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On power series statistical convergence and new uniform integrability of double sequences

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Abstract. In the present paper, we mostly focus on P_p^2-statistical convergence. We will look into the uniform integrability via the power series method and its characterizations for double sequences. Also, the notions of P_p^2-statistically Cauchy sequence, P_p^2-statistical boundedness and core for double sequences will be described in addition to these findings.

§1 Introduction

In studying the fundamental theory of functional analysis and in particular sequence space, a convergence of sequences has an important place because it constructs many valuable and significant results, identities, and theorems. Although the answer may have become more apparent over time, many problems remain complex. It is therefore important to study new types of convergence and also, their properties. The concept of statistical convergence of sequences of real numbers was given in [16] and [24], independently. Extensive research has been conducted on statistical convergence by numerous authors from different angles, leading to the discovery of different types of convergence [1, 2, 5, 8, 10, 12–15, 19]. One of the notions of the statistical type of convergence, named, power series statistical convergence has recently been given in [27]. The use of this convergence had a great impulse, for example, the properties of J_p-statistical convergence were given in [25] and the concept of strong J_p-convergence via a modulus function was given in [4], Şahin Bayram gave some criteria in [23]. Also, Demirci et al. [9] and Cabrera et al. [6] introduced some concepts of variation. Ünver and Bayram established a relationship between the concepts of P-statistical convergence and P-uniform integrability [26]. With the help of this convergence, Demirci et al. [11] gave a new type of power series statistical convergence, P-statistical relative uniform convergence of sequences of functions at a point. Then, they established an approximation theorem. More recently, this notion of convergence was extended to double sequences in [28]. Motivated by the above research, we mostly focus on P_p^2-statistical convergence. We will look into the uniform integrability via the power series method and its characterizations for double sequences. Also, the notions of P_p^2-statistically Cauchy sequence, P_p^2-statistical boundedness and core for double sequences will be described in addition to these findings.

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§2 Definitions and Preliminaries

The symbol $\mathbb{N}$ is commonly used to represent a set of natural numbers. A sequence $x=\{x_{m,v}\}$ is called Pringsheim convergent if, for every $\varepsilon>0$, there exists $M=M(\varepsilon)\in\mathbb{N}$ such that $|x_{m,v}-L|<\varepsilon$ whenever $m,v>M$. Here, L is called the Pringsheim limit of x and we denote this convergence by writing $P-\lim_{m,v}x_{m,v}=L$ (see [21]). A double sequence is said to be bounded if there exists a positive number K such that $|x_{m,v}|\leq K$ for all $(m,v)\in\mathbb{N}^2=\mathbb{N}\times\mathbb{N}$. Let l_∞^2, c_2 denote the sets of all real bounded double sequences and P-convergent double sequences, respectively. If $x\in c_2$ then, either $x\in l_\infty^2$ or $x\notin l_\infty^2$.

The following is a summability of the double sequence generating an infinite matrix:

Let $A=[a_{k,l,m,v}]$, $k,l,m,v\in\mathbb{N}$, be a four-dimensional infinite matrix. The A-transform of x for a given sequence $x=\{x_{m,v}\}$, denoted by $Ax:=((Ax)_{k,l})$, is given by

$$(Ax)_{k,l}=\sum_{(m,v)\in\mathbb{N}^2}a_{k,l,m,v}x_{m,v},\quad k,l\in\mathbb{N},$$

provided the double series converges in Pringsheim's sense for every $(k,l)\in\mathbb{N}^2$. According to the definition, a double sequence x is considered $A-$summable to L if the A-transform of x exists for all values of k and l in the set of natural numbers, and converges in Pringsheim's sense.

In our study, we shall need the following.

Now, let c_2^∞ stand for the set of all real bounded and P-convergent double sequences. Recall that we say a $A=[a_{k,l,m,v}]$ is *RH−regular* if $Ax\in c_2^\infty$ and $P-\lim_{k,l}(Ax)_{k,l}=P-\lim_{m,v}x_{m,v}$ for $x\in c_2^\infty$. The characterization of $RH-$regularity known as Robison-Hamilton conditions (see also [17, 22]) and a four-dimensional matrix $A=[a_{k,l,m,v}]$ is *RH−regular* $(A\in(c_2^\infty,c_2^\infty;P))$ if and only if

(i) $P-\lim_{k,l}a_{k,l,m,v}=0$ for each m and v,

(ii) $P-\lim_{k,l}\sum\limits_{m,v=1}^{\infty}a_{k,l,m,v}=1$,

(iii) $P-\lim_{k,l}\sum\limits_{m=1}^{\infty}|a_{k,l,m,v}|=0$ for each $v\in\mathbb{N}$,

(iv) $P-\lim_{k,l}\sum\limits_{v=1}^{\infty}|a_{k,l,m,v}|=0$ for each $m\in\mathbb{N}$,

(v) $\sum\limits_{m,v=1}^{\infty}|a_{k,l,m,v}|$ is $P-$convergent,

(vi) There are finite positive integers M_1 and M_2, such that for every $(k,l)\in\mathbb{N}^2$, the following inequality holds:

$$\sum_{m,v>M_1}|a_{k,l,m,v}|<M_2.$$

Prior to moving forward, it would be helpful to revisit the principles of natural density and statistical convergence for single and double sequences.

Let K be a subset of $\mathbb{N}_0$ and the symbol $\#\{.\}$ represents the number of elements in a set, which is called its cardinality. The natural density of K, denoted by $\delta(K)$ is given by

$$\delta(K):=\lim_v\frac{1}{v+1}\#\{k\leq v:k\in K\}$$

whenever the limit exists. A sequence $x=\{x_v\}$ is considered statistically convergent if for any value of ε greater than zero,

$$\delta(\{v\in\mathbb{N}_0:|x_v-L|\geq\varepsilon\})=0.$$

In this case, we write $st - \lim_v x_v = L$.

If $E \subset \mathbb{N}_0^2 = \mathbb{N}_0 \times \mathbb{N}_0$, then $E_{j,k} := \{(m,v) \in E : m \leq j, v \leq k\}$. The double natural density of E, denoted by $\delta_2(E)$ is given by

$$\delta_2(E) := P - \lim_{j,k} \frac{1}{(j+1)(k+1)} \# E_{j,k},$$

whenever the limit exists ([18]). Let $x = \{x_{m,v}\}$ be a number sequence. Then $x = \{x_{m,v}\}$ is statistically convergent to L if for every $\varepsilon > 0$, the set

$$E := E_{j,k}(\varepsilon) := \{m \leq j, v \leq k : \ |x_{m,v} - L| \geq \varepsilon\}$$

has natural density zero; in that case we write $st_2 - \lim_{m,v} x_{m,v} = L$ ([18]).

In what follows, a non-negative real double sequence $\{p_{m,v}\}$ will be a given such that $p_{00} > 0$ and the corresponding power series

$$p(t,u) := \sum_{m,v=0}^{\infty} p_{m,v} t^m u^v$$

has a radius of convergence R with $R \in (0, \infty]$ and $t, u \in (0, R)$. It is said that $x = \{x_{m,v}\}$ is convergent in the sense of power series method and this is denoted by $P_p^2 - \lim x_{m,v} = L$ if for all $t, u \in (0, R)$, the limit

$$\lim_{t,u \to R^-} \frac{1}{p(t,u)} \sum_{m,v=0}^{\infty} p_{m,v} t^m u^v x_{m,v} = L$$

exists ([3]). Note that the method is regular iff

$$\lim_{t,u \to R^-} \frac{\sum_{m=0}^{\infty} p_{m,\kappa} t^m}{p(t,u)} = 0 \text{ and } \lim_{t,u \to R^-} \frac{\sum_{v=0}^{\infty} p_{\mu,v} u^v}{p(t,u)} = 0, \text{ for any } \mu, \kappa, \tag{1}$$

hold (see [3]).

Definition 1. *Let $x = \{x_{m,v}\}$ be a double sequence. Then the double sequence x is said to be P_p^2-strongly convergent to L if*

$$\lim_{t,u \to R^-} \frac{1}{p(t,u)} \sum_{m,v=0}^{\infty} p_{m,v} t^m u^v |x_{m,v} - L| = 0.$$

First of all, let us emphasize that from this point onward, we work under the assumption that the power series method is regular.

The notions of statistical convergence of sequence with respect to the power series method, P_p-statistical convergence, and P_p-density of $E \subset \mathbb{N}_0$ were introduced by Ünver and Orhan [27]. The key point here is that, these convergence methods are incompatible. Motivated by this work, the definitions of P_p^2-density of $F \subset \mathbb{N}_0^2 = \mathbb{N}_0 \times \mathbb{N}_0$ and P_p^2-statistical convergence have been more recently introduced by Yıldız, Demirci and Dirik [28] as follows:

Definition 2. *[28] Let $F \subset \mathbb{N}_0^2$. If the limit*

$$\delta_{P_p}^2(F) := \lim_{t,u \to R^-} \frac{1}{p(t,u)} \sum_{(m,v) \in F} p_{m,v} t^m u^v$$

exists, then $\delta_{P_p}^2(F)$ is said to be the P_p^2-density of F.

Notice that, it is not difficult to see from the definition of a power series method and P_p^2–density that $0 \leq \delta_{P_p}^2(F) \leq 1$ if it exists.

Definition 3. *[28] Let $x = \{x_{m,v}\}$ be a double sequence. Then $\{x_{m,v}\}$ is called statistically convergent with respect to the power series method (P_p^2-statistically convergent) to L if for any $\varepsilon > 0$*

$$\lim_{t,u \to R^-} \frac{1}{p(t,u)} \sum_{(m,v) \in F_\varepsilon} p_{m,v} t^m u^v = 0 \tag{2}$$

where $F_\varepsilon = \left\{(m,v) \in \mathbb{N}_0^2 : |x_{m,v} - L| \geq \varepsilon\right\}$, that is $\delta_{P_p}^2(F_\varepsilon) = 0$ for any $\varepsilon > 0$. This is denoted $st_{P_p}^2 - \lim x_{m,v} = L$.

Example 1. *Let $\{p_{m,v}\}$ be defined as follows*

$$p_{m,v} = \begin{cases} 0, & m \text{ and } v \text{ odd} \\ 1, & m \text{ or } v \text{ even} \end{cases},$$

and take the sequence $\{s_{m,v}\}$ defined by

$$s_{m,v} = \begin{cases} mv, & m \text{ and } v \text{ odd} \\ 0, & m \text{ or } v \text{ even} \end{cases}. \tag{3}$$

We get that, since for any $\varepsilon > 0$,

$$\lim_{t,u \to R^-} \frac{1}{p(t,u)} \sum_{(m,v):|u_{m,v}| \geq \varepsilon} p_{m,v} t^m u^v = 0,$$

$\{s_{m,v}\}$ is P_p^2-statistically convergent to 0. However, the sequence $\{s_{m,v}\}$ is neither Pringsheim convergent nor statistically convergent to 0.

In view of the above definitions, we obtain the following propositions:

Proposition 1. *The $st_{P_p}^2$-limit of a double sequence $x = \{x_{m,v}\}$ is unique.*

Proposition 2. *Let $x = \{x_{m,v}\}$ and $y = \{y_{m,v}\}$ be two double sequences. If $st_{P_p}^2 - \lim x_{m,v} = L_1$ and $st_{P_p}^2 - \lim y_{m,v} = L_2$, then the following statements hold:*

(i) $st_{P_p}^2 - \lim \{x_{m,v} + y_{m,v}\} = L_1 + L_2$,

(ii) $st_{P_p}^2 - \lim c x_{m,v} = cL_1$ $(c \in \mathbb{R})$.

Definition 4. *[28]The P_p^2-statistical superior limit of $x = \{x_{m,v}\}$ is*

$$st_{P_p}^2 - \limsup x_{m,v} = \begin{cases} \sup G_x, & \text{if } G_x \neq \varnothing, \\ -\infty, & \text{if } G_x = \varnothing, \end{cases}$$

where $G_x := \left\{c \in \mathbb{R} : \delta_{P_p}^2(\{(m,v) : x_{m,v} > c\}) > 0 \text{ or does not exist in } \mathbb{R}\right\}$ and $\varnothing$ denotes the empty set. Similarly, the P_p^2-statistical inferior limit of x is

$$st_{P_p}^2 - \liminf x_{m,v} = \begin{cases} \inf H_x, & \text{if } H_x \neq \varnothing, \\ \infty, & \text{if } H_x = \varnothing, \end{cases}$$

where $H_x := \left\{d \in \mathbb{R} : \delta_{P_p}^2(\{(m,v) : x_{m,v} < d\}) > 0 \text{ or does not exist in } \mathbb{R}\right\}$.

By exploiting the relation between the above definitions, the following theorem can be given as a consequence:

Theorem 1. *(i) $st_{P_p}^2 - \limsup x_{m,v} = s$ if and only if for any $\varepsilon > 0$, (a) $\delta_{P_p}^2\left(\left\{(m,v) : x_{m,v} > s - \varepsilon\right\}\right) \neq 0$; (b) $\delta_{P_p}^2\left(\left\{(m,v) : x_{m,v} > s + \varepsilon\right\}\right) = 0$.*

(ii) $st_{P_p}^2 - \liminf x_{m,v} = t$ if and only if for any $\varepsilon > 0$, (a) $\delta_{P_p}^2(\{(m,v) : x_{m,v} < t + \varepsilon\}) \neq 0$; (b) $\delta_{P_p}^2(\{(m,v) : x_{m,v} < t - \varepsilon\}) = 0$.

Remark 1. *It is easy to see that*

(i) For any real double sequence $x = \{x_{m,v}\}$, $st^2_{P_p} - \liminf x_{m,v} \leq st^2_{P_p} - \limsup x_{m,v}$ *and*

(ii) $P - \liminf x_{m,v} \leq st^2_{P_p} - \liminf x_{m,v} \leq st^2_{P_p} - \limsup x_{m,v} \leq P - \limsup x_{m,v}$ *(see also [9, 28]).*

§3 Main Results

In this section and its subsections, we characterize the properties of the above notions and establish some if and only if results. In the continuation, we give our new definitions then we characterize the properties of these notions. Then, we will look into the uniform integrability via the power series method and its characterizations. Also, the notions of P_p^2-statistically Cauchy sequence, P_p^2-statistical boundedness, and the core for double sequences will be described in addition to these findings.

Theorem 2. *Let* $x = \{x_{m,v}\}$ *be a double sequence. Then, the following expressions are equivalent:*

(i) $st^2_{P_p} - \lim x_{m,v} = L$.

(ii) There exists a subset F *of* $\mathbb{N}_0^2$ *such that* $\delta^2_{P_p}(F) = 1$ *and* $P - \lim_{\substack{m,v \\ (m,v)\in F}} x_{m,v} = L$.

(iii) There exist two double sequences $y = \{y_{m,v}\}$ *and* $z = \{z_{m,v}\}$ *such that* $x = y + z$ *and* $P - \lim_{m,v} y_{m,v} = L$ *and* $st^2_{P_p} - \lim z_{m,v} = 0$.

Proof. $(i) \Longrightarrow (ii)$: Let $st^2_{P_p} - \lim x_{m,v} = L$. Define the following sets:

$$F_k^1 \;:\; = \left\{(m,v) : |x_{m,v} - L| \geq \frac{1}{k}\right\},$$
$$F_k^2 \;:\; = \left\{(m,v) : |x_{m,v} - L| < \frac{1}{k}\right\},$$

$k = 1, 2, \ldots.$ Observe that, $\delta^2_{P_p}\left(F_k^1\right) = 0$ and $\delta^2_{P_p}\left(F_k^2\right) = 1$, $k = 1, 2, \ldots.$ Also, $F_1^2 \supset F_2^2 \supset \ldots.$ If we show that $P - \lim_{m,v} x_{m,v} = L$ for every $(m,v) \in F_k^2$ then we get the desired result. Now, suppose that x is not convergent to L. Hence, there is $\varepsilon > 0$ such that, $|x_{m,v} - L| \geq \varepsilon$ for infinitely many terms. Put

$$F_\varepsilon := \{(m,v) : |x_{m,v} - L| < \varepsilon\} \text{ and } \varepsilon > \frac{1}{k},$$

$k = 1, 2, \ldots.$ Then $\delta^2_{P_p}(F_\varepsilon) = 0$ and $F_k^2 \subset F_\varepsilon$. Therefore $\delta^2_{P_p}\left(F_k^2\right) = 0$ which is a contradiction. Hence, x is convergent to L.

$(ii) \Longrightarrow (iii)$: There exists a subset F of $\mathbb{N}_0^2$ such that $\delta^2_{P_p}(F) = 1$ and $P - \lim_{\substack{m,v \\ (m,v)\in F}} x_{m,v} = L$. Now, let define two double sequences $y = \{y_{m,v}\}$ and $z = \{z_{m,v}\}$ as follows:

$$y_{m,v} = \begin{cases} x_{m,v}, & \text{if } (m,v) \in F \\ L, & \text{otherwise} \end{cases} \text{ and } z_{m,v} = \begin{cases} 0, & \text{if } (m,v) \in F, \\ x_{m,v} - L, & \text{otherwise.} \end{cases}$$

Then it is easy to see that $P - \lim_{m,v} y_{m,v} = L$ and $st^2_{P_p} - \lim z_{m,v} = 0$ and $x = y + z$.

$(iii) \Longrightarrow (i)$: There exist two double sequences $y = \{y_{m,v}\}$ and $z = \{z_{m,v}\}$ such that $x = y + z$ and $P - \lim_{m,v} y_{m,v} = L$ and $st^2_{P_p} - \lim z_{m,v} = 0$. For any $\varepsilon > 0$, let

$$S_1 = \left\{(m,v) : |y_{m,v} - L| \geq \frac{\varepsilon}{2}\right\} \text{ and}$$
$$S_2 = \left\{(m,v) : |z_{m,v} - 0| \geq \frac{\varepsilon}{2}\right\}.$$

Then, clearly $\delta^2_{P_p}(S_1) = 0$ and $\delta^2_{P_p}(S_2) = 0$. Hence, we have

$$\begin{aligned} & \delta^2_{P_p}\left(\{(m,v) : |x_{m,v} - L| \geq \varepsilon\}\right) \\ \leq \quad & \delta^2_{P_p}(S_1) + \delta^2_{P_p}(S_2) = 0, \end{aligned}$$

whence the result. □

Remark 2. *(i) If $st^2_{P_p} - \lim x_{m,v} = L$, then there exists a sequence $\{y_{m,v}\}$ such that $P - \lim_{m,v} y_{m,v} = L$ and $\delta^2_{P_p}(\{(m,v) : x_{m,v} = y_{m,v}\}) = 1$, i.e., $x_{m,v} = y_{m,v}$ for almost all m, v.*

(ii) If $st^2_{P_p} - \lim x_{m,v} = L$, then there is a subsequence $\{x_{m_k,v_l}\}$ of $\{x_{m,v}\}$ such that $P - \lim_{k,l} x_{m_k,v_l} = L$.

Theorem 3. *(i) If a sequence $x = \{x_{m,v}\}$ is P^2_p-strongly convergent to L, then it is P^2_p-statistically convergent to L.*

(ii) If $x = \{x_{m,v}\}$ is P^2_p-statistically convergent to L and bounded, then it is P^2_p-strongly convergent to L.

Proof. (i) Let a double sequence $x = \{x_{m,v}\}$ be P^2_p-strongly convergent to L. Put $\varepsilon > 0$. Then

$$\begin{aligned} \frac{1}{p(t,u)} \sum_{m,v=0}^{\infty} p_{m,v} t^m u^v |x_{m,v} - L| &= \frac{1}{p(t,u)} \sum_{\substack{m,v=0 \\ (m,v)\in\mathbb{N}_0^2 : |x_{m,v}-L|\geq\varepsilon}}^{\infty} p_{m,v} t^m u^v |x_{m,v} - L| \\ &\quad + \frac{1}{p(t,u)} \sum_{\substack{m,v=0 \\ (m,v)\in\mathbb{N}_0^2 : |x_{m,v}-L|<\varepsilon}}^{\infty} p_{m,v} t^m u^v |x_{m,v} - L| \\ &\geq \varepsilon \frac{1}{p(t,u)} \sum_{(m,v)\in\mathbb{N}_0^2 : |x_{m,v}-L|\geq\varepsilon} p_{m,v} t^m u^v. \end{aligned}$$

This implies that $\delta^2_{P_p}\left(\left\{(m,v) \in \mathbb{N}_0^2 : |x_{m,v} - L| \geq \varepsilon\right\}\right) = 0$, so that $\{x_{m,v}\}$ is P^2_p-statistically convergent to L.

(ii) Let $x = \{x_{m,v}\}$ be bounded and P^2_p-statistically convergent to L. Put $\varepsilon > 0$. Then, $\delta^2_{P_p}\left(\left\{(m,v) \in \mathbb{N}_0^2 : |x_{m,v} - L| \geq \varepsilon\right\}\right) = 0$ and there exists a positive number B such that $|x_{m,v}| \leq B$ for all $(m,v) \in \mathbb{N}_0^2$. We have

$$\begin{aligned} \frac{1}{p(t,u)} \sum_{m,v=0}^{\infty} p_{m,v} t^m u^v |x_{m,v} - L| &= \frac{1}{p(t,u)} \sum_{\substack{m,v=0 \\ (m,v)\in\mathbb{N}_0^2 : |x_{m,v}-L|\geq\varepsilon}}^{\infty} p_{m,v} t^m u^v |x_{m,v} - L| \\ &\quad + \frac{1}{p(t,u)} \sum_{\substack{m,v=0 \\ (m,v)\in\mathbb{N}_0^2 : |x_{m,v}-L|<\varepsilon}}^{\infty} p_{m,v} t^m u^v |x_{m,v} - L| \\ &\leq N \frac{1}{p(t,u)} \sum_{(m,v)\in\mathbb{N}_0^2 : |x_{m,v}-L|\geq\varepsilon} p_{m,v} t^m u^v + \varepsilon \end{aligned}$$

where $N = B + |L|$. This gives

$$\lim_{t,u\to R^-} \frac{1}{p(t,u)} \sum_{m,v=0}^{\infty} p_{m,v} t^m u^v |x_{m,v} - L| = 0.$$

□

3.1 P_p^2-Uniform Integrability

In this subsection, we introduce the P_p^2-uniform integrability of a double sequence and get multidimensional analogs of the findings of Ünver and Bayram [26] presented in 2022 (See also, [27]).

Definition 5. *A double sequence $x = \{x_{m,v}\}$ is P_p^2-uniformly integrable if for any $\varepsilon > 0$ there exist $t_0 \in (0, R)$ and $A > 0$ such that*

$$\sup_{t_0 \le t,u < R} \frac{1}{p(t,u)} \sum_{(m,v):|x_{m,v}| \ge a} |x_{m,v}| \, p_{m,v} t^m u^v < \varepsilon,$$

whenever $a \ge A$.

We now begin the characterization of P_p^2-uniform integrability.

Theorem 4. *Let $x = \{x_{m,v}\}$ be a double sequence. Then the following are equivalent:*
(i) x is P_p^2-uniformly integrable,

(ii) there exists $t_0 \in (0, R)$ such that: $(*)$ $\sup_{t_0 \le t,u < R} \frac{1}{p(t,u)} \sum_{m,v=0}^{\infty} |x_{m,v}| \, p_{m,v} t^m u^v < \infty$, $(**)$ for any $\varepsilon > 0$ there exists $\delta > 0$ such that for any subset F with

$$\sup_{t_0 \le t,u < R} \frac{1}{p(t,u)} \sum_{(m,v) \in F} p_{m,v} t^m u^v < \delta, \tag{4}$$

we have

$$\sup_{t_0 \le t,u < R} \frac{1}{p(t,u)} \sum_{(m,v) \in F} |x_{m,v}| \, p_{m,v} t^m u^v < \varepsilon. \tag{5}$$

Proof. $(i) \implies (ii)$: Let $\varepsilon > 0$. Then, from the hypothesis, there exists $t_0 \in (0, R)$ and $A > 0$ such that

$$\sup_{t_0 \le t,u < R} \frac{1}{p(t,u)} \sum_{(m,v):|x_{m,v}| \ge a} |x_{m,v}| \, p_{m,v} t^m u^v < \frac{\varepsilon}{2},$$

whenever $a \ge A$. Hence, we get

$$\begin{aligned} & \sup_{t_0 \le t,u < R} \frac{1}{p(t,u)} \sum_{m,v=0}^{\infty} |x_{m,v}| \, p_{m,v} t^m u^v \\ \le & \sup_{t_0 \le t,u < R} \frac{1}{p(t,u)} \sum_{(m,v):|x_{m,v}| < A} |x_{m,v}| \, p_{m,v} t^m u^v \\ + & \sup_{t_0 \le t,u < R} \frac{1}{p(t,u)} \sum_{(m,v):|x_{m,v}| \ge A} |x_{m,v}| \, p_{m,v} t^m u^v \\ < & A \sup_{t_0 \le t,u < R} \frac{1}{p(t,u)} \sum_{m,v=0}^{\infty} p_{m,v} t^m u^v + \frac{\varepsilon}{2} = A + \frac{\varepsilon}{2}, \end{aligned}$$

which proves $(*)$. Now, take $\delta = \frac{\varepsilon}{2A}$ and given F with (4), we have

$$\begin{aligned} & \sup_{t_0 \le t,u < R} \frac{1}{p(t,u)} \sum_{(m,v) \in F} |x_{m,v}| \, p_{m,v} t^m u^v \\ \le & \sup_{t_0 \le t,u < R} \frac{1}{p(t,u)} \sum_{(m,v) \in F:|x_{m,v}| \ge A} |x_{m,v}| \, p_{m,v} t^m u^v \end{aligned}$$

$$
\begin{aligned}
&+ \sup_{t_0 \le t,u<R} \frac{1}{p(t,u)} \sum_{(m,v)\in F:|x_{m,v}|<A} |x_{m,v}| \, p_{m,v} t^m u^v \\
&\le \sup_{t_0 \le t,u<R} \frac{1}{p(t,u)} \sum_{(m,v):|x_{m,v}|\ge A} |x_{m,v}| \, p_{m,v} t^m u^v + A \sup_{t_0 \le t,u<R} \frac{1}{p(t,u)} \sum_{(m,v)\in F} p_{m,v} t^m u^v \\
&< \sup_{t_0 \le t,u<R} \frac{1}{p(t,u)} \sum_{(m,v):|x_{m,v}|\ge A} |x_{m,v}| \, p_{m,v} t^m u^v + A \frac{\varepsilon}{2A} < \varepsilon,
\end{aligned}
$$

which gives $(**)$.

$(ii) \implies (i)$: There exists $t_0 \in (0,R)$ such that $(*)$ and $(**)$ hold. From $(*)$ let $B := \sup_{t_0 \le t,u<R} \frac{1}{p(t,u)} \sum_{m,v=0}^{\infty} |x_{m,v}| \, p_{m,v} t^m u^v < \infty$. By $(**)$, given $\varepsilon > 0$ there exists $\delta > 0$ with (4) implies (5). Hence, take $A = \frac{B}{\delta}$ and consider the set $F = F(a) := \{(m,v) : |x_{m,v}| \ge a\}$. Then, we get for any fixed $a \ge A$ that

$$
\frac{1}{p(t,u)} \sum_{(m,v)\in F} p_{m,v} t^m u^v \le \frac{1}{a} \frac{1}{p(t,u)} \sum_{m,v=0}^{\infty} |x_{m,v}| \, p_{m,v} t^m u^v .
$$

So, taking supremum over $t, u \in [t_0, R)$, we obtain that

$$
\sup_{t_0 \le t,u<R} \frac{1}{p(t,u)} \sum_{(m,v)\in F} p_{m,v} t^m u^v \le \frac{B}{a} \le \frac{B}{A} = \delta.
$$

Which gives for the set F, thanks to $(**)$, that

$$
\sup_{t_0 \le t,u<R} \frac{1}{p(t,u)} \sum_{\{(m,v):|x_{m,v}|\ge a\}} \sum_{m,v=0}^{\infty} |x_{m,v}| \, p_{m,v} t^m u^v < \varepsilon
$$

for $a \ge A$ whence the result. □

With the following theorem, we show that the condition for x, for characterizing P_p^2-strong convergence via P_p^2-statistical convergence, is P_p^2-uniform integrable.

Theorem 5. *Let $x = \{x_{m,v}\}$ be a double sequence. Then x is P_p^2-strongly convergent to zero if and only if x is P_p^2-statistically convergent to zero and P_p^2-uniformly integrable.*

Proof. Let x be P_p^2-strongly convergent to zero. Then we can write that

$$
\lim_{t,u\to R^-} \frac{1}{p(t,u)} \sum_{m,v=0}^{\infty} p_{m,v} t^m u^v \, |x_{m,v}| = 0. \tag{6}
$$

Also, for any $\varepsilon > 0$, we get

$$
\frac{1}{p(t,u)} \sum_{\{(m,v):|x_{m,v}|\ge \varepsilon\}} p_{m,v} t^m u^v \le \frac{1}{\varepsilon} \frac{1}{p(t,u)} \sum_{m,v=0}^{\infty} p_{m,v} t^m u^v \, |x_{m,v}| . \tag{7}
$$

Then, thanks to (6) and (7) that (2) is satisfied. Hence, x is P_p^2-statistically convergent to zero. Also, from P_p^2-strong convergence of x, we can write, for any $\varepsilon > 0$, there exists $t_0 \in (0,R)$ such that

$$
\sup_{t_0 \le t,u<R} \frac{1}{p(t,u)} \sum_{m,v=0}^{\infty} p_{m,v} t^m u^v \, |x_{m,v}| \le \varepsilon.
$$

Therefore, for any $A > 0$ and $a \ge A$, we have

$$
\sup_{t_0 \le t,u<R} \frac{1}{p(t,u)} \sum_{\{(m,v):|x_{m,v}|\ge a\}} p_{m,v} t^m u^v \, |x_{m,v}| \le \sup_{t_0 \le t,u<R} \frac{1}{p(t,u)} \sum_{m,v=0}^{\infty} p_{m,v} t^m u^v \, |x_{m,v}| \le \varepsilon.
$$

Hence, x is P_p^2-uniformly integrable.

Conversely, let x be P_p^2-statistically convergent to zero and P_p^2-uniformly integrable. Then there exist $t_0 \in (0,R)$ and $A>0$ such that

$$\sup_{t_0\le t,u<R} \frac{1}{p(t,u)} \sum_{\{(m,v):|x_{m,v}|\ge a\}} p_{m,v} t^m u^v |x_{m,v}| < \frac{\varepsilon}{3}$$

whenever $a \ge A$ and

$$\frac{1}{p(t,u)} \sum_{\left\{(m,v):|x_{m,v}|\ge \frac{\varepsilon}{3}\right\}} p_{m,v} t^m u^v < \frac{\varepsilon}{3A}.$$

For $t_0 \le t,u<R$, we obtain

$$\begin{aligned} & \frac{1}{p(t,u)} \sum_{\{(m,v):|x_{m,v}|\le A\}} p_{m,v} t^m u^v |x_{m,v}| \\ \le\ & \frac{1}{p(t,u)} \sum_{\left\{(m,v):\frac{\varepsilon}{3}\le|x_{m,v}|\le A\right\}} p_{m,v} t^m u^v |x_{m,v}| + \frac{1}{p(t,u)} \sum_{\left\{(m,v):|x_{m,v}|\le \min\left\{A,\frac{\varepsilon}{3}\right\}\right\}} p_{m,v} t^m u^v |x_{m,v}| \\ \le\ & A\frac{1}{p(t,u)} \sum_{\left\{(m,v):|x_{m,v}|\ge \frac{\varepsilon}{3}\right\}} p_{m,v} t^m u^v + \frac{\varepsilon}{3}\frac{1}{p(t,u)} \sum_{m,v=0}^{\infty} p_{m,v} t^m u^v < A\frac{\varepsilon}{3A} + \frac{\varepsilon}{3} = \frac{2\varepsilon}{3}. \end{aligned}$$

Using the above inequality and for $t_0 \le t,u<R$, we have

$$\begin{aligned} & \frac{1}{p(t,u)} \sum_{m,v=0}^{\infty} p_{m,v} t^m u^v |x_{m,v}| \\ =\ & \frac{1}{p(t,u)} \sum_{\{(m,v):|x_{m,v}|\le A\}} p_{m,v} t^m u^v |x_{m,v}| + \frac{1}{p(t,u)} \sum_{\{(m,v):|x_{m,v}|> A\}} p_{m,v} t^m u^v |x_{m,v}| \\ <\ & \frac{2\varepsilon}{3} + \sup_{t_0\le t,u<R} \frac{1}{p(t,u)} \sum_{\{(m,v):|x_{m,v}|> A\}} p_{m,v} t^m u^v |x_{m,v}| < \frac{2\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon, \end{aligned}$$

hence x is P_p^2-strongly convergent to zero. □

3.2 P_p^2-Statistically Cauchy Sequence

Definition 6. *A double sequence $x=\{x_{m,v}\}$ is P_p^2-statistically Cauchy sequence provided that for every $\varepsilon>0$ there exist numbers $M,N\in\mathbb{N}$ such that*

$$\lim_{t,u\to R^-} \frac{1}{p(t,u)} \sum_{m,v=0}^{\infty} p_{m,v} t^m u^v \chi\left(\left\{(m,v)\in\mathbb{N}_0^2 : |x_{m,v}-x_{M,N}|\ge\varepsilon\right\}\right)=0.$$

Theorem 6. *A double sequence $x=\{x_{m,v}\}$ is P_p^2-statistically convergent if and only if x is P_p^2-statistically Cauchy.*

Proof. Let x be P_p^2-statistically convergent to L. In this case, for every $\varepsilon>0$,

$$\delta_{P_p}^2\left(\left\{(m,v)\in\mathbb{N}_0^2 : |x_{m,v}-L|\ge\tfrac{\varepsilon}{2}\right\}\right)=0.$$

We can choose $M,N\in\mathbb{N}_0$ such that $|x_{M,N}-L|<\frac{\varepsilon}{2}$ holds. Hence, thanks to the inequality

$$|x_{m,v}-x_{M,N}|\le|x_{m,v}-L|+|x_{M,N}-L|$$

we can get that

$$\delta_{P_p}^2\left(\left\{(m,v)\in\mathbb{N}_0^2 : |x_{m,v}-x_{M,N}|\ge\varepsilon\right\}\right)=0$$

and x is P_p^2-statistically Cauchy sequence. Conversely, let x be P_p^2-statistically Cauchy. For $\varepsilon = 1$ there exist $M_1, N_1 \in \mathbb{N}_0$ such that $\delta_{P_p}^2 \left(\left\{ (m,v) \in \mathbb{N}_0^2 : |x_{m,v} - x_{M_1,N_1}| \geq 1 \right\} \right) = 0$. For $\varepsilon = \frac{1}{2}$ there exist $M_2, N_2 > \max\{M_1, N_1\}$ such that $\delta_{P_p}^2 \left(\left\{ (m,v) \in \mathbb{N}_0^2 : |x_{m,v} - x_{M_2,N_2}| \geq \frac{1}{2} \right\} \right) = 0$. Inductively we obtain two sequences $\{M_k\}$ and $\{N_k\}$ such that $M_{k+1}, N_{k+1} > \max\{M_k, N_k\}$ and

$$\delta_{P_p}^2 \left(\left\{ (m,v) \in \mathbb{N}_0^2 : |x_{m,v} - x_{M_k,N_k}| \geq \frac{1}{k} \right\} \right) = 0.$$

Clearly, given $i, j \in \mathbb{N}$ there exist $(m_0, v_0) \in \mathbb{N}_0^2$ such that $|x_{m_0,v_0} - x_{M_i,N_i}| < \frac{1}{i}$ and $\left| x_{m_0,v_0} - x_{M_j,N_j} \right| < \frac{1}{j}$, thus

$$\left| x_{M_i,N_i} - x_{M_j,N_j} \right| < \frac{1}{i} + \frac{1}{j} \to 0 \text{ as } i, j \to \infty,$$

that is, the ordinary (single) sequence $\{x_{M_k,N_k}\}_k$ is a Cauchy sequence, so it is convergent in a limit L. Hence, given $\varepsilon > 0$, there exists $k_0 \in \mathbb{N}$ such that $k_0 > \frac{2}{\varepsilon}$ and $|x_{M_k,N_k} - L| < \frac{\varepsilon}{2}$. Observe that

$$\begin{aligned} \left\{ (m,v) \in \mathbb{N}_0^2 : |x_{m,v} - L| \geq \varepsilon \right\} &\subseteq \left\{ (m,v) \in \mathbb{N}_0^2 : \left| x_{m,v} - x_{M_{k_0},N_{k_0}} \right| \geq \frac{\varepsilon}{2} \right\} \\ &\subseteq \left\{ (m,v) \in \mathbb{N}_0^2 : \left| x_{m,v} - x_{M_{k_0},N_{k_0}} \right| \geq \frac{1}{k} \right\}, \end{aligned}$$

so, it is clear that

$$\delta_{P_p}^2 \left(\left\{ (m,v) \in \mathbb{N}_0^2 : |x_{m,v} - L| \geq \varepsilon \right\} \right) = 0,$$

whence the result. □

3.3 P_p^2-Statistical Boundedness and Core Theorems

Definition 7. *(i) A real double sequence $x = \{x_{m,v}\}$ is P_p^2-statistically bounded above if for an $M \in \mathbb{R}$ it holds that $\delta_{P_p}^2 \left(\left\{ (m,v) \in \mathbb{N}_0^2 : x_{m,v} > M \right\} \right) = 0$.*

(ii) A real double sequence $x = \{x_{m,v}\}$ is P_p^2-statistically bounded below if for an $N \in \mathbb{R}$ it holds that $\delta_{P_p}^2 \left(\left\{ (m,v) \in \mathbb{N}_0^2 : x_{m,v} < N \right\} \right) = 0$.

If a real double sequence $x = \{x_{m,v}\}$ is P_p^2-statistically bounded above and below, then we say that it is P_p^2-statistically bounded. We can say that any bounded double sequence is also P_p^2-statistically bounded.

Definition 8. *For any $M, N \in \mathbb{R}$ and a real double sequence $x = \{x_{m,v}\}$, the P_p^2-statistical superior of x is $\inf \left\{ M : \delta_{P_p}^2 \left(\left\{ (m,v) \in \mathbb{N}_0^2 : x_{m,v} > M \right\} \right) = 0 \right\}$ and the statistical inferior of x is $\sup \left\{ N : \delta_{P_p}^2 \left(\left\{ (m,v) \in \mathbb{N}_0^2 : x_{m,v} < N \right\} \right) = 0 \right\}$.*

Theorem 7. *A P_p^2-statistically bounded double sequence $x = \{x_{m,v}\}$ is P_p^2-statistically convergent if and only if*

$$st_{P_p}^2 - \liminf x_{m,v} = st_{P_p}^2 - \limsup x_{m,v}.$$

Proof. Let $st_{P_p}^2 - \lim x_{m,v} = L$. For any $\varepsilon > 0$, $\delta_{P_p}^2 \left(\{ (m,v) : |x_{m,v} - L| \geq \varepsilon \} \right) = 0$,and thanks to $\delta_{P_p}^2 \left(\{ (m,v) : x_{m,v} > L + \varepsilon \} \right) = 0$, and $\delta_{P_p}^2 \left(\{ (m,v) : x_{m,v} < L - \varepsilon \} \right) = 0$, we have $st_{P_p}^2 - \limsup x_{m,v} \leq L$ and $L \leq st_{P_p}^2 - \liminf x_{m,v}$. Therefore, $st_{P_p}^2 - \limsup x_{m,v} \leq st_{P_p}^2 - \liminf x_{m,v}$. From Remark 1-$(i)$ we get

$$st_{P_p}^2 - \liminf x_{m,v} = st_{P_p}^2 - \limsup x_{m,v}.$$

Conversely, let $st^2_{P_p} - \liminf x_{m,v} = st^2_{P_p} - \limsup x_{m,v} = L$. For $\varepsilon > 0$, thanks to Theorem 1, we have

$$\delta^2_{P_p}\left(\left\{(m,v) : x_{m,v} > L + \frac{\varepsilon}{2}\right\}\right) = \delta^2_{P_p}\left(\left\{(m,v) : x_{m,v} < L - \frac{\varepsilon}{2}\right\}\right) = 0,$$

whence the result. □

Definition 9. *The P^2_p-statistical core of x, any P^2_p-statistically bounded real double sequence, is the closed interval*

$$\left[st^2_{P_p} - \liminf x, st^2_{P_p} - \limsup x\right].$$

The P^2_p-statistical core is either $\left(-\infty, st^2_{P_p} - \limsup x\right]$, $\left[st^2_{P_p} - \liminf x, \infty\right)$ or $(-\infty,\infty)$ whenever x is not P^2_p-statistically bounded.

It can be written from Remark 1-(ii) that P^2_p-statistical core $(x) \subseteq P$-core (x).

Consider $c_2^{\infty,0}$, $st^2_{P_p}$ as the sets of bounded double sequences that converge to zero as P-convergent and P^2_p-statistically convergent double sequences, respectively.

Lemma 1. *Let $A = [a_{k,l,m,v}]$ be a four-dimensional matrix. $A \in \left(st^2_{P_p} \cap l^2_\infty, c_2^\infty; P\right)$ if and only if (i) A is RH-regular, (ii) $P - \lim_{k,l} \sum\limits_{m,v \in F} |a_{k,l,m,v}| = 0$ for every $F \subset \mathbb{N}_0^2$ with $\delta^2_{P_p}(F) = 0$.*

Proof. Let $A \in \left(st^2_{P_p} \cap l^2_\infty, c_2^\infty; P\right)$. Because of $c_2^\infty \subset st^2_{P_p} \cap l^2_\infty$, we get $A \in (c_2^\infty, c_2^\infty; P)$. Hence, (i) holds. Let $x = \{x_{m,v}\} \in l^2_\infty$ and define $y = \{y_{m,v}\}$ by

$$y_{m,v} = \begin{cases} x_{m,v}, & \text{if } (m,v) \in F \\ 0, & \text{otherwise.} \end{cases}$$

Then, we have $st^2_{P_p} - \lim y_{m,v} = 0$ and $y \in st^2_{P_p}$. Also, since $Ay = \sum\limits_{m,v \in F} a_{k,l,m,v} x_{m,v}$, then define the matrix $B = [b_{k,l,m,v}]$ by

$$b_{k,l,m,v} = \begin{cases} a_{k,l,m,v}, & \text{if } (m,v) \in F \\ 0, & \text{otherwise.} \end{cases}$$

Thus $B \in \left(l^2_\infty, c_2^{\infty,0}\right)$. From Lemma 3.2 of [7], we get (ii).

Conversely, let (i) and (ii) hold and also, $x \in st^2_{P_p} \cap l^2_\infty$ with $st^2_{P_p} - \lim x_{m,v} = L$. Then, for any $\varepsilon > 0$, $\delta^2_{P_p}(F) = \delta^2_{P_p}(\{(m,v) : |x_{m,v} - L| \geq \varepsilon\}) = 0$ and $|x_{m,v} - L| < \varepsilon$ for $(m,v) \notin F$. Observe that

$$\sum_{m,v} a_{k,l,m,v} x_{m,v} = \sum_{m,v} a_{k,l,m,v}(x_{m,v} - L) + L \sum_{m,v} a_{k,l,m,v},$$

then by RH-regularity of A,

$$P - \lim_{k,l} \sum_{m,v} a_{k,l,m,v} x_{m,v} = P - \lim_{k,l} \sum_{m,v} a_{k,l,m,v}(x_{m,v} - L) + L. \tag{8}$$

Since

$$\begin{aligned} \left|\sum_{m,v} a_{k,l,m,v}(x_{m,v} - L)\right| &= \left|\sum_{(m,v) \in F} a_{k,l,m,v}(x_{m,v} - L) + \sum_{(m,v) \notin F} a_{k,l,m,v}(x_{m,v} - L)\right| \\ &\leq |x_{m,v} - L| \sum_{m,v \in F} |a_{k,l,m,v}| + \varepsilon \sum_{m,v=1}^{\infty} |a_{k,l,m,v}|, \end{aligned}$$

thanks to RH-regularity of A and the hypothesis (ii) that

$$P-\lim_{k,l}\sum_{m,v} a_{k,l,m,v}\left(x_{m,v}-L\right)=0.$$

From equation (8), we get $P-\lim_{k,l}(Ax)_{k,l}=L=st^2_{P_p}-\lim x_{m,v}$, which leads us to the required result. □

Theorem 8. *Let* $\sup_{k,l}\sum_{m,v=1}^{\infty}|a_{k,l,m,v}|<\infty$ *and* $x\in l^2_\infty$. *Then,* $P-\limsup Ax\le st^2_{P_p}-\limsup x$ *if and only if* $A\in\left(st^2_{P_p}\cap l^2_\infty, c^\infty_2; P\right)$ *and* $P-\lim_{k,l}\sum_{m,v=1}^{\infty}|a_{k,l,m,v}|=1$.

Proof. Let $P-\limsup Ax\le st^2_{P_p}-\limsup x$. Then, we can easily get the following inequality

$$st^2_{P_p}-\liminf x\le P-\liminf Ax\le P-\limsup Ax\le st^2_{P_p}-\limsup x.$$

With the arbitrariness of x and since $st^2_{P_p}\cap l^2_\infty\subset l^2_\infty$, it can be considered any $x\in st^2_{P_p}\cap l^2_\infty$ and $st^2_{P_p}-\lim x=L$. Then thanks to above inequality, we get $A\in\left(st^2_{P_p}\cap l^2_\infty, c^\infty_2; P\right)$. Now, because of $st^2_{P_p}-\limsup x\le P-\limsup x$ by Remark 1-(ii), it follows from Theorem 3.2. of [20] that $P-\lim_{k,l}\sum_{m,v=1}^{\infty}|a_{k,l,m,v}|=1$.

Conversely, let $A\in\left(st^2_{P_p}\cap l^2_\infty, c^\infty_2; P\right)$ and $P-\lim_{k,l}\sum_{m,v=1}^{\infty}|a_{k,l,m,v}|=1$. Also let $x\in l^2_\infty$. Then $Ax\in l^2_\infty$ and $st^2_{P_p}-\limsup x$ is finite. Hence, thanks to Theorem 1, for any $\varepsilon>0$, $\delta^2_{P_p}(F)=\delta^2_{P_p}\left(\left\{(m,v):x_{m,v}\ge st^2_{P_p}-\limsup x+\varepsilon\right\}\right)=0$ and $x_{m,v}<st^2_{P_p}-\limsup x+\varepsilon$ with $(m,v)\notin F$. Observe that

$$\begin{aligned}
&\sum_{m,v} a_{k,l,m,v}x_{m,v}\\
\le\ & \left|\sum_{m,v}\frac{|a_{k,l,m,v}x_{m,v}|+a_{k,l,m,v}x_{m,v}}{2}+\sum_{m,v}\frac{|a_{k,l,m,v}x_{m,v}|-a_{k,l,m,v}x_{m,v}}{2}\right|\\
\le\ & \left|\sum_{m,v} a_{k,l,m,v}x_{m,v}\right|+\sum_{m,v}\left(|a_{k,l,m,v}|-a_{k,l,m,v}\right)|x_{m,v}|\\
\le\ & \left|\sum_{(m,v)\in F} a_{k,l,m,v}x_{m,v}+\sum_{(m,v)\notin F} a_{k,l,m,v}x_{m,v}\right|+\sup_{m,v}|x_{m,v}|\sum_{m,v}\left(|a_{k,l,m,v}|-a_{k,l,m,v}\right)\\
\le\ & \sup_{m,v}|x_{m,v}|\sum_{(m,v)\in F}|a_{k,l,m,v}|+\left(st^2_{P_p}-\limsup x+\varepsilon\right)\sum_{(m,v)\notin F}|a_{k,l,m,v}|\\
& ++\sup_{m,v}|x_{m,v}|\sum_{m,v}\left(|a_{k,l,m,v}|-a_{k,l,m,v}\right).
\end{aligned}$$

Since A is RH-regular and thanks to $P-\lim_{k,l}\sum_{m,v=1}^{\infty}|a_{k,l,m,v}|=1$ that

$$P-\limsup Ax\le st^2_{P_p}-\limsup x+\varepsilon,$$

thus from the arbitrariness of x and ε, we reach the required result. □

Declarations

Conflict of interest The authors declare no conflict of interest.

References

[1] K J Ansari, I Ahmad, M Mursaleen, I Hussain. *On some statistical approximation by (p,q)-Bleimann, Butzer and Hahn operators*, Symmetry, 2018, 10(12), https://doi.org-/10.3390/sym10120731.

[2] C Bardaro, A Boccuto, K Demirci, I Mantellini, S Orhan. *Triangular A-statistical approximation by double sequences of positive linear operators*, Results in Mathematics, 2015, 68: 271-291.

[3] S Baron, U Stadtmüller. *Tauberian theorems for power series methods applied to double sequences*, Journal of Mathematical Analysis and Applications, 1997, 211(2): 574-589.

[4] C Belen, M Yıldırım, C Sümbül. *On statistical and strong convergence with respect to a modulus function and a power series method*, Filomat, 2020, 34(12): 3981-3993.

[5] Q B Cai, K J Ansari, M T Ersoy, F Özger. *Statistical blending-type approximation by a class of operators that includes shape parameters λ and α*, Mathematics, 2022, 10(7), https://doi.org/10.3390/math10071149.

[6] M O Cabrera, A Rosalsky, M Ünver, A Volodin. *A New Version of Uniform Integrability via Power Series Summability Methods*, Theory of Probability and Its Applications, 2022, 67(1): 89-104.

[7] C Çakan, B Altay. *Statistically boundedness and statistical core of double sequences*, Journal of Mathematical Analysis and Applications, 2006, 317(2): 690-697.

[8] K Demirci. *On lacunary statistical limit points*, Demonstratio Mathematica, 2002, 35(1): 93-102.

[9] K Demirci, D Djurcic, L Kocinac, S Yıldız. *A theory of variations via P-statistical convergence*, Publications de I'Institut Mathématique, 2021, 110(124): 11-27.

[10] K Demirci, F Dirik, S Yıldız. *Approximation via equi-statistical convergence in the sense of power series method*, Revista de la Real Academia de Ciencias Exactas, Físicas y Naturales Serie A Matemáticas, 2022, 116, https://doi.org/10.1007/s13398-021-01191-4.

[11] K Demirci, F Dirik, S Yıldız. *Approximation via statistical relative uniform convergence of sequences of functions at a point with respect to power series method*, Afrika Matematika, 2023, 34, https://doi.org/10.1007/s13370-023-01078-0.

[12] K Demirci, S Orhan. *Statistically relatively uniform convergence of positive linear operators*, Results in Mathematics, 2016, 69: 359-367.

[13] O Duman, C Orhan. *μ-statistically convergent function sequences*, Czechoslovak Mathematical Journal, 2004, 54: 413-422.

[14] J A Fridy, C Orhan. *Lacunary statistical convergence*, Pacific Journal of Mathematics, 1993, 160(1): 43-51.

[15] S Karakuş, K Demirci, O Duman. *Equi-statistical convergence of positive linear operators*, Journal of Mathematical Analysis and Applications, 2008, 339(2): 1065-1072.

[16] H Fast. *Sur la convergence statistique*, Colloquium Mathematicum, 1951, 2(3-4): 241-244.

[17] H J Hamilton. *Transformations of multiple sequences*, Duke Mathematical Journal, 1936, 2: 29-60.

[18] F Moricz. *Statistical convergence of multiple sequences*, Arch Math (Basel), 2004, 81: 82-89.

[19] F Özger, K J Ansari. *Statistical convergence of bivariate generalized Bernstein operators via four-dimensional infinite matrices*, Filomat, 2022, 36(2): 507-525.

[20] R F Patterson. *Double sequence core theorems*, International Journal of Mathematics and Mathematical Sciences, 1999, 22(4): 785-793.

[21] A Pringsheim. *Zur theorie der zweifach unendlichen zahlenfolgen*, Mathematische Annalen, 1900, 53: 289-321.

[22] G M Robinson. *Divergent double sequences and series*, American Mathematical Society Translations, 1926, 28, 50-73.

[23] N Şahin Bayram. *Criteria for statistical convergence with respect to power series methods*, Positivity, 2021, 25(3): 1097-1105.

[24] H Steinhaus. *Sur la convergence ordinaire et la convergence asymtotique*, Colloquium Mathematicum, 1951, 2: 73-74.

[25] C Sümbül, C Belen, M Yıldırım. *Properties of J_p-Statistical Convergence*, Cumhuriyet Science Journal, 2022, 43(2): 294-298.

[26] M Ünver, N Ş Bayram. *On statistical convergence with respect to power series methods*, Positivity, 2022, 26, https://doi.org/10.1007/s11117-022-00921-1.

[27] M Ünver, C Orhan. *Statistical convergence with respect to power series methods and applications to approximation theory*, Numerical Functional Analysis and Optimization, 2019, 40(5): 535-547.

[28] S Yıldız, K Demirci, F Dirik. *Korovkin theory via P_p–statistical relative modular convergence for double sequences*, Rendiconti del Circolo Matematico di Palermo Series 2, 2023, 72: 1125-1141.

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