Appl. Math. J. Chinese Univ.
2024, 39(3): 545-553

A Cauchy integral formula for (p, q)-monogenic functions with α-weight

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Abstract. Firstly, some properties for (p, q)-monogenic functions with α-weight in Clifford analysis are given. Then, the Cauchy-Pompeiu formula is proved. Finally, the Cauchy integral formula and the Cauchy integral theorem for (p, q)-monogenic functions with α-weight are given.

§1 Introduction

Clifford algebra [3] was proposed by Clifford in 1878. In 2002, Malonek and Ren [10] introduced Dirac operators with α-weight. In 2017, a Cauchy integral formula for inframonogenic functions in Clifford analysis was obtained [6]. In 2018, Kang and Wang [9] studied k-monogenic functions over 6-dimensional Euclidean space, Yang et al. [12] gave the Cauchy integral formula for k-monogenic functions with α-weight. In 2020, Dinh $[4-5]$ discussed the representation of Weinstein k-monogenic functions and the generalized (k_i)-monogenic functions. García et al. [7] studied the decomposition of inframonogenic functions with applications in Elasticity theory. Blaya et al. [1] derived the Cauchy integral theorem for infrapolymonogenic functions. In 2021, Santiesteban et al. [11] discussed (φ, ψ)-inframonogenic functions in Clifford analysis. On the basis of the above works, the Cauchy-Pompeiu formula, the Cauchy integral formula and the Cauchy integral theorem for (p, q)-monogenic functions with α-weight are obtained.

§2 Preliminaries

Let $Cl_{0,n}(\mathbf{R})$ be the real Clifford algebra generated by $\{e_0, e_1, \ldots, e_n\}$, $e_0 = e_\phi = 1$ is its identity and $e_ie_j + e_je_i = -2\delta_{ij}(i, j = 1, \ldots, n)$, where δ_{ij} is the Kronecker delta. The

Received: 2021-08-11. Revised: 2021-10-01.
MR Subject Classification: 30E20, 30E25, 45E05.
Keywords: (p, q)-monogenic functions with α-weight, Cauchy-Pompeiu formula, Cauchy integral formula, Cauchy integral theorem.
Digital Object Identifier(DOI): https://doi.org/10.1007/s11766-024-4530-9.
Supported by the National Natural Science Foundation of China(11871191), the Science Foundation of Hebei Province(A2023205006, A2019106037), the Key Development Foundation of Hebei Normal University in 2024(L2024ZD08), the Graduate Student Innovation Project Fund of Hebei Province(CXZZBS2022066), and the Key Foundation of Hebei Normal University(L2018Z01).
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element $a \in Cl_{0,n}(\mathbf{R})$ has the form $a = \sum\limits_{A} a_A e_A$, where $A = \phi$ or $A = \{\alpha_1, \alpha_2, \ldots, \alpha_h\} \in \{1, 2, \ldots, n\}(1 \leq \alpha_1 < \alpha_2 < \cdots < \alpha_h \leq n)$, $a_A \in \mathbf{R}$ and $e_A = e_{\alpha_1} e_{\alpha_2} \cdots e_{\alpha_h}$. The elements $x = \sum\limits_{i=1}^{n} x_i e_i (x_i \in \mathbf{R}, i = 1, 2, \ldots, n)$ are called vectors. The set $\mathbf{R}^n$ is identified with the set of vectors. Then $|x| = -x^2$ and $x^{-1} = -\dfrac{x}{|x|^2}$, where $x \in \mathbf{R}^n \backslash \{\mathbf{0}\}$.

In this paper, let $\Omega \subset \mathbf{R}^n \backslash \{\mathbf{0}\}$ be a nonempty connected open set and its boundary $\partial\Omega$ be a differentiable, oriented, compact Liapunov surface [8]. $\Omega^* = \{x | x + x_0 \in \Omega, x_0 \in \overline{\Omega}\}$. The function $f : \Omega \to Cl_{0,n}(\mathbf{R})$ is denoted by $f = \sum\limits_{A} f_A e_A$, where f_A is a real-valued function. A function f is continuous in Ω means that each component of f is continuous in Ω.

Let $C^r(\Omega, Cl_{0,n}(\mathbf{R})) = \{f | f : \Omega \to Cl_{0,n}(\mathbf{R}), f = \sum\limits_{A} f_A e_A$, where f_A is r-times continuously differentiable in Ω, $r \in N^*$, N^* is the set of positive integers$\}$.

For $f \in C^1(\Omega, Cl_{0,n}(\mathbf{R}))$, we introduce Dirac operators with α-weight as follows [5]:

$$D^\alpha f(x) = |x|^{-\alpha} x (Df(x)), \;\; f(x) D^\alpha = (f(x) D) x |x|^{-\alpha},$$

where $Df(x) = \sum\limits_{j=1}^{n} e_j \dfrac{\partial f(x)}{\partial x_j}$, $f(x)D = \sum\limits_{j=1}^{n} \dfrac{\partial f(x)}{\partial x_j} e_j, \alpha \in \mathbf{R} \backslash \{0\}$.

Definition 2.1. [2] If $f \in C^1(\Omega, Cl_{0,n}(\mathbf{R}))$ satisfies $Df(x) = 0 (f(x)D = 0)$ in Ω, we say that f is a left(right) monogenic function in Ω.

Definition 2.2. [12] If $f \in C^k(\Omega, Cl_{0,n}(\mathbf{R}))$ satisfies $(D^\alpha)^k f(x) = 0 (f(x)(D^\alpha)^k = 0)$ in Ω, where $k \in N^*$, we say that f is a left(right) k-monogenic function with α-weight in Ω.

Lemma 2.1. [8] If $f, g \in C^1(\Omega, Cl_{0,n}(\mathbf{R}))$, then

$$D(f(x)g(x)) = (Df(x))g(x) + \sum_{j=1}^{n} e_j f(x) \frac{\partial g(x)}{\partial x_j},$$

$$(f(x)g(x))D = \sum_{j=1}^{n} \frac{\partial f(x)}{\partial x_j} g(x) e_j + f(x)(g(x)D).$$

Lemma 2.2. [8] Let Γ be an arbitrary n-chain satisfying $\overline{\Gamma} \subset \Omega$, $f, g \in C^r(\Omega, Cl_{0,n}(\mathbf{R}))$, $r \geq 1$, then we have

$$\int_{\partial\Gamma} f(x) d\sigma_x g(x) = \int_{\Gamma} [(f(x)D)g(x) + f(x)(Dg(x))] dx^n,$$

where $d\widehat{x_i} = dx_1 \wedge \cdots \wedge dx_{i-1} \wedge dx_{i+1} \wedge \cdots \wedge dx_n, i = 1, 2, \ldots, n$, $d\sigma_x = \sum\limits_{i=1}^{n} (-1)^{i-1} e_i d\widehat{x_i}$, $dx^n = dx_1 \wedge \cdots \wedge dx_n$.

Let $H_k(x) = \dfrac{A_k}{|x|^{n-k\alpha}}$, where $A_k = \dfrac{(-1)^{k-1}}{\omega_n \alpha^{k-1}(k-1)!}$, $k \geq 1$, $k \in N^*$, $\alpha \in \mathbf{R} \backslash \{0\}$, ω_n is the area of the unit sphere in $\mathbf{R}^n$.

Lemma 2.3. [12] When $k > 1$, we have

$$D(H_k(x)|x|^{-\alpha} x) = (H_k(x)|x|^{-\alpha} x) D = H_{k-1}(x).$$

By reference [12] or similar to reference [12], we have the following three propositions.

Proposition 2.1. If $f \in C^k(\Omega, Cl_{0,n}(\mathbf{R}))$ is a left and right monogenic function in Ω, then

$$(D^\alpha)^k (|x|^{k\alpha} f(x)) = (|x|^{k\alpha} f(x))(D^\alpha)^k = (-1)^k k! \alpha^k f(x).$$

Proposition 2.2. If $f \in C^k(\Omega, Cl_{0,n}(\mathbf{R}))$ is a left(right) monogenic function in Ω, then $|x|^{(k-j)\alpha} f$ is a left(right) k-monogenic function with α-weight in Ω, where $j \le k,\ k, j \in N^*$.

Proposition 2.3. $H_k(x)|x|^{-j\alpha}x$ is a left(right) k-monogenic function with α-weight in Ω, where $j \le k,\ k, j \in N^*$.

Lemma 2.4. [12] (Cauchy-Pompeiu formula) If $f \in C^r(\overline{\Omega}, Cl_{0,n}(\mathbf{R})),\ r \ge k,\ n \ge k, 0 < \alpha < \dfrac{1}{k}$, then for any $x_0 \in \Omega$, we have

$$f(x_0) = \sum_{j=1}^{k}(-1)^j \int_{\partial\Omega^*} H_j(x)|x|^{-\alpha} x d\sigma_x((D^\alpha)^{j-1} f(x+x_0)) - (-1)^k \int_{\Omega^*} H_k(x)((D^\alpha)^k f(x+x_0))dx.$$

Lemma 2.5. [12] (Cauchy theorem) Let $f \in C^r(\overline{\Omega}, Cl_{0,n}(\mathbf{R})),\ r \ge k,\ n \ge k,\ 0 < \alpha < \dfrac{1}{k}$. If $f(x+x_0)$ is a left k-monogenic function with α-weight in Ω^*, then

$$\sum_{j=1}^{k}(-1)^j \int_{\partial\Omega^*} H_j(x)|x|^{-\alpha} x d\sigma_x((D^\alpha)^{j-1} f(x+x_0)) = \begin{cases} f(x_0), & x_0 \in \Omega; \\ 0, & x_0 \in \mathbf{R}^n \backslash \overline{\Omega}. \end{cases}$$

Similar to the proofs of Lemma 2.4 and Lemma 2.5, we have the following theorems.

Theorem 2.4. (Cauchy-Pompeiu formula) Let $f \in C^r(\overline{\Omega}, Cl_{0,n}(\mathbf{R})),\ r \ge k,\ n \ge k, 0 < \alpha < \dfrac{1}{k}$. Then for any $x_0 \in \Omega$, we have

$$f(x_0) = \sum_{j=1}^{k}(-1)^j \int_{\partial\Omega^*} (f(x+x_0)(D^\alpha)^{j-1}) d\sigma_x(H_j(x)|x|^{-\alpha}x) - (-1)^k \int_{\Omega^*} H_k(x)(f(x+x_0)(D^\alpha)^k)dx.$$

Theorem 2.5. (Cauchy theorem) Let $f \in C^r(\overline{\Omega}, Cl_{0,n}(\mathbf{R})),\ r \ge k,\ n \ge k,\ 0 < \alpha < \dfrac{1}{k}$. If $f(x+x_0)$ is a right k-monogenic function with α-weight in Ω^*, then

$$\sum_{j=1}^{k}(-1)^j \int_{\partial\Omega^*} (f(x+x_0)(D^\alpha)^{j-1}) d\sigma_x(H_j(x)|x|^{-\alpha}x) = \begin{cases} f(x_0), & x_0 \in \Omega; \\ 0, & x_0 \in \mathbf{R}^n \backslash \overline{\Omega}. \end{cases}$$

§3 Main results

Definition 3.1. If $f \in C^{p+q}(\Omega, Cl_{0,n}(\mathbf{R}))$ satisfies $((D^\alpha)^p f(x))(D^\alpha)^q = 0$ in Ω, we say that f is a (p,q)-monogenic function with α-weight in Ω, where $p, q \in N^*$. Especially when $p = q$, we say that f is an infrapolymonogenic function with α-weight in Ω.

Obviously, if $f \in C^{p+q}(\Omega, Cl_{0,n}(\mathbf{R}))$ is a left p-monogenic function with α-weight in Ω, then

f is a (p,q)-monogenic function with α-weight in Ω.

Proposition 3.1. If f is a left and right monogenic function in Ω, then $|x|^{(p+q-j)\alpha}f$ is a (p,q)-monogenic function with α-weight in Ω, where $j \leq (p+q)$, $j \in N^*$.

proof. *(i)* When $q < j \leq (p+q)$, that is, $1 \leq (j-q) \leq p$, $|x|^{(p-(j-q))\alpha}f$ is a p-monogenic function with α-weight by Proposition 2.2, so $|x|^{(p+q-j)\alpha}f$ is a (p,q)-monogenic function with α-weight in Ω.

(ii) When $1 \leq j \leq q$, as f is a left monogenic function and $x^2 = -|x|^2$,

$$\begin{aligned}
&D^\alpha(|x|^{(p+q-j)\alpha}f(x))\\
=&|x|^{-\alpha}x\{(D|x|^{(p+q-j)\alpha})f(x) + |x|^{(p+q-j)\alpha}(Df(x))\}\\
=&|x|^{-\alpha}x(p+q-j)\alpha|x|^{(p+q-j)\alpha-2}xf(x)\\
=&-(p+q-j)\alpha|x|^{(p+q-j-1)\alpha}f(x),
\end{aligned}$$

thus

$$\begin{aligned}
&(D^\alpha)^2(|x|^{(p+q-j)\alpha}f(x))\\
=&-(p+q-j)\alpha|x|^{-\alpha}x(D(|x|^{(p+q-j-1)\alpha}f(x)))\\
=&-(p+q-j)\alpha|x|^{-\alpha}x(p+q-j-1)\alpha|x|^{(p+q-j-1)\alpha-2}xf(x)\\
=&(-1)^2\frac{(p+q-j)!}{(p+q-j-2)!}\alpha^2|x|^{(p+q-j-2)\alpha}f(x).
\end{aligned}$$

Suppose $(D^\alpha)^{p-1}(|x|^{(p+q-j)\alpha}f(x)) = (-1)^{p-1}\frac{(p+q-j)!}{(q-j+1)!}\alpha^{p-1}|x|^{(q-j+1)\alpha}f(x)$, then

$$\begin{aligned}
&(D^\alpha)^p(|x|^{(p+q-j)\alpha}f(x))\\
=&D^\alpha((-1)^{p-1}\frac{(p+q-j)!}{(q-j+1)!}\alpha^{p-1}(|x|^{(q-j+1)\alpha}f(x)))\\
=&(-1)^{p-1}\frac{(p+q-j)!}{(q-j+1)!}\alpha^{p-1}|x|^{-\alpha}x(D(|x|^{(q-j+1)\alpha}f(x)))\\
=&(-1)^{p-1}\frac{(p+q-j)!}{(q-j+1)!}\alpha^{p-1}|x|^{-\alpha}x(q-j+1)\alpha|x|^{(q-j+1)\alpha-2}xf(x)\\
=&(-1)^p\frac{(p+q-j)!}{(q-j)!}\alpha^p(|x|^{(q-j)\alpha}f(x)).
\end{aligned}$$

So

$$(D^\alpha)^p(|x|^{(p+q-j)\alpha})f(x) = (-1)^p\frac{(p+q-j)!}{(q-j)!}\alpha^p(|x|^{(q-j)\alpha}f(x)).$$

As f is a right monogenic function, $|x|^{(q-j)\alpha}f$ is a right q-monogenic function with α-weight by Proposition 2.2, then

$$\begin{aligned}
&((D^\alpha)^p(|x|^{(p+q-j)\alpha}f(x)))(D^\alpha)^q\\
=&(-1)^p\frac{(p+q-j)!}{(q-j)!}\alpha^p((|x|^{(q-j)\alpha}f(x))(D^\alpha)^q) = 0,
\end{aligned}$$

that is, $|x|^{(p+q-j)\alpha}f$ is a (p,q)-monogenic function with α-weight in Ω.

Corollary 3.2. $H_{p+q}(x)|x|^{-j\alpha}x$ is a (p,q)-monogenic function with α-weight in Ω, where $j \leq (p+q)$, $j \in N^*$.

proof. As

$$H_{p+q}(x)|x|^{-j\alpha}x=\frac{A_{p+q}}{|x|^{n-(p+q)\alpha}}|x|^{-j\alpha}x=|x|^{(p+q-j)\alpha}A_{p+q}\frac{x}{|x|^n},$$

$\frac{x}{\omega_n|x|^n}$ is the Cauchy kernel, then $D(\frac{x}{|x|^n})=(\frac{x}{|x|^n})D=0$. By Proposition 3.1 we draw the conclusion.

Theorem 3.3. Let $f\in C^r(\overline{\Omega},Cl_{0,n}(\mathbf{R})),r\geq(p+q)$. Then for any $x_0\in\overline{\Omega}$, $((D^\alpha)^pf(x+x_0))(D^\alpha)^q$ is bounded in $\overline{\Omega^*}$ when $0<\alpha<\frac{1}{p+q}$.

proof. Let

$$\begin{aligned}
f_1(x)=&Df(x+x_0);\\
f_2(x)=&\alpha f_1(x)+D(xf_1(x));\\
f_3(x)=&2\alpha f_2(x)+D(xf_2(x));\\
&\cdots;\\
f_p(x)=&(p-1)\alpha f_{p-1}(x)+D(xf_{p-1}(x));\\
f_{p+1}(x)=&-p\alpha|x|^{-2}xf_p(x)x+(xf_p(x))D;\\
f_{p+2}(x)=&(p+1)\alpha f_{p+1}(x)+(f_{p+1}(x)x)D;\\
f_{p+3}(x)=&(p+2)\alpha f_{p+2}(x)+(f_{p+2}(x)x)D;\\
&\cdots;\\
f_{p+q}(x)=&(p+q-1)\alpha f_{p+q-1}(x)+(f_{p+q-1}(x)x)D.
\end{aligned}$$

Then $f_1(x),f_2(x),\ldots$, and $f_{p+q}(x)$ are bounded in $\overline{\Omega^*}$.
Following [12], we have $(D^\alpha)^pf(x+x_0)=|x|^{-p\alpha}xf_p(x)$, then

$$\begin{aligned}
&((D^\alpha)^pf(x+x_0))D^\alpha\\
=&((|x|^{-p\alpha}xf_p(x))D)x|x|^{-\alpha}\\
=&\{(-p\alpha)|x|^{-p\alpha-2}xf_p(x)x+|x|^{-p\alpha}((xf_p(x))D)\}x|x|^{-\alpha}\\
=&\{-p\alpha|x|^{-2}xf_p(x)x+(xf_p(x))D\}x|x|^{-(p+1)\alpha}\\
=&f_{p+1}(x)x|x|^{-(p+1)\alpha},
\end{aligned}$$

thus

$$\begin{aligned}
&((D^\alpha)^pf(x+x_0))(D^\alpha)^2\\
=&((f_{p+1}(x)x|x|^{-(p+1)\alpha})D)x|x|^{-\alpha}\\
=&\{f_{p+1}(x)x(-(p+1)\alpha)|x|^{-(p+1)\alpha-2}x+((f_{p+1}(x)x)D)|x|^{-(p+1)\alpha}\}x|x|^{-\alpha}\\
=&\{(p+1)\alpha f_{p+1}(x)|x|^{-(p+1)\alpha}+((f_{p+1}(x)x)D)|x|^{-(p+1)\alpha}\}x|x|^{-\alpha}\\
=&\{(p+1)\alpha f_{p+1}(x)+(f_{p+1}(x)x)D\}x|x|^{-(p+2)\alpha}\\
=&f_{p+2}(x)x|x|^{-(p+2)\alpha}.
\end{aligned}$$

Suppose $((D^\alpha)^pf(x+x_0))(D^\alpha)^{q-1}=f_{p+q-1}(x)x|x|^{-(p+q-1)\alpha}$, then

$$\begin{aligned}
&((D^\alpha)^pf(x+x_0))(D^\alpha)^q\\
=&((f_{p+q-1}(x)x|x|^{-(p+q-1)\alpha})D)x|x|^{-\alpha}
\end{aligned}$$

$$
\begin{aligned}
&=\{f_{p+q-1}(x)x(-(p+q-1)\alpha)|x|^{-(p+q-1)\alpha-2}x\\
&\quad+((f_{p+q-1}(x)x)D)|x|^{-(p+q-1)\alpha}\}x|x|^{-\alpha}\\
&=\{(p+q-1)\alpha f_{p+q-1}(x)+(f_{p+q-1}(x)x)D\}x|x|^{-(p+q)\alpha}\\
&=f_{p+q}(x)x|x|^{-(p+q)\alpha}.
\end{aligned}
$$

So

$$((D^\alpha)^p f(x+x_0))(D^\alpha)^q = f_{p+q}(x)x|x|^{-(p+q)\alpha}.$$

Hence, when $(p+q)\alpha < 1$, that is, $0<\alpha<\dfrac{1}{p+q}$, $((D^\alpha)^p f(x+x_0))(D^\alpha)^q$ is bounded in $\overline{\Omega^*}$ for any $x_0 \in \Omega^*$.

Theorem 3.4. (Cauchy-Pompeiu formula) If $f \in C^r(\overline{\Omega}, Cl_{0,n}(\mathbf{R}))$, where $r \geq (p+q)$, $n \geq (p+q)$, $0<\alpha<\dfrac{1}{p+q}$, then for any $x_0 \in \Omega$, we have

$$
\begin{aligned}
f(x_0)=&\sum_{j=1}^{q}(-1)^{p+j}\int_{\partial\Omega^*}((D^\alpha)^p f(x+x_0))(D^\alpha)^{j-1}d\sigma_x(x|x|^{-\alpha}H_{p+j}(x))\\
&+\sum_{j=1}^{p}(-1)^{j}\int_{\partial\Omega^*}H_j(x)|x|^{-\alpha}x d\sigma_x((D^\alpha)^{j-1}f(x+x_0))\\
&-(-1)^{p+q}\int_{\Omega^*}H_{p+q}(x)\{((D^\alpha)^p f(x+x_0))(D^\alpha)^q\}dx.
\end{aligned}
$$

proof. *(i)* Using Lemma 2.4, in order to prove Theorem 3.4, we need to show that

$$
\begin{aligned}
&\sum_{j=1}^{q}(-1)^{p+j}\int_{\partial\Omega^*}((D^\alpha)^p f(x+x_0))(D^\alpha)^{j-1}d\sigma_x(x|x|^{-\alpha}H_{p+j}(x))\\
&-(-1)^{p+q}\int_{\Omega^*}H_{p+q}(x)\{((D^\alpha)^p f(x+x_0))(D^\alpha)^q\}dx\\
&=-(-1)^{p}\int_{\Omega^*}H_p(x)((D^\alpha)^p f(x+x_0))dx.
\end{aligned}
$$

(ii) For any $x_0 \in \Omega$, $0+x_0 \in \Omega$. Hence, $0 \in \Omega^*$. Let $\delta > 0$, $B_\delta = \{x : |x| < \delta\}$, $\overline{B_\delta} \subset \Omega^*$. By Lemma 2.2 and Lemma 2.3, when $j = 1, 2, \ldots, q$, we have

$$
\begin{aligned}
&\int_{\partial\Omega^*}((D^\alpha)^p f(x+x_0))(D^\alpha)^{j-1}d\sigma_x(x|x|^{-\alpha}H_{p+j}(x))\\
&-\lim_{\delta\to 0}\int_{\partial B_\delta}((D^\alpha)^p f(x+x_0))(D^\alpha)^{j-1}d\sigma_x(x|x|^{-\alpha}H_{p+j}(x))\\
=&\lim_{\delta\to 0}\int_{\Omega^*/B_\delta}\{(((D^\alpha)^p f(x+x_0))(D^\alpha)^j)H_{p+q}(x)+(((D^\alpha)^p f(x+x_0))(D^\alpha)^{j-1})H_{p+q-1}(x)\}dx\\
=&\int_{\Omega^*}\{(((D^\alpha)^p f(x+x_0))(D^\alpha)^j)H_{p+q}(x)+(((D^\alpha)^p f(x+x_0))(D^\alpha)^{j-1})H_{p+q-1}(x)\}dx.
\end{aligned}
$$

By Theorem 3.3, $|((D^\alpha)^p f(x+x_0))(D^\alpha)^j|$ is bounded in Ω^*, then $|((D^\alpha)^p f(x+x_0))(D^\alpha)^j| < M$, where $j = 1, 2, \ldots, q$, M is a positive constant. So

$$
\begin{aligned}
&\left|\int_{\partial B_\delta}((D^\alpha)^p f(x+x_0))(D^\alpha)^{j-1}d\sigma_x(x|x|^{-\alpha}H_{p+j}(x))\right|\\
\leq& M_1\int_{\partial B_\delta}|H_{p+j}(x)||x|^{1-\alpha}|d\sigma_x| = \frac{M_2}{\alpha^{p+j-1}(p+j-1)!}\delta^{(p+j-1)\alpha},
\end{aligned}
$$

where M_1 and M_2 are positive constants. So

$$\lim_{\delta\to 0}\int_{\partial B_\delta}((D^\alpha)^p f(x+x_0))(D^\alpha)^{j-1}d\sigma_x(x|x|^{-\alpha}H_{p+j}(x))=0.$$

Hence

$$\begin{aligned}
&\sum_{j=1}^{q}(-1)^{p+j}\int_{\partial\Omega^*}((D^\alpha)^p f(x+x_0))(D^\alpha)^{j-1}d\sigma_x(x|x|^{-\alpha}H_{p+j}(x))\\
=&(-1)^{p+q}\int_{\Omega^*}\{(((D^\alpha)^p f(x+x_0))(D^\alpha)^q)H_{p+q}(x)+(((D^\alpha)^p f(x+x_0))(D^\alpha)^{q-1})H_{p+q-1}(x)\}dx\\
&+(-1)^{p+q-1}\int_{\Omega^*}\{(((D^\alpha)^p f(x+x_0))(D^\alpha)^{q-1})H_{p+q-1}(x)\\
&+(((D^\alpha)^p f(x+x_0))(D^\alpha)^{q-2})H_{p+q-2}(x)\}dx\\
&+\cdots\\
&+(-1)^{p+1}\int_{\Omega^*}\{(((D^\alpha)^p f(x+x_0))D^\alpha)H_{p+q-1}(x)+((D^\alpha)^p f(x+x_0))H_p(x)\}dx\\
=&(-1)^{p+q}\int_{\Omega^*}H_{p+q}(x)\{((D^\alpha)^p f(x+x_0))(D^\alpha)^q\}dx-(-1)^p\int_{\Omega^*}H_p(x)\{((D^\alpha)^p f(x+x_0))\}dx.
\end{aligned}$$

Theorem 3.5. (Cauchy integral formula) Suppose $f\in C^r(\overline{\Omega},Cl_{0,n}(\mathbf{R}))$, where $r\geq(p+q)$, $n\geq(p+q)$, $0<\alpha<\dfrac{1}{p+q}$, if $f(x+x_0)$ is a (p,q)-monogenic function with α-weight in Ω^*, then for any $x_0\in\Omega$, we have

$$\begin{aligned}
f(x_0)=&\sum_{j=1}^{q}(-1)^{p+j}\int_{\partial\Omega^*}((D^\alpha)^p f(x+x_0))(D^\alpha)^{j-1}d\sigma_x(x|x|^{-\alpha}H_{p+j}(x))\\
&+\sum_{j=1}^{p}(-1)^{j}\int_{\partial\Omega^*}H_j(x)|x|^{-\alpha}xd\sigma_x((D^\alpha)^{j-1}f(x+x_0)).
\end{aligned}$$

proof. As $f(x+x_0)$ is a (p,q)-monogenic function with α-weight in Ω^*, $((D^\alpha)^p f(x+x_0))(D^\alpha)^q=0$, by Theorem 3.4 we draw the conclusion.

Theorem 3.6. (Cauchy integral theorem) Suppose $f\in C^r(\overline{\Omega},Cl_{0,n}(\mathbf{R}))$, where $r\geq(p+q)$, $n\geq(p+q)$, $0<\alpha<\dfrac{1}{p+q}$, if $f(x+x_0)$ is a (p,q)-monogenic function with α-weight in Ω^*, then for any $x_0\in\mathbf{R}^n\backslash\overline{\Omega}$, we have

$$\begin{aligned}
&\sum_{j=1}^{q}(-1)^{p+j}\int_{\partial\Omega^*}((D^\alpha)^p f(x+x_0))(D^\alpha)^{j-1}d\sigma_x(x|x|^{-\alpha}H_{p+j}(x))\\
&+\sum_{j=1}^{p}(-1)^{j}\int_{\partial\Omega^*}H_j(x)|x|^{-\alpha}xd\sigma_x((D^\alpha)^{j-1}f(x+x_0))=0.
\end{aligned}$$

proof. When $x_0\in\mathbf{R}^n/\overline{\Omega}$, by Lemma 2.2, Lemma 2.3, Lemma 2.5 and $((D^\alpha)^p f(x+x_0))(D^\alpha)^q=0$ in Ω^*, we have

$$\begin{aligned}
0=&(-1)^{p+q}\int_{\Omega^*}H_{p+q}(x)\{((D^\alpha)^p f(x+x_0))(D^\alpha)^q\}dx\\
=&(-1)^{p+q}\int_{\Omega^*}\{(((D^\alpha)^p f(x+x_0))(D^\alpha)^{q-1})D\}x|x|^{-\alpha}H_{p+q}(x)dx
\end{aligned}$$

$$
\begin{aligned}
=&(-1)^{p+q}\int_{\partial\Omega^*}((D^\alpha)^p f(x+x_0))(D^\alpha)^{q-1}d\sigma_x(x|x|^{-\alpha}H_{p+q}(x))\\
&-(-1)^{p+q}\int_{\Omega^*}\{((D^\alpha)^p f(x+x_0))(D^\alpha)^{q-1}\}\{D(x|x|^{-\alpha}H_{p+q}(x))\}dx\\
=&(-1)^{p+q}\int_{\partial\Omega^*}((D^\alpha)^p f(x+x_0))(D^\alpha)^{q-1}d\sigma_x(x|x|^{-\alpha}H_{p+q}(x))\\
&+(-1)^{p+q-1}\int_{\Omega^*}\{((D^\alpha)^p f(x+x_0))(D^\alpha)^{q-1}\}H_{p+q-1}(x)dx\\
=&(-1)^{p+q}\int_{\partial\Omega^*}((D^\alpha)^p f(x+x_0))(D^\alpha)^{q-1}d\sigma_x(x|x|^{-\alpha}H_{p+q}(x))\\
&+(-1)^{p+q-1}\int_{\partial\Omega^*}((D^\alpha)^p f(x+x_0))(D^\alpha)^{q-2}d\sigma_x(x|x|^{-\alpha}H_{p+q-1}(x))\\
&+(-1)^{p+q-2}\int_{\Omega^*}\{((D^\alpha)^p f(x+x_0))(D^\alpha)^{q-2}\}H_{p+q-2}(x)dx\\
=&\cdots\\
=&\sum_{j=1}^{q}(-1)^{p+j}\int_{\partial\Omega^*}((D^\alpha)^p f(x+x_0))(D^\alpha)^{j-1}d\sigma_x(x|x|^{-\alpha}H_{p+j}(x))\\
&+(-1)^{p}\int_{\Omega^*}((D^\alpha)^p f(x+x_0))H_{p}(x)dx\\
=&\sum_{j=1}^{q}(-1)^{p+j}\int_{\partial\Omega^*}((D^\alpha)^p f(x+x_0))(D^\alpha)^{j-1}d\sigma_x(x|x|^{-\alpha}H_{p+j}(x))\\
&+\sum_{j=1}^{p}(-1)^{j}\int_{\partial\Omega^*}H_j(x)|x|^{-\alpha}x d\sigma_x((D^\alpha)^{j-1}f(x+x_0)).
\end{aligned}
$$

Corollary 3.7. (Cauchy integral formula) Suppose $f\in C^r(\overline{\Omega},Cl_{0,n}(\mathbf{R}))$, where $r\geq 2k$, $n\geq 2k$, $0<\alpha<\frac{1}{2k}$, if $f(x+x_0)$ is an infrapolymonogenic function with α-weight in Ω^*, then for any $x_0\in\Omega$, we have

$$
\begin{aligned}
f(x_0)=&\sum_{j=1}^{k}(-1)^{k+j}\int_{\partial\Omega^*}((D^\alpha)^k f(x+x_0))(D^\alpha)^{j-1}d\sigma_x(x|x|^{-\alpha}H_{k+j}(x))\\
&+\sum_{j=1}^{k}(-1)^{j}\int_{\partial\Omega^*}H_j(x)|x|^{-\alpha}x d\sigma_x((D^\alpha)^{j-1}f(x+x_0)).
\end{aligned}
$$

Corollary 3.8. (Cauchy integral theorem) Suppose $f\in C^r(\overline{\Omega},Cl_{0,n}(\mathbf{R}))$, where $r\geq 2k$, $n\geq 2k$, $0<\alpha<\frac{1}{2k}$, if $f(x+x_0)$ is an infrapolymonogenic function with α-weight in Ω^*, then then for any $x_0\in\mathbf{R}^n\backslash\overline{\Omega}$, we have

$$
\begin{aligned}
&\sum_{j=1}^{k}(-1)^{k+j}\int_{\partial\Omega^*}((D^\alpha)^k f(x+x_0))(D^\alpha)^{j-1}d\sigma_x(x|x|^{-\alpha}H_{k+j}(x))\\
&+\sum_{j=1}^{k}(-1)^{j}\int_{\partial\Omega^*}H_j(x)|x|^{-\alpha}x d\sigma_x((D^\alpha)^{j-1}f(x+x_0))=0.
\end{aligned}
$$

Acknowledgements

Thanks to the reviewers for their valuable comments on my manuscript.

Declarations

Conflict of interest The authors declare no conflict of interest.

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