod_{i=1}^{2}|t_i|_p^{-\frac{n}{q_i}-\alpha_i-\frac{1}{p^m}}\varphi(t)dt.\end{aligned}$$

when $m\to\infty$,

$$\infty>\|U^2_{\varphi,p}\|_{K_{q_1}^{\alpha_1,l_1}(\mathbb{Q}_p^n)\times K_{q_2}^{\alpha_2,l_2}(\mathbb{Q}_p^n)\to K_q^{\alpha,l}(\mathbb{Q}_p^n)}\ge\int_{(\mathbb{Z}_p^*)^2}|t_1|_p^{-\frac{n}{q_1}-\alpha_1}|t_2|_p^{-\frac{n}{q_2}-\alpha_2}\varphi(t)dt.$$

Combine the above formula,

$$\|U^2_{\varphi,p}\|_{K_{q_1}^{\alpha_1,l_1}(\mathbb{Q}_p^n)\times K_{q_2}^{\alpha_2,l_2}(\mathbb{Q}_p^n)\to K_q^{\alpha,l}(\mathbb{Q}_p^n)}=\int_{(\mathbb{Z}_p^*)^2}|t_1|_p^{-\frac{n}{q_1}-\alpha_1}|t_2|_p^{-\frac{n}{q_2}-\alpha_2}\varphi(t)dt.$$

By a similar argument, we also obtain the norm of $V^2_{\varphi,p}$.

4.2 Proof of Theorem 3.2

By similarity, we only consider the case in which $m=2$, By using Minkowski's inequality and Hölder's inequality, we have

$$\|U^2_{\varphi,p}(f_1,f_2)\|_{MK_{l,q}^{\alpha,\lambda}(\mathbb{Q}_p^n)}$$

$$= \sup_{k_0\in\mathbb{Z}} p^{-k_0\lambda}\bigg(\sum_{k=-\infty}^{k_0} p^{k\alpha l}\|U^2_{\varphi,p}(f_1,f_2)\chi_k\|^l_{L^q(\mathbb{Q}_p^n)}\bigg)^{\frac{1}{l}}$$

$$= \sup_{k_0\in\mathbb{Z}} p^{-k_0\lambda}\bigg[\sum_{k=-\infty}^{k_0} p^{k\alpha l}\bigg(\int_{(\mathbb{Z}_p^*)^2}\|f_1\chi_{k+n_1}\|_{L^{q_1}(\mathbb{Q}_p^n)}\|f_2\chi_{k+n_2}\|_{L^{q_2}}|t_1|_p^{-\frac{n}{q_1}}|t_2|_p^{-\frac{n}{q_2}}\varphi(t)dt\bigg)^l\bigg]^{\frac{1}{l}}$$

$$\le \int_{(\mathbb{Z}_p^*)^2}\sup_{k_0\in\mathbb{Z}} p^{-k_0\lambda}\bigg(\sum_{k=-\infty}^{k_0} p^{k\alpha l}\|f_1\chi_{k+n_1}\|^l_{L^{q_1}(\mathbb{Q}_p^n)}\|f_2\chi_{k+n_2}\|^l_{L^{q_2}(\mathbb{Q}_p^n)}\bigg)^{\frac{1}{l}}|t_1|_p^{-\frac{n}{q_1}}|t_2|_p^{-\frac{n}{q_2}}\varphi(t)dt$$

$$\le \int_{(\mathbb{Z}_p^*)^2}\sup_{k_0\in\mathbb{Z}} p^{-k_0\lambda_1}\bigg(\sum_{k=-\infty}^{k_0} p^{k\alpha_1 l_1}\|f_1\chi_{k+n_1}\|^{l_1}_{L^{q_1}(\mathbb{Q}_p^n)}\bigg)^{\frac{1}{l_1}}$$

$$\times \sup_{k_0\in\mathbb{Z}} p^{-k_0\lambda_2}\bigg(\sum_{k=-\infty}^{k_0} p^{k\alpha_2 l_2}\|f_2\chi_{k+n_2}\|^{l_2}_{L^{q_2}(\mathbb{Q}_p^n)}\bigg)^{\frac{1}{l_2}}|t_1|_p^{-\frac{n}{q_1}}|t_2|_p^{-\frac{n}{q_2}}\varphi(t)dt$$

$$= \|f_1\|_{MK^{\alpha_1,\lambda_1}_{l_1,q_1}(\mathbb{Q}_p^n)}\|f_2\|_{MK^{\alpha_2,\lambda_2}_{l_2,q_2}(\mathbb{Q}_p^n)}\int_{(\mathbb{Z}_p^*)^2}|t_1|_p^{\lambda_1-\alpha_1-\frac{n}{q_1}}|t_2|_p^{\lambda_2-\alpha_2-\frac{n}{q_2}}\varphi(t)dt.$$

$$\|U^2_{\varphi,p}\|_{MK^{\alpha_1,\lambda_1}_{l_1,q_1}(\mathbb{Q}_p^n)\times MK^{\alpha_2,\lambda_2}_{l_2,q_2}(\mathbb{Q}_p^n)\to K^{\alpha,\lambda}_{l,q}(\mathbb{Q}_p^n)} \le \int_{(\mathbb{Z}_p^*)^2}|t_1|_p^{\lambda_1-\frac{n}{q_1}-\alpha_1}|t_2|_p^{\lambda_2-\frac{n}{q_2}-\alpha_2}\varphi(t)dt.$$

Next, the opposite of the proof is given and we shall consider the following function:

$$f_1(x)=|x|_p^{\lambda_1-\frac{n}{q_1}-\alpha_1}, \qquad f_2(x)=|x|_p^{\lambda_2-\frac{n}{q_2}-\alpha_2},$$

$$\|f_1\chi_k\|^{q_1}_{L^{q_1}(\mathbb{Q}_p^n)} = \int_{S_k}|x|_p^{\lambda_1 q_1-\alpha_1 q_1-n}dx = (1-p^{-n})p^{k(\lambda_1 q_1-\alpha_1 q_1)},$$

$$\|f_1\|_{MK^{\alpha_1,\lambda_1}_{l_1,q_1}(\mathbb{Q}_p^n)} = (1-p^{-n})^{\frac{1}{q_1}}\frac{p^{\lambda_1}}{(p^{\lambda_1 l_1}-1)^{\frac{1}{l_1}}}.$$

Similarly,

$$\|f_2\|_{MK^{\alpha_2,\lambda_2}_{l_2,q_2}(\mathbb{Q}_p^n)} = (1-p^{-n})^{\frac{1}{q_2}}\frac{p^{\lambda_2}}{(p^{\lambda_2 l_2}-1)^{\frac{1}{l_2}}},$$

$$U^2_{\varphi,p}(f_1,f_2)(x) = \int_{(\mathbb{Z}_p^*)^2}|x|_p^{\lambda-\alpha-\frac{n}{q}}|t_1|_p^{\lambda_1-\frac{n}{q_1}-\alpha_1}|t_2|_p^{\lambda_2-\frac{n}{q_2}-\alpha_2}\varphi(t)dt,$$

$$\|U^2_{\varphi,p}(f_1,f_2)\chi_k\|^q_{L^q(\mathbb{Q}_p^n)}$$
$$= \int_{S_k}|x|_p^{\lambda q-\alpha q-n}dx\bigg(\int_{(\mathbb{Z}_p^*)^2}|t_1|_p^{\lambda_1-\frac{n}{q_1}-\alpha_1}|t_2|_p^{\lambda_2-\frac{n}{q_2}-\alpha_2}\varphi(t)dt\bigg)^q$$
$$= (1-p^{-n})p^{k(\lambda q-\alpha q)}\bigg(\int_{(\mathbb{Z}_p^*)^2}|t_1|_p^{\lambda_1-\frac{n}{q_1}-\alpha_1}|t_2|_p^{\lambda_2-\frac{n}{q_2}-\alpha_2}\varphi(t)dt\bigg)^q.$$

$$\|U^2_{\varphi,p}(f_1,f_2)\|_{MK^{\alpha,\lambda}_{l,q}(\mathbb{Q}_p^n)}$$
$$= \sup_{k_0\in\mathbb{Z}} p^{-k_0\lambda}\bigg(\sum_{k=-\infty}^{k_0}p^{kl\lambda}\bigg)^{\frac{1}{l}}(1-p^{-n})^{\frac{1}{q}}\int_{(\mathbb{Z}_p^*)^2}|t_1|_p^{\lambda_1-\frac{n}{q_1}-\alpha_1}|t_2|_p^{\lambda_2-\frac{n}{q_2}-\alpha_2}\varphi(t)dt$$
$$= \frac{p^\lambda}{(p^{l\lambda}-1)^{\frac{1}{l}}}(1-p^{-n})^{\frac{1}{q}}\int_{(\mathbb{Z}_p^*)^2}|t_1|_p^{\lambda_1-\frac{n}{q_1}-\alpha_1}|t_2|_p^{\lambda_2-\frac{n}{q_2}-\alpha_2}\varphi(t)dt.$$

When $l_1=l_2=2l$ and $\lambda_1=\lambda_2=\frac{1}{2}\lambda$.

$$\begin{aligned}&\|U^2_{\varphi,p}(f_1,f_2)\|_{MK^{\alpha,\lambda}_{l,q}(\mathbb{Q}^n_p)}\\&=\frac{p^{\lambda_1}}{(p^{l_1\lambda_1}-1)^{\frac{1}{l_1}}}(1-p^{-n})^{\frac{1}{q_1}}\frac{p^{\lambda_2}}{(p^{l_2\lambda_2}-1)^{\frac{1}{l_2}}}(1-p^{-n})^{\frac{1}{q_2}}\int_{(\mathbb{Z}^*_p)^2}|t_1|_p^{\lambda_1-\frac{n}{q_1}-\alpha_1}|t_2|_p^{\lambda_2-\frac{n}{q_2}-\alpha_2}\varphi(t)dt\\&=\|f_1\|_{MK^{\alpha_1,\lambda_1}_{l_1,q_1}(\mathbb{Q}^n_p)}\|f_2\|_{MK^{\alpha_2,\lambda_2}_{l_2,q_2}(\mathbb{Q}^n_p)}\int_{(\mathbb{Z}^*_p)^2}|t_1|_p^{\lambda_1-\alpha_1-\frac{n}{q_1}}|t_2|_p^{\lambda_2-\alpha_2-\frac{n}{q_2}}\varphi(t)dt,\end{aligned}$$

$$\infty>\|U^2_{\varphi,p}\|_{MK^{\alpha_1,\lambda_1}_{l_1,q_1}(\mathbb{Q}^n_p)\times MK^{\alpha_2,\lambda_2}_{l_2,q_2}(\mathbb{Q}^n_p)\to MK^{\alpha,\lambda}_{l,q}(\mathbb{Q}^n_p)}\geq\int_{(\mathbb{Z}^*_p)^2}|t_1|_p^{\lambda_1-\frac{n}{q_1}-\alpha_1}|t_2|_p^{\lambda_2-\frac{n}{q_2}-\alpha_2}\varphi(t)dt.$$

Hense, we can know

$$\|U^2_{\varphi,p}\|_{MK^{\alpha_1,\lambda_1}_{l_1,q_1}(\mathbb{Q}^n_p)\times MK^{\alpha_2,\lambda_2}_{l_2,q_2}(\mathbb{Q}^n_p)\to MK^{\alpha,\lambda}_{l,q}(\mathbb{Q}^n_p)}=\int_{(\mathbb{Z}^*_p)^2}|t_1|_p^{\lambda_1-\frac{n}{q_1}-\alpha_1}|t_2|_p^{\lambda_2-\frac{n}{q_2}-\alpha_2}\varphi(t)dt.$$

The proof methods for $V^2_{\varphi,p}$ and $U^2_{\varphi,p}$ are the same.

4.3 Proof of Theorem 3.3

We only consider the case in which $m=2$. By Minkowski's inequality

$$\begin{aligned}&\|U^{2,\vec{b}}_{\varphi,p}(f_1,f_2)\chi_k\|_{L^q(\mathbb{Q}^n_p)}\\&=\left(\int_{S_k}\left|\int_{(\mathbb{Z}^*_p)^2}f_1(t_1x)f_2(t_2x)\prod_{i=1}^2\big(b_i(x)-b_i(t_ix)\big)\varphi(t)dt\right|^q dx\right)^{\frac{1}{q}}\\&\leq\int_{(\mathbb{Z}^*_p)^2}\left(\int_{S_k}\left|f_1(t_1x)f_2(t_2x)\prod_{i=1}^2\big(b_i(x)-b_i(t_ix)\big)\right|^q dx\right)^{\frac{1}{q}}\varphi(t)dt\\&=I.\end{aligned}$$

$$\begin{aligned}&\prod_{i=1}^2\big|b_i(x)-b_i(t_ix)\big|\\&\leq\prod_{i=1}^2\big|b_i(x)-b_{i,B_k}\big|+\prod_{i=1}^2\big|b_i(t_ix)-b_{i,t_iB_k}\big|+\prod_{i=1}^2\big|b_{i,B_k}-b_{i,t_iB_k}\big|\\&+\sum_{1\leq i\neq j\leq 2}\big|b_i(x)-b_{i,B_k}\big|\big|b_{j,B_k}-b_{j,t_jB_k}\big|\\&+\sum_{1\leq i\neq j\leq 2}\big|b_i(x)-b_{i,B_k}\big|\big|b_j(t_jx)-b_{j,t_jB_k}\big|\\&+\sum_{1\leq i\neq j\leq 2}\big|b_{i,B_k}-b_{i,t_iB_k}\big|\big|b_j(t_jx)-b_{j,t_jB_k}\big|.\end{aligned}$$

First we estimate I,

$$I\leq I_1+I_2+I_3+I_4+I_5+I_6.$$

We use Hölder's inequality,

$$\begin{aligned}I_1&=\int_{(\mathbb{Z}^*_p)^2}\left(\int_{S_k}\left(\big|f_1(t_1x)f_2(t_2x)\big|\prod_{i=1}^2\big|b_i(x)-b_{i,B_k}\big|\right)^q dx\right)^{\frac{1}{q}}\varphi(t)dt\\&\leq\int_{(\mathbb{Z}^*_p)^2}\prod_{i=1}^2\left(\int_{S_k}\big|f_i(t_ix)\big|^{q_i}dx\right)^{\frac{1}{q_i}}\prod_{j=1}^2\left(\int_{S_k}\big|b_j(x)-b_{j,B_k}\big|^{p_i}dx\right)^{\frac{1}{p_i}}\varphi(t)dt.\end{aligned}$$

There exist n_1, n_2 such that $|t_1|_p = p^{n_1}$, $|t_2|_p = p^{n_2}$, and by changing variables, we know that

$$I_1 \leq \prod_{j=1}^{2} \|b_j\|_{CMO^{p_j}(\mathbb{Q}_p^n)} \int_{(\mathbb{Z}_p^*)^2} \|f_1\chi_{k+n_1}\|_{L^{q_1}(\mathbb{Q}_p^n)} \|f_2\chi_{k+n_2}\|_{L^{q_2}(\mathbb{Q}_p^n)} \prod_{i=1}^{2} |t_i|_p^{-\frac{n}{q_i}} p^{\frac{kn}{p_i}} \varphi(t)dt.$$

$$\begin{aligned}
I_2 &= \int_{(\mathbb{Z}_p^*)^2} \left(\int_{S_k} \left(\left| f_1(t_1x) f_2(t_2x) \right| \prod_{i=1}^{2} \left| b_i(t_ix) - b_{i,t_iB_k} \right| \right)^q dx \right)^{\frac{1}{q}} \varphi(t)dt \\
&\leq \int_{(\mathbb{Z}_p^*)^2} \prod_{i=1}^{2} \left(\int_{S_k} \left| f_i(t_ix) \right|^{q_i} dx \right)^{\frac{1}{q_i}} \prod_{j=1}^{2} \left(\int_{S_k} \left| b_j(t_jx) - b_{j,t_jB_k} \right|^{p_i} dx \right)^{\frac{1}{p_i}} \varphi(t)dt \\
&= \prod_{j=1}^{2} \|b_j\|_{CMO^{p_j}(\mathbb{Q}_p^n)} \int_{(\mathbb{Z}_p^*)^2} \|f_1\chi_{k+n_1}\|_{L^{q_1}(\mathbb{Q}_p^n)} \|f_2\chi_{k+n_2}\|_{L^{q_2}(\mathbb{Q}_p^n)} \prod_{i=1}^{2} |t_i|_p^{-\frac{n}{q_i}} p^{\frac{kn}{p_i}} \varphi(t)dt,
\end{aligned}$$

$$\begin{aligned}
I_3 &= \int_{(\mathbb{Z}_p^*)^2} \left(\int_{S_k} \left(\left| f_1(t_1x) f_2(t_2x) \right| \prod_{i=1}^{2} \left| b_{i,B_k} - b_{i,t_iB_k} \right| \right)^q dx \right)^{\frac{1}{q}} \varphi(t)dt \\
&\leq \int_{(\mathbb{Z}_p^*)^2} \prod_{i=1}^{2} \left(\int_{S_k} \left| f_i(t_ix) \right|^{q_i} dx \right)^{\frac{1}{q_i}} \prod_{j=1}^{2} \left(\int_{S_k} \left| b_{j,B_k} - b_{j,t_jB_k} \right|^{p_i} dx \right)^{\frac{1}{p_i}} \varphi(t)dt \\
&= \int_{(\mathbb{Z}_p^*)^2} \|f_1\chi_{k+n_1}\|_{L^{q_1}(\mathbb{Q}_p^n)} \|f_2\chi_{k+n_2}\|_{L^{q_2}(\mathbb{Q}_p^n)} \prod_{j=1}^{2} \left| b_{j,B_k} - b_{j,t_jB_k} \right| \prod_{i=1}^{2} |t_i|_p^{-\frac{n}{q_i}} p^{\frac{kn}{p_i}} \varphi(t)dt \\
&\leq \sum_{m_1=0}^{\infty} \sum_{m_2=0}^{\infty} \int_{p^{-m_1-1}<|t_1|_p\leq p^{-m_1}} \int_{p^{-m_2-1}<|t_2|_p\leq p^{-m_2}} \|f_1\chi_{k+n_1}\|_{L^{q_1}(\mathbb{Q}_p^n)} \|f_2\chi_{k+n_2}\|_{L^{q_2}(\mathbb{Q}_p^n)} \\
&\times \left(\sum_{j=0}^{m_1} \left| b_{1,p^{-j}B_k} - b_{1,p^{-j-1}B_k} \right| \right) \left(\sum_{j=0}^{m_2} \left| b_{2,p^{-j}B_k} - b_{2,p^{-j-1}B_k} \right| \right) \prod_{i=1}^{2} |t_i|_p^{-\frac{n}{q_i}} p^{\frac{kn}{p_i}} \varphi(t)dt \\
&\leq C \prod_{j=1}^{2} \|b_j\|_{CMO^{p_j}(\mathbb{Q}_p^n)} \int_{(\mathbb{Z}_p^*)^2} \|f_1\chi_{k+n_1}\|_{L^{q_1}(\mathbb{Q}_p^n)} \|f_2\chi_{k+n_2}\|_{L^{q_2}(\mathbb{Q}_p^n)} \\
&\times \prod_{i=1}^{2} |t_i|_p^{-\frac{n}{q_i}} p^{\frac{kn}{p_i}} \log_p \frac{p}{|t_i|_p} \varphi(t)dt.
\end{aligned}$$

Based on the fact,

$$\begin{aligned}
&\left| b_{1,B_k} - b_{1,t_1B_k} \right| \\
&\leq \sum_{j=0}^{m_1} \left| b_{1,p^{-j}B_k} - b_{1,p^{-j-1}B_k} \right| \\
&\leq C(m_1+1) \|b_1\|_{CMO^{p_1}(\mathbb{Q}_p^n)} \\
&\leq C \log_p \frac{p}{|t_1|_p} \|b_1\|_{CMO^{p_1}(\mathbb{Q}_p^n)},
\end{aligned}$$

and

$$\left| b_{2,B_k} - b_{2,t_2B_k} \right| \leq C \log_p \frac{p}{|t_2|_p} \|b_2\|_{CMO^{p_2}(\mathbb{Q}_p^n)}.$$

Similarly, we obtain

$$I_4=\int_{(\mathbb{Z}_p^*)^2}\Big(\int_{S_k}\big(|f_1(t_1x)f_2(t_2x)|\sum_{1\le i\ne j\le 2}|b_i(x)-b_{i,B_k}||b_{j,B_k}-b_{j,t_jB_k}|\big)^q dx\Big)^{\frac{1}{q}}\varphi(t)dt$$

$$\le\int_{(\mathbb{Z}_p^*)^2}\Big(\int_{S_k}\big(|f_1(t_1x)f_2(t_2x)||b_1(x)-b_{1,B_k}||b_{2,B_k}-b_{2,t_2B_k}|\big)^q dx\Big)^{\frac{1}{q}}\varphi(t)dt$$

$$+\int_{(\mathbb{Z}_p^*)^2}\Big(\int_{S_k}\big(|f_1(t_1x)f_2(t_2x)||b_2(x)-b_{2,B_k}||b_{1,B_k}-b_{1,t_1B_k}|\big)^q dx\Big)^{\frac{1}{q}}\varphi(t)dt$$

$$\le\prod_{j=1}^2\|b_j\|_{CMO^{p_j}(\mathbb{Q}_p^n)}\int_{(\mathbb{Z}_p^*)^2}\|f_1\chi_{k+n_1}\|_{L^{q_1}(\mathbb{Q}_p^n)}\|f_2\chi_{k+n_2}\|_{L^{q_2}(\mathbb{Q}_p^n)}$$

$$\times\prod_{i=1}^2|t_i|_p^{-\frac{n}{q_i}}p^{\frac{kn}{p_i}}\big(\log_p\frac{p}{|t_1|_p}+\log_p\frac{p}{|t_2|_p}\big)\varphi(t)dt.$$

$$I_5=\int_{(\mathbb{Z}_p^*)^2}\Big(\int_{S_k}\big(|f_1(t_1x)f_2(t_2x)|\sum_{1\le i\ne j\le 2}|b_i(x)-b_{i,B_k}||b_j(t_jx)-b_{j,t_jB_k}|\big)^q dx\Big)^{\frac{1}{q}}\varphi(t)dt$$

$$\le\int_{(\mathbb{Z}_p^*)^2}\Big(\int_{S_k}\big(|f_1(t_1x)f_2(t_2x)||b_1(x)-b_{1,B_k}||b_2(t_2x)-b_{2,t_2B_k}|\big)^q dx\Big)^{\frac{1}{q}}\varphi(t)dt$$

$$+\int_{(\mathbb{Z}_p^*)^2}\Big(\int_{S_k}\big(|f_1(t_1x)f_2(t_2x)||b_2(x)-b_{2,B_k}||b_1(t_1x)-b_{1,t_1B_k}|\big)^q dx\Big)^{\frac{1}{q}}\varphi(t)dt$$

$$\le C\prod_{j=1}^2\|b_j\|_{CMO^{p_j}(\mathbb{Q}_p^n)}\int_{(\mathbb{Z}_p^*)^2}\|f_1\chi_{k+n_1}\|_{L^{q_1}(\mathbb{Q}_p^n)}\|f_2\chi_{k+n_2}\|_{L^{q_2}(\mathbb{Q}_p^n)}\prod_{i=1}^2|t_i|_p^{-\frac{n}{q_i}}p^{\frac{kn}{p_i}}\varphi(t)dt.$$

$$I_6=\int_{(\mathbb{Z}_p^*)^2}\Big(\int_{S_k}\big(|f_1(t_1x)f_2(t_2x)|\sum_{1\le i\ne j\le 2}|b_{i,B_k}-b_{i,t_iB_k}||b_j(t_jx)-b_{j,t_jB_k}|\big)^q dx\Big)^{\frac{1}{q}}\varphi(t)dt$$

$$\le\int_{(\mathbb{Z}_p^*)^2}\Big(\int_{S_k}\big(|f_1(t_1x)f_2(t_2x)||b_{1,B_k}-b_{1,t_1B_k}||b_2(t_2x)-b_{2,t_2B_k}|\big)^q dx\Big)^{\frac{1}{q}}\varphi(t)dt$$

$$+\int_{(\mathbb{Z}_p^*)^2}\Big(\int_{S_k}\big(|f_1(t_1x)f_2(t_2x)||b_{2,B_k}-b_{2,t_2B_k}||b_1(t_1x)-b_{1,t_1B_k}|\big)^q dx\Big)^{\frac{1}{q}}\varphi(t)dt$$

$$\le\prod_{j=1}^2\|b_j\|_{CMO^{p_j}(\mathbb{Q}_p^n)}\int_{(\mathbb{Z}_p^*)^2}\|f_1\chi_{k+n_1}\|_{L^{q_1}(\mathbb{Q}_p^n)}\|f_2\chi_{k+n_2}\|_{L^{q_2}(\mathbb{Q}_p^n)}$$

$$\times\prod_{i=1}^2|t_i|_p^{-\frac{n}{q_i}}p^{\frac{kn}{p_i}}\big(\log_p\frac{p}{|t_1|_p}+\log_p\frac{p}{|t_2|_p}\big)\varphi(t)dt.$$

We know the facts

$$\|U_{\varphi,p}^{2,\vec{b}}(f_1,f_2)\|_{MK_{l,q}^{\alpha,\lambda}(\mathbb{Q}_p^n)}=\sup_{k_0\in\mathbb{Z}}p^{-k_0\lambda}\Big(\sum_{k=-\infty}^{k_0}p^{k\alpha l}\|U_{\varphi,p}^{2,\vec{b}}(f_1,f_2)\chi_k\|_{L^q(\mathbb{Q}_p^n)}^l\Big)^{\frac{1}{l}},$$

$$\|U_{\varphi,p}^{2,\vec{b}}(f_1,f_2)\chi_k\|_{L^q(\mathbb{Q}_p^n)}\le I_1+I_2+I_3+I_4+I_5+I_6,$$

$$\|U_{\varphi,p}^{2,\vec{b}}(f_1,f_2)\|_{MK_{l,q}^{\alpha,\lambda}(\mathbb{Q}_p^n)}\le\|I_1\|_{MK_{l,q}^{\alpha,\lambda}(\mathbb{Q}_p^n)}+\|I_2\|_{MK_{l,q}^{\alpha,\lambda}(\mathbb{Q}_p^n)}+\|I_3\|_{MK_{l,q}^{\alpha,\lambda}(\mathbb{Q}_p^n)}$$

$$+\|I_4\|_{MK^{\alpha,\lambda}_{l,q}(\mathbb{Q}_p^n)}+\|I_5\|_{MK^{\alpha,\lambda}_{l,q}(\mathbb{Q}_p^n)}+\|I_6\|_{MK^{\alpha,\lambda}_{l,q}(\mathbb{Q}_p^n)}.$$

We consider two cases about l, that is, $1\le l<\infty$ and $\lambda\ge 0$ or $0<l<1$ and $\lambda>0$.

Case 1, When $1\le l<\infty$ and $\lambda\ge 0$.

$$\begin{aligned}
&\|I_1\|_{MK^{\alpha,\lambda}_{l,q}(\mathbb{Q}_p^n)}\\
&\le \prod_{j=1}^{2}\|b_j\|_{CMO^{p_j}(\mathbb{Q}_p^n)}\sup_{k_0\in\mathbb{Z}}p^{-k_0\lambda}\Big[\sum_{k=-\infty}^{k_0}p^{k\alpha l}\Big(\int_{(\mathbb{Z}_p^*)^2}\|f_1\chi_{k+n_1}\|_{L^{q_1}(\mathbb{Q}_p^n)}\\
&\times\|f_2\chi_{k+n_2}\|_{L^{q_2}(\mathbb{Q}_p^n)}\prod_{i=1}^{2}|t_i|_p^{-\frac{n}{q_i}}p^{\frac{kn}{p_i}}\varphi(t)dt\Big)^l\Big]^{\frac{1}{l}}\\
&\le \prod_{j=1}^{2}\|b_j\|_{CMO^{p_j}(\mathbb{Q}_p^n)}\sup_{k_0\in\mathbb{Z}}p^{-k_0\lambda}\int_{(\mathbb{Z}_p^*)^2}\Big(\sum_{k=-\infty}^{k_0}p^{k\alpha l}\|f_1\chi_{k+n_1}\|^l_{L^{q_1}(\mathbb{Q}_p^n)}\\
&\times\|f_2\chi_{k+n_2}\|^l_{L^{q_2}(\mathbb{Q}_p^n)}p^{kn(\frac{1}{p_1}+\frac{1}{p_2})l}\Big)^{\frac{1}{l}}\prod_{i=1}^{2}|t_i|_p^{-\frac{n}{q_i}}\varphi(t)dt\\
&\le \prod_{j=1}^{2}\|b_j\|_{CMO^{p_j}(\mathbb{Q}_p^n)}\int_{(\mathbb{Z}_p^*)^2}\sup_{k_0\in\mathbb{Z}}p^{-k_0\lambda}\Big(\sum_{k=-\infty}^{k_0}\big(p^{k(\alpha_1+\frac{n}{p_1})}\|f_1\chi_{k+n_1}\|_{L^{q_1}(\mathbb{Q}_p^n)}\big)^l\\
&\times\big(p^{k(\alpha_2+\frac{n}{p_2})}\|f_2\chi_{k+n_2}\|_{L^{q_2}(\mathbb{Q}_p^n)}\big)^l\Big)^{\frac{1}{l}}\prod_{i=1}^{2}|t_i|_p^{-\frac{n}{q_i}}\varphi(t)dt\\
&\le \prod_{j=1}^{2}\|b_j\|_{CMO^{p_j}(\mathbb{Q}_p^n)}\int_{(\mathbb{Z}_p^*)^2}\sup_{k_0\in\mathbb{Z}}p^{-k_0\lambda_1}\Big(\sum_{k=-\infty}^{k_0}\Big(p^{k(\alpha_1+\frac{n}{p_1})l_1}\|f_1\chi_{k+n_1}\|^{l_1}_{L^{q_1}(\mathbb{Q}_p^n)}\Big)\Big)^{\frac{1}{l_1}}\\
&\times\sup_{k_0\in\mathbb{Z}}p^{-k_0\lambda_2}\Big(\sum_{k=-\infty}^{k_0}\Big(p^{k(\alpha_2+\frac{n}{p_2})l_2}\|f_2\chi_{k+n_2}\|^{l_2}_{L^{q_2}(\mathbb{Q}_p^n)}\Big)\Big)^{\frac{1}{l_2}}\prod_{i=1}^{2}|t_i|_p^{-\frac{n}{q_i}}\varphi(t)dt\\
&\le \prod_{j=1}^{2}\|b_j\|_{CMO^{p_j}(\mathbb{Q}_p^n)}\|f_j\|_{MK^{\alpha_j+\frac{n}{p_j},\lambda_j}_{l_j,q_j}(\mathbb{Q}_p^n)}\int_{(\mathbb{Z}_p^*)^2}\prod_{i=1}^{2}|t_i|_p^{\lambda_i-\alpha_i-\frac{n}{p_i}-\frac{n}{q_i}}\varphi(t)dt.
\end{aligned}$$

Similarly, we can get

$$\|I_2\|_{MK^{\alpha,\lambda}_{l,q}(\mathbb{Q}_p^n)}\le \prod_{j=1}^{2}\|b_j\|_{CMO^{p_j}(\mathbb{Q}_p^n)}\|f_j\|_{MK^{\alpha_j+\frac{n}{p_j},\lambda_j}_{l_j,q_j}(\mathbb{Q}_p^n)}\int_{(\mathbb{Z}_p^*)^2}\prod_{i=1}^{2}|t_i|_p^{\lambda_i-\alpha_i-\frac{n}{p_i}-\frac{n}{q_i}}\varphi(t)dt,$$

$$\begin{aligned}
\|I_3\|_{MK^{\alpha,\lambda}_{l,q}(\mathbb{Q}_p^n)}&\le \prod_{j=1}^{2}\|b_j\|_{CMO^{p_j}(\mathbb{Q}_p^n)}\|f_j\|_{MK^{\alpha_j+\frac{n}{p_j},\lambda_j}_{l_j,q_j}(\mathbb{Q}_p^n)}\int_{(\mathbb{Z}_p^*)^2}\prod_{i=1}^{2}|t_i|_p^{\lambda_i-\alpha_i-\frac{n}{p_i}-\frac{n}{q_i}}\\
&\quad\times\log_p\frac{p}{|t_i|_p}\varphi(t)dt,\\
\|I_4\|_{MK^{\alpha,\lambda}_{l,q}(\mathbb{Q}_p^n)}&\le \prod_{j=1}^{2}\|b_j\|_{CMO^{p_j}(\mathbb{Q}_p^n)}\|f_j\|_{MK^{\alpha_j+\frac{n}{p_j},\lambda_j}_{l_j,q_j}(\mathbb{Q}_p^n)}\int_{(\mathbb{Z}_p^*)^2}\prod_{i=1}^{2}|t_i|_p^{\lambda_i-\alpha_i-\frac{n}{p_i}-\frac{n}{q_i}}\\
&\quad\times\big(\log_p\frac{p}{|t_1|_p}+\log_p\frac{p}{|t_2|_p}\big)\varphi(t)dt,
\end{aligned}$$

$$\|I_5\|_{MK^{\alpha,\lambda}_{l,q}(\mathbb{Q}_p^n)}\le \prod_{j=1}^{2}\|b_j\|_{CMO^{p_j}(\mathbb{Q}_p^n)}\|f_j\|_{MK^{\alpha_j+\frac{n}{p_j},\lambda_j}_{l_j,q_j}(\mathbb{Q}_p^n)}\int_{(\mathbb{Z}_p^*)^2}\prod_{i=1}^{2}|t_i|_p^{\lambda_i-\alpha_i-\frac{n}{p_i}-\frac{n}{q_i}}\varphi(t)dt,$$

$$\|I_6\|_{MK^{\alpha,\lambda}_{l,q}(\mathbb{Q}_p^n)} \leq \prod_{j=1}^{2}\|b_j\|_{CMO^{p_j}(\mathbb{Q}_p^n)}\|f_j\|_{MK^{\alpha_j+\frac{n}{p_j},\lambda_j}_{l_j,q_j}(\mathbb{Q}_p^n)}\int_{(\mathbb{Z}_p^*)^2}\prod_{i=1}^{2}|t_i|_p^{\lambda_i-\alpha_i-\frac{n}{p_i}-\frac{n}{q_i}} \times\big(\log_p\tfrac{p}{|t_1|_p}+\log_p\tfrac{p}{|t_2|_p}\big)\varphi(t)dt.$$

Therefore, We completed proof of case 1.

Case 2, When $\lambda>0$ and $0<l<1$.

$$\begin{aligned}&\|f_1\chi_{k+n_1}\|_{L^{q_1}(\mathbb{Q}_p^n)}\\ &\leq p^{-(k+n_1)(\alpha_1+\frac{n}{p_1})}\left(\sum_{j=-\infty}^{k+n_1}p^{j(\alpha_1+\frac{n}{p_1})l_1}\|f_1\chi_j\|^{l_1}_{L^{q_1}(\mathbb{Q}_p^n)}\right)^{\frac{1}{l_1}}\\ &=p^{-k(\alpha_1+\frac{n}{p_1}-\lambda_1)}p^{-n_1(\alpha_1+\frac{n}{p_1}-\lambda_1)}\left(p^{-(k+n_1)\lambda_1}\sum_{j=-\infty}^{k+n_1}p^{j(\alpha_1+\frac{n}{p_1})l_1}\|f_1\chi_j\|^{l_1}_{L^{q_1}(\mathbb{Q}_p^n)}\right)^{\frac{1}{l_1}}\\ &=p^{-k(\alpha_1+\frac{n}{p_1}-\lambda_1)}p^{-n_1(\alpha_1+\frac{n}{p_1}-\lambda_1)}\|f_1\|_{MK^{\alpha_1+\frac{n}{p_1},\lambda_1}_{l_1,q_1}(\mathbb{Q}_p^n)}.\end{aligned}$$

$$\|f_2\chi_{k+n_2}\|_{L^{q_2}(\mathbb{Q}_p^n)}\leq p^{-k(\alpha_2+\frac{n}{p_2}-\lambda_2)}p^{-n_2(\alpha_2+\frac{n}{p_2}-\lambda_2)}\|f_2\|_{MK^{\alpha_2+\frac{n}{p_2},\lambda_2}_{l_2,q_2}(\mathbb{Q}_p^n)}.$$

$$\begin{aligned}I_1&\leq\prod_{j=1}^{2}\|b_j\|_{CMO^{p_j}(\mathbb{Q}_p^n)}\int_{(\mathbb{Z}_p^*)^2}\|f_1\chi_{k+n_1}\|_{L^{q_1}(\mathbb{Q}_p^n)}\|f_2\chi_{k+n_2}\|_{L^{q_2}(\mathbb{Q}_p^n)}\prod_{i=1}^{2}|t_i|_p^{-\frac{n}{q_i}}p^{\frac{kn}{p_i}}\varphi(t)dt\\ &\leq\prod_{j=1}^{2}\|b_j\|_{CMO^{p_j}(\mathbb{Q}_p^n)}\|f_j\|_{MK^{\alpha_j+\frac{n}{p_j},\lambda_j}_{l_j,q_j}(\mathbb{Q}_p^n)}\int_{(\mathbb{Z}_p^*)^2}\prod_{i=1}^{2}|t_i|_p^{\lambda_i-\alpha_i-\frac{n}{p_i}-\frac{n}{q_i}}p^{(\lambda-\alpha)k}\varphi(t)dt,\end{aligned}$$

$$\begin{aligned}&\|I_1\|_{MK^{\alpha,\lambda}_{l,q}(\mathbb{Q}_p^n)}\\ &\leq\sup_{k_0\in\mathbb{Z}}p^{-k_0\lambda}\left(\sum_{k=-\infty}^{k_0}p^{k\alpha l}p^{(\lambda-\alpha)kl}\right)^{\frac{1}{l}}\prod_{j=1}^{2}\|b_j\|_{CMO^{p_j}(\mathbb{Q}_p^n)}\|f_j\|_{MK^{\alpha_j+\frac{n}{p_j},\lambda_j}_{l_j,q_j}(\mathbb{Q}_p^n)}\\ &\times\int_{(\mathbb{Z}_p^*)^2}\prod_{i=1}^{2}|t_i|_p^{\lambda_i-\alpha_i-\frac{n}{p_i}-\frac{n}{q_i}}\varphi(t)dt\\ &=\left(\frac{1}{1-p^{-\lambda l}}\right)^{\frac{1}{l}}\prod_{j=1}^{2}\|b_j\|_{CMO^{p_j}(\mathbb{Q}_p^n)}\|f_j\|_{MK^{\alpha_j+\frac{n}{p_j},\lambda_j}_{l_j,q_j}(\mathbb{Q}_p^n)}\\ &\times\int_{(\mathbb{Z}_p^*)^2}\prod_{i=1}^{2}|t_i|_p^{\lambda_i-\alpha_i-\frac{n}{p_i}-\frac{n}{q_i}}\varphi(t)dt\\ &=C\prod_{j=1}^{2}\|b_j\|_{CMO^{p_j}(\mathbb{Q}_p^n)}\|f_j\|_{MK^{\alpha_j+\frac{n}{p_j},\lambda_j}_{l_j,q_j}(\mathbb{Q}_p^n)}\int_{(\mathbb{Z}_p^*)^2}\prod_{i=1}^{2}|t_i|_p^{\lambda_i-\alpha_i-\frac{n}{p_i}-\frac{n}{q_i}}\varphi(t)dt.\end{aligned}$$

Similarly, we can get norm estimates of I_2, I_3, I_4, I_5, I_6. So, we get the norm estimate of $U^{2,\vec{b}}_{\varphi,p}$.

By a similar argument, we also obtain the norm estimate of $V^{2,\vec{b}}_{\varphi,p}$.

4.4 Proof of Theorem 3.4

For any $x \in S_k$ and $b_i \in Lip^{\beta_i}$, it is easy to see that

$$\prod_{j=1}^{2}\left|b_j(x)-b_j(t_jx)\right| \le \prod_{j=1}^{2}\|b_j\|_{Lip^{\beta_j}}|1-t_j|_p^{\beta_j}p^{k(\beta_1+\beta_2)},$$

$$\begin{aligned}
&\|U^{2,\vec{b}}_{\varphi,p}(f_1,f_2)\chi_k\|_{L^q(\mathbb{Q}_p^n)}\\
&=\Big(\int_{S_k}\Big|\int_{(\mathbb{Z}_p^*)^2} f_1(t_1x)f_2(t_2x)\prod_{j=1}^{2}\big(b_j(x)-b_j(t_jx)\big)\varphi(t)dt\Big|^q dx\Big)^{\frac{1}{q}}\\
&\le \int_{(\mathbb{Z}_p^*)^2}\Big(\int_{S_k}\Big|f_1(t_1x)f_2(t_2x)\prod_{j=1}^{2}\big(b_j(x)-b_j(t_jx)\big)\Big|^q dx\Big)^{\frac{1}{q}}\varphi(t)dt\\
&\le \int_{(\mathbb{Z}_p^*)^2}\prod_{i=1}^{2}\Big(\int_{S_k}|f_i(t_ix)|^{q_i}dx\Big)^{\frac{1}{q_i}}\prod_{j=1}^{2}\Big(\int_{S_k}|b_j(x)-b_j(t_jx)|^{p_i}dx\Big)^{\frac{1}{p_i}}\varphi(t)dt\\
&\le \prod_{j=1}^{2}\|b_j\|_{Lip^{\beta_j}(\mathbb{Q}_p^n)}\int_{(\mathbb{Z}_p^*)^2}\|f_1\chi_{k+n_1}\|_{L^{q_1}(\mathbb{Q}_p^n)}\|f_2\chi_{k+n_2}\|_{L^{q_2}(\mathbb{Q}_p^n)}\\
&\times\prod_{i=1}^{2}|t_i|_p^{-\frac{n}{q_i}}p^{k(\beta_i+\frac{n}{p_i})}|1-t_i|_p^{\beta_i}\varphi(t)dt.
\end{aligned}$$

We consider two cases about l, that is, $1\le l<\infty$ and $\lambda\ge 0$ or $0<l<1$ and $\lambda>0$.

Case 1, When $1\le l<\infty$ and $\lambda\ge 0$, we use Minkowski's inequality and Hölder's inequality, we can get

$$\begin{aligned}
&\|U^{2,\vec{b}}_{\varphi,p}(f_1,f_2)\|_{MK^{\alpha,\lambda}_{l,q}(\mathbb{Q}_p^n)}\\
&\le \prod_{j=1}^{2}\|b_j\|_{Lip^{\beta_j}(\mathbb{Q}_p^n)}\sup_{k_0\in\mathbb{Z}}p^{-k_0\lambda}\Big[\sum_{k=-\infty}^{k_0}p^{k\alpha l}\Big(\int_{(\mathbb{Z}_p^*)^2}\|f_1\chi_{k+n_1}\|_{L^{q_1}(\mathbb{Q}_p^n)}\|f_2\chi_{k+n_2}\|_{L^{q_2}(\mathbb{Q}_p^n)}\\
&\times\prod_{i=1}^{2}|t_i|_p^{-\frac{n}{q_i}}p^{k(\beta_i+\frac{n}{p_i})}|1-t_i|_p^{\beta_i}\varphi(t)dt\Big)^l\Big]^{\frac{1}{l}}\\
&\le \prod_{j=1}^{2}\|b_j\|_{Lip^{\beta_j}(\mathbb{Q}_p^n)}\int_{(\mathbb{Z}_p^*)^2}\sup_{k_0\in\mathbb{Z}}p^{-k_0\lambda}\Big(\sum_{k=-\infty}^{k_0}p^{k\alpha l}\|f_1\chi_{k+n_1}\|^l_{L^{q_1}(\mathbb{Q}_p^n)}\|f_2\chi_{k+n_2}\|^l_{L^{q_2}(\mathbb{Q}_p^n)}\\
&\times p^{k(\beta_1+\frac{n}{p_1}+\beta_2+\frac{n}{p_2})l}\Big)^{\frac{1}{l}}\prod_{i=1}^{2}|t_i|_p^{-\frac{n}{q_i}}|1-t_i|_p^{\beta_i}\varphi(t)dt\\
&\le \prod_{j=1}^{2}\|b_j\|_{Lip^{\beta_j}(\mathbb{Q}_p^n)}\int_{(\mathbb{Z}_p^*)^2}\sup_{k_0\in\mathbb{Z}}p^{-k_0\lambda_1}\Big(\sum_{k=-\infty}^{k_0}p^{k(\alpha_1+\beta_1+\frac{n}{p_1})l_1}\|f_1\chi_{k+n_1}\|^{l_1}_{L^{q_1}(\mathbb{Q}_p^n)}\Big)^{\frac{1}{l_1}}\\
&\times\sup_{k_0\in\mathbb{Z}}p^{-k_0\lambda_2}\Big(\sum_{k=-\infty}^{k_0}p^{k(\alpha_2+\beta_2+\frac{n}{p_2})l_2}\|f_2\chi_{k+n_2}\|^{l_2}_{L^{q_2}(\mathbb{Q}_p^n)}\Big)^{\frac{1}{l_2}}\prod_{i=1}^{2}|t_i|_p^{-\frac{n}{q_i}}|1-t_i|_p^{\beta_i}\varphi(t)dt\\
&=\prod_{j=1}^{2}\|b_j\|_{Lip^{\beta_j}(\mathbb{Q}_p^n)}\|f_j\|_{MK^{\alpha_j+\beta_j+\frac{n}{p_j},\lambda_j}_{l_j,q_j}}\int_{(\mathbb{Z}_p^*)^2}\prod_{i=1}^{2}|t_i|_p^{\lambda_i-\alpha_i-\beta_i-\frac{n}{p_i}-\frac{n}{q_i}}|1-t_i|_p^{\beta_i}\varphi(t)dt.
\end{aligned}$$

Case 2, When $\lambda > 0$ and $0 < l < 1$.

$$\|f_1\chi_{k+n_1}\|_{L^{q_1}(\mathbb{Q}_p^n)}$$
$$\le p^{-(k+n_1)(\alpha_1+\beta_1+\frac{n}{p_1})}\Big(\sum_{j=-\infty}^{k+n_1} p^{j(\alpha_1+\beta_1+\frac{n}{p_1})l_1}\|f_1\chi_j\|^{l_1}_{L^{q_1}(\mathbb{Q}_p^n)}\Big)^{\frac{1}{l_1}}$$
$$= p^{-k(\alpha_1+\frac{n}{p_1}+\beta_1-\lambda_1)}p^{-n_1(\alpha_1+\frac{n}{p_1}+\beta_1-\lambda_1)}\Big(p^{-(k+n_1)\lambda_1}\sum_{j=-\infty}^{k+n_1} p^{j(\alpha_1+\beta_1+\frac{n}{p_1})l_1}\|f_1\chi_j\|^{l_1}_{L^{q_1}(\mathbb{Q}_p^n)}\Big)^{\frac{1}{l_1}}$$
$$= p^{-k(\alpha_1+\frac{n}{p_1}+\beta_1-\lambda_1)}p^{-n_1(\alpha_1+\frac{n}{p_1}+\beta_1-\lambda_1)}\|f_1\|_{MK^{\alpha_1+\beta_1+\frac{n}{p_1},\lambda_1}_{l_1,q_1}(\mathbb{Q}_p^n)},$$

$$\|f_2\chi_{k+n_2}\|_{L^{q_2}(\mathbb{Q}_p^n)} \le p^{-k(\alpha_2+\beta_2+\frac{n}{p_2}-\lambda_2)}p^{-n_2(\alpha_2+\beta_2+\frac{n}{p_2}-\lambda_2)}\|f_2\|_{MK^{\alpha_2+\beta_2+\frac{n}{p_2},\lambda_2}_{l_2,q_2}(\mathbb{Q}_p^n)},$$

$$\|U^{2,\vec{b}}_{\varphi,p}(f_1,f_2)\|_{MK^{\alpha,\lambda}_{l,q}(\mathbb{Q}_p^n)}$$
$$\le \sup_{k_0\in\mathbb{Z}} p^{-k_0\lambda}\Big(\sum_{k=-\infty}^{k_0} p^{k\alpha l}p^{(\lambda-\alpha)kl}\Big)^{\frac{1}{l}}\prod_{j=1}^{2}\|b_j\|_{Lip^{\beta_j}(\mathbb{Q}_p^n)}\|f_j\|_{MK^{\alpha_j+\beta_j+\frac{n}{p_j},\lambda_j}_{l_j,q_j}(\mathbb{Q}_p^n)}$$
$$\times\int_{(\mathbb{Z}_p^*)^2}\prod_{i=1}^{2}|t_i|_p^{\lambda_i-\alpha_i-\beta_i-\frac{n}{p_i}-\frac{n}{q_i}}|1-t_i|_p^{\beta_i}\varphi(t)dt$$
$$= \Big(\frac{1}{1-p^{-\lambda l}}\Big)^{\frac{1}{l}}\prod_{j=1}^{2}\|b_j\|_{Lip^{\beta_j}(\mathbb{Q}_p^n)}\|f_j\|_{MK^{\alpha_j+\beta_j+\frac{n}{p_j},\lambda_j}_{l_j,q_j}(\mathbb{Q}_p^n)}$$
$$\times\int_{(\mathbb{Z}_p^*)^2}\prod_{i=1}^{2}|t_i|_p^{\lambda_i-\alpha_i-\beta_i-\frac{n}{p_i}-\frac{n}{q_i}}|1-t_i|_p^{\beta_i}\varphi(t)dt$$
$$= C\prod_{j=1}^{2}\|b_j\|_{Lip^{\beta_j}(\mathbb{Q}_p^n)}\|f_j\|_{MK^{\alpha_j+\beta_j+\frac{n}{p_j},\lambda_j}_{l_j,q_j}(\mathbb{Q}_p^n)}\int_{(\mathbb{Z}_p^*)^2}\prod_{i=1}^{2}|t_i|_p^{\lambda_i-\alpha_i-\beta_i-\frac{n}{p_i}-\frac{n}{q_i}}|1-t_i|_p^{\beta_i}\varphi(t)dt.$$

By a similar argument, we also obtain the norm estimate of $V^{2,\vec{b}}_{\varphi,p}$.
This fnishes the proof of Theorem 3.4.

Declarations

Conflict of interest The authors declare no conflict of interest.

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