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# Travelling wave solutions of nonlinear conformable Bogoyavlenskii equations via two powerful analytical approaches

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**Abstract**. The presented study deals with the investigation of nonlinear Bogoyavlenskii equations with conformable time-derivative which has great importance in plasma physics and non-inspectoral scattering problems. Travelling wave solutions of this nonlinear conformable model are constructed by utilizing two powerful analytical approaches, namely, the modified auxiliary equation method and the Sardar sub-equation method. Many novel soliton solutions are extracted using these methods. Furthermore, 3D surface graphs, contour plots and parametric graphs are drawn to show dynamical behavior of some obtained solutions with the aid of symbolic software such as Mathematica. The constructed solutions will help to understand the dynamical framework of nonlinear Bogoyavlenskii equations in the related physical phenomena.

## §1 Introduction

In recent decades, the mathematical modeling and numerical simulation of physical phenomena appearing in various fields such as hydrodynamics, optics, biology, fluid mechanics, physics, and many others, has been achieved by utilizing the nonlinear partial differential equations (NLPDEs) [1-4]. Exploring the traveling wave solutions and the effects due to different differential operators has been an active area of research. In recent decades, numerical and the experimental study of seismic wave propagation in complex media, from ultrasonic (MHz) to seismic scale (Hz) has become an active area of research [5-6]. Bouchaala et al. discussed the compressional and shear wave attenuations from walkway VSP and sonic data in an offshore Abu Dhabi oilfield [7]. Aslanova discussed a comparative study of the hardness and force analysis methods used in truss optimization with metaheuristic algorithms and under dynamic loading [8]. Moghadam and Ebrahimi presented an analysis of a torsional mode MEMS disk resonator for RF applications [9]. Kumar et al. discussed performance analysis of $1\times4$ RMPA

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array using step cut and DGS techniques with different feeds techniques for LTE, wi-fi, wlan and military communications [10]. Among the traveling wave solutions, solitary waves with nonlinear dispersive effects, known as solitons, are of great significance due to their additional property of holding a permanent shape [11-13]. Solitons play a key role in the telecommunication industry as signals can proceed far away without any distortion in the form of solitons [14-16].

Many powerful techniques are applied to study and discuss the explicit solutions as well as the physical behaviour of these models. Some of the techniques are the extended tanh-function method [17], the modified kudryashov method [18], the simplified bilinear method [19], the sub-equation method [20], the first integral method [21], the new transfer function method [22], the new version of the trial equation method [23] and so on [24-30]. Over the past few years, many researchers have discussed the modified auxiliary equation method, which is powerful and effective, to seek novel soliton solutions for different kinds of NLPDEs [31-34]. Moreover, a powerful and direct method called the Sardar sub-equation method is utilized to get exact soliton solutions of various PDE [35-37].

The (2 + 1)-dimensional Bogoyavlenskii equation is an important nonlinear mathematical model to elucidate the hydrodynamic model of shallow water wave, plasma physics, the wave of leading fluid flow etc [38]. Kudryashov and Pickering [39] proposed the nonlinear Bogoyavlenskii equations as a member of a (2 + 1) Schwarzian breaking soliton hierarchy. Clarkson et al. [40] examined Bogoyavlenskii equations as one of the equations associated to nonisospectral scattering problems. Peng and Shen applied the singular manifold method to extract the shock wave solution as well as the complex solitary wave solution for Bogoyavlenskii equations [41]. In 2016, Yu and Sun [42] used the modified simplest equation method to derive few of the analytical traveling wave solutions for Bogoyavlenskii equations. Li and Ziao [43] used the improved fractional sub-equation method to derive some solitary wave solutions of the (2+1)-dimensional space-time fractional Bogoyavlenskiis breaking soliton equation. In 2018, Liu et al. [44] used new B cklund transformation and found the residual symmetry of the (2+1)-dimensional Bogoyavlenskii equation. In 2018, Feng [45] used specific fractional transformation and the Jacobi elliptic equation to get the analytical solutions of the above equation. In 2020, Alam and Tunc [46] successfully applied the generalized $(G'/G)$-expansion method to construct some new solitary wave phenomena. Nisar et al. [47] derived analytical solutions of fractional Bogoyavlenskii equations by $\exp(-K(\phi))-$expansion method and rational $\tan(K(\phi))-$expansion method.

In this study, we consider the nonlinear Bogoyavlenskii equations model with respect to the conformable derivative, as

$$\begin{aligned} 4D_t^k p + p_{xxy} - 4p^2 p_y - 4p_x q = 0, \\ q_x = pp_y, \end{aligned} \tag{1}$$

where $p(x,y,t)$ and $q(x,y,t)$ are the unknown functions. $D_t^k$ represents the conformable time-derivative.

The primary goal of this work is to retrieve the new solitary wave solutions to nonlinear time conformable Bogoyavlenskii equations by utilizing two efficient and powerful analytical approaches, the modified auxiliary equation method and the Sardar sub-equation method. The proposed techniques are utilized for the first time to obtain a new type of solution of the considered conformable model. Comparing these two methods to the previous results, it can be noticed that our approaches are more effective. The obtained results will help in understanding the mechanism of the suggested problem, which is one of the cardinal focuses.

The strategy of the given article is given as, some preliminaries of conformable derivative are given in section 2. The fundamental steps of the proposed methods are described in section 3. Exact solutions for the concerned equation are extracted in Section 4. The results and discussion given are discussed in section 5. The modulation instability analysis of the provided model is represented in section 6. Finally, some conclusions are drawn in the last section.

## §2 Basic Preliminaries of Conformable Derivative

In recent decades, various non-integer derivatives have been introduced such as Caputo, RiemannLiouville and Grunwald Letnikov to understand the nonlinear phenomena [48-50]. Recently, the Khalil et al. introduced a new simple and intriguing definition of the non-integer derivative called conformable derivative [51]. The definition of the conformable derivative of order $k$ is given below:

**Definition 2.1.**

The conformable derivative for a function $f:(0,\infty)\to R$ of order $0<k\le 1$ at $t>0$ is defined as

$$D^k(f)(t)=\lim_{\epsilon\to 0}\frac{f(t+\epsilon t^{(1-k)})}{\epsilon}.$$

Note that if the function $f$ is $k$-differentiable in $(0,a)$ then $D^k(f)(t)=(t)^{1-k}f'(t)$.

The definition of conformable derivative satisfies the properties stated in the theorem given below [52-53]:

**Theorem 2.2.**

Let $g,h$ be the $k$-differentiable at a point $t$ and $0<k\le 1$, then

$$(i)\ D^k(ag+bh)=aD_k(g)+bD_k(h),\ for\ a,b\in R,$$

$$(ii)\ D^k(t^\eta)=\eta t^{\eta-k},\ \ \forall\ \eta\ \in\ R,$$

$$(iii)\ D^k(gh)=g_tD_k(h)+h_tD_k(g),$$

$$(iv)\ D^k(\frac{g}{h})=\frac{g_tD_k(h)-h_tD_k(g)}{h^2}.$$

**Theorem 2.3.**

A function $f:(0,\infty)\to R$ of order $0<k\le 1$ at $t>0$, so that it is differentiable and also it is $\alpha$-differentiable. Let $h$ be a differentiable function described in the range of $f$, then, the following rule holds

$$_tD^k(foh)(t)=(_tD^kf)(h(t))(_tD^kh)(t)h(t)^{k-1}.$$

when $t=0$,

$$_tD^k(foh)(0)=\lim_{t\to 0^+}(_tD^kf)(h(t))(_tD^kh)(t)h(t)^{k-1}.$$

## §3 The General Structure of The Proposed Methods

The main focus of the presented study is to investigate the nonlinear Bogoyavlenskii equations using the concept of conformable derivative for the first time with the modified auxiliary

equation method and the Sardar sub-equation method. The suggested methodologies are proficient, easy to proceed, and reliable for retrieving new-wave solutions of NLPDEs. These approaches can generate a variety of closed-form solutions, including trigonometric, hyperbolic, and rational functions. Theoretically, some remarkable kind of wave solutions can be extracted by assigning arbitrary values to free parameters appearing in the aforementioned techniques for the proposed conformable model. In the laboratory, these solutions can be used as prior knowledge to generate desired possible soliton pulses in fluids.
The general structure of The proposed methods is given below:
By letting a nonlinear conformable partial differential equation having the form given below:

$$U(D_t^k p, p_x, p_y, D_t^{2k} p, p_{xx}, p_{yy}, p_{xy}, ...) = 0, \tag{2}$$

where $U$ is a polynomial in $p(x,y,t)$ and its higher-order partial derivatives. $D_t^k$ represents the conformable time derivative.
Using the wave transformation:

$$p(x,y,t) = P(\xi), \qquad where \;\; \xi = x + y - c\frac{t^k}{k}. \tag{3}$$

Using the above wave transformation on Eq. (3), the NFPDE is being converted into an ODE having the form:

$$U(p, p', p'', ...) = 0. \tag{4}$$

## 3.1 The Modified Auxiliary Equation Method

The fundamental steps of the suggested method are given below:
**Step 1:** By letting the solutions of the above ODE (4) have the following form:

$$p(x,y,t) = \sum_{i=0}^{Z} a_i K^{if(\xi)} + a_0 + \sum_{i=0}^{Z} b_i K^{-if(\xi)}, \tag{5}$$

where the arbitrary constants $a_i, a_0$ and $b_i$ will be evaluate afterward, while $f(\xi)$ satisfies the ODE:

$$f'(\zeta) = \frac{1}{\ln K}(\alpha K^{-f(\xi)} + \beta + \sigma K^{f(\xi)}). \tag{6}$$

**Step 2:** Evaluate the positive integer $Z$ in Eq. (5) by the balancing method of the highest order derivative terms and the nonlinear terms.
**Step 3:** Using Eq. (5) and Eq. (6) in Eq. (4) and then collecting the terms having same powers of $(K^{if(\xi)})$, where $i = -Z, ..., Z$ and by equating these terms to zero, a system of algebraic equations is obtained that will be simplified with the aid of wolfram Mathematica-9 or maple to get the values of $\alpha, \beta, \sigma$, $a_i$ and $b_i$.
**Step 4:** Substitute gained values with solutions of Eq. (5) into Eq. (4), the analytical solutions for Eq. (1) are obtained.

## 3.2 The Sardar Sub-Equation Method

It is supposed that the Equation (4) has a formal solution of the form given below:

$$p(x,y,t) = \sum_{i=0}^{Z} \nu_i \psi^i(\xi), \tag{7}$$

where $\nu_i$, $(i = 0, 1, 2..., Z.)$ are the coefficients to be determined later and $\psi^{'}(\xi)$ satisfies the following ODE.

$$(\psi'(\xi))^2 = \alpha + g\psi^2 + \psi^4, \tag{8}$$

where $\alpha$ and $g$ are arbitrary constants. The solutions of the ODE are

**Case 1:**

If $g > 0$ and $\alpha = 0$, then

$$\psi_1^{\pm}(\xi) = \pm\sqrt{-rsg}\,\mathrm{sech}_{rs}(\sqrt{g}\xi),$$
$$\psi_2^{\pm}(\xi) = \pm\sqrt{-rsg}\,\mathrm{csch}_{rs}(\sqrt{g}\xi),$$

where

$$\mathrm{sech}_{rs}(\xi) = \frac{2}{re^{\xi} + se^{-\xi}}, \qquad \mathrm{csch}_{rs}(\xi) = \frac{2}{re^{\xi} - se^{-\xi}}.$$

**Case 2:**

If $g < 0$ and $\alpha = 0$, then

$$\psi_3^{\pm}(\xi) = \pm\sqrt{-rsg}\,\sec_{rs}(\sqrt{-g}\xi),$$
$$\psi_4^{\pm}(\xi) = \pm\sqrt{-rsg}\,\csc_{rs}(\sqrt{-g}\xi),$$

where

$$\sec_{rs}(\xi) = \frac{2}{re^{i\xi} + se^{-i\xi}}, \qquad \csc_{rs}(\xi) = \frac{2}{re^{i\xi} - se^{-i\xi}}.$$

**Case 3:**

If $g < 0$ and $\alpha = \frac{g^2}{4}$, then

$$\psi_5^{\pm}(\xi) = \pm\sqrt{\frac{-g}{2}}\tanh_{rs}\left(\sqrt{\frac{-g}{2}}\xi\right),$$
$$\psi_6^{\pm}(\xi) = \pm\sqrt{\frac{-g}{2}}\coth_{rs}\left(\sqrt{\frac{-g}{2}}\xi\right),$$
$$\psi_7^{\pm}(\xi) = \pm\sqrt{\frac{-g}{2}}\left(\tanh_{rs}\left(\sqrt{-2g}\xi\right) \pm i\sqrt{rs}\,\mathrm{sech}_{rs}\left(\sqrt{-2g}\xi\right)\right),$$
$$\psi_8^{\pm}(\xi) = \pm\sqrt{\frac{-g}{2}}\left(\coth_{rs}\left(\sqrt{-2g}\xi\right) \pm \sqrt{rs}\,\mathrm{csch}_{rs}\left(\sqrt{-2g}\xi\right)\right),$$
$$\psi_9^{\pm}(\xi) = \pm\sqrt{\frac{-g}{8}}\left(\tanh_{rs}\left(\sqrt{\frac{-g}{8}}\xi\right) \pm \sqrt{rs}\coth_{rs}\left(\sqrt{\frac{-g}{8}}\xi\right)\right),$$

where

$$\tanh_{rs}(\xi) = \frac{re^{\xi} - se^{-\xi}}{re^{\xi} + se^{-\xi}}, \qquad \coth_{rs}(\xi) = \frac{re^{\xi} + se^{-\xi}}{re^{\xi} - se^{-\xi}}.$$

**Case 4:**

If $g > 0$ and $\alpha = \frac{g^2}{4}$, then

$$\psi_{10}^{\pm}(\xi) = \pm\sqrt{\frac{g}{2}}\tan_{rs}\left(\sqrt{\frac{-g}{2}}\xi\right),$$
$$\psi_{11}^{\pm}(\xi) = \pm\sqrt{\frac{g}{2}}\cot_{rs}\left(\sqrt{\frac{-g}{2}}\xi\right),$$
$$\psi_{12}^{\pm}(\xi) = \pm\sqrt{\frac{g}{2}}\left(\tan_{rs}\left(\sqrt{2g}\xi\right) \pm i\sqrt{rs}\sec_{rs}\left(\sqrt{2g}\xi\right)\right),$$
$$\psi_{13}^{\pm}(\xi) = \pm\sqrt{\frac{g}{2}}\left(\cot_{rs}\left(\sqrt{2g}\xi\right) \pm \sqrt{rs}\csc_{rs}\left(\sqrt{2g}\xi\right)\right),$$

$$\psi_{14}^{\pm}(\xi) = \pm\sqrt{\frac{g}{8}}\left(\tan_{rs}\left(\sqrt{\frac{g}{8}}\xi\right) \pm \sqrt{rs}\cot_{rs}\left(\sqrt{\frac{g}{8}}\xi\right)\right),$$

where

$$\tan_{rs}(\xi) = -i\frac{re^{i\xi} - se^{-i\xi}}{re^{i\xi} + se^{-i\xi}}, \qquad \cot_{rs}(\xi) = i\frac{re^{i\xi} + se^{-i\xi}}{re^{i\xi} - se^{-i\xi}}.$$

By balancing method value of $Z$ will be evaluated. After finding the value of $Z$, the predicted solutions as well as necessary derivatives will be substituted to Eq. (7). All the coefficients of power of $\psi(\xi)$ are further equated to zero and an algebraic system of equations is obtained. The resultant algebraic system is solved on maple or wolfram mathematica for values of $\nu_i$'s and $g$. At the last, put ($\xi = x + y - c\frac{t^k}{k}$) into the obtained solutions.

## §4 Application to The Nonlinear Bogoyavlenskii Equations Model

In the nonlinear wave theory, the most important aspect is the investigation of a kind of analytical solution, that is, the traveling wave solutions which are the solutions of some constant forms moving by a constant velocity. Here, consider the traveling wave solutions for Eq. (1) having the following form

$$p(x, y, t) = P(\xi), \quad q(x, y, t) = Q(\xi), \quad where, \ \xi = x + y - c\frac{t^k}{k}. \tag{9}$$

By substituting Eq. (9) into Eq. (1) and then integrating it, and for simplicity letting the constant of integration equal to zero, the following system obtains,

$$\begin{aligned} -4cP_\xi + P_{\xi\xi\xi} - 4P^2P_\xi - 4P_\xi Q = 0, \\ Q_\xi = PP_\xi. \end{aligned} \tag{10}$$

Integrate Eq. (10) w.r.t. $\xi$, the second equation of the above system becomes

$$Q = \frac{1}{2}P^2. \tag{11}$$

Substitute obtained value of $Q$ in the first integrated equation of the above system, following ODE is obtained

$$P_{\xi\xi} - 2P^3 - 4cP = 0. \tag{12}$$

### 4.1 Analysis of Solutions via Modified Auxiliary Equation Method

The modified auxiliary equation method is being implemented on the nonlinear Bogoyavlenskii equations to construct various different and novel solitary wave solutions in this section. Firstly, by applying the homogenous balancing principle between $P_{\xi\xi}$ and $P^3$, the value of $Z$ is obtained as $Z = 1$. According to the proposed method, Eq.(6) becomes

$$P(\xi) = a_0 + a_1K^{g(\xi)} + b_1K^{-g(\xi)}, \tag{13}$$

where

$$g'(\xi) = \frac{1}{\ln K}(\alpha + \beta K^{-g(\xi)} + \gamma K^{g(\xi)}). \tag{14}$$

Substitute Eq.(13) and its derivative in Eq.(12) and collect all coefficients of the same terms for $K^{ig(\xi)}$, where $i = -Z, ..., Z$. Further, by equating these terms to zero, we get the following system of algebraic equations:

$$
\begin{aligned}
K^g &: \alpha^2 a_1 + 2a_1\beta\gamma - 6a_0^2 a_1 - 6a_1^2 b_1 - 4ca_1 = 0\\
K^{2g} &: 3a_1\alpha\gamma - 6a_0 a_1^2 = 0\\
K^{3g} &: 2a_1\gamma^2 - 2a_1^3 = 0\\
K^{-g} &: \alpha^2 b_1 + 2b_1\beta\gamma - 6b_0^2 a_1 - 6a_0^2 b_1 - 4cb_1 = 0\\
K^{-2g} &: 3b_1\alpha\beta - 6a_0 b_1^2 = 0\\
K^{-3g} &: 2b_1\beta^2 - 2b_1^3 = 0\\
Cont. &: a_1\alpha\beta + b_1\alpha\gamma - 2a_0^3 - 4a_0 c = 0
\end{aligned} \tag{15}
$$

The following families are obtained by solving above system using computer software wolfram mathematica 9 or maple.

**Family 1:**

$$a_1 \to \frac{-a_0^2 - 2c}{\beta},\ b_1 \to 0,\ \alpha \to -2a_0,\ \beta \to \beta,\ \gamma \to \frac{a_0^2 + 2c}{\beta}. \tag{16}$$

As a consequence, the solutions will have the following forms:

Whenever $\beta^2 - 4\alpha\gamma < 0$ and $\gamma \neq 0$,

$$P(\xi) = a_0 + \frac{-a_0^2 - 2c}{\beta}\left(\frac{-\alpha + \sqrt{-\alpha^2 + 4\beta\gamma}\tan(0.5\sqrt{-\alpha^2 + 4\beta\gamma}\xi)}{2\gamma}\right), \tag{17}$$

or

$$P(\xi) = a_0 + \frac{-a_0^2 - 2c}{\beta}\left(\frac{-\alpha + \sqrt{-\alpha^2 + 4\beta\gamma}\cot(0.5\sqrt{-\alpha^2 + 4\beta\gamma}\xi)}{2\gamma}\right), \tag{18}$$

where $\xi = x + y - c\frac{t^k}{k}$.

Whenever, $\beta^2 - 4\alpha\gamma > 0$ and $\gamma \neq 0$,

$$P(\xi) = a_0 + \frac{-a_0^2 - 2c}{\beta}\left(\frac{-\alpha + \sqrt{-\alpha^2 + 4\beta\gamma}\tanh(0.5\sqrt{-\alpha^2 + 4\beta\gamma}\xi)}{2\gamma}\right), \tag{19}$$

or

$$P(\xi) = a_0 + \frac{-a_0^2 - 2c}{\beta}\left(\frac{-\alpha + \sqrt{-\alpha^2 + 4\beta\gamma}\coth(0.5\sqrt{-\alpha^2 + 4\beta\gamma}\xi)}{2\gamma}\right), \tag{20}$$

where $\xi = x + y - c\frac{t^k}{k}$.

Whenever, $\beta^2 - 4\alpha\gamma = 0$ and $\gamma \neq 0$,

$$P(\xi) = a_0 + \frac{-a_0^2 - 2c}{\beta}\left(\frac{2 - \alpha\xi}{2\gamma\xi}\right), \tag{21}$$

where $\xi = x + y - c\frac{t^k}{k}$.

**Family 2:**

$$a_0 \to -i\sqrt{2}\sqrt{c},\ a_1 \to 0,\ b_1 \to \beta,\ \alpha \to -i2\sqrt{2}\sqrt{c}\ \gamma \to \frac{2c}{\beta},\ \beta = \beta. \tag{22}$$

As a consequence, the soliton solutions have the following forms:

Whenever, $\beta^2 - 4\alpha\gamma < 0$ and $\gamma \neq 0$,

$$P(\xi) = -i\sqrt{2}\sqrt{c} + \beta\left(\frac{2\gamma}{-\alpha + \sqrt{-\alpha^2 + 4\beta\gamma}\tan(0.5\sqrt{-\alpha^2 + 4\beta\gamma}\xi)}\right), \tag{23}$$

or

$$P(\xi) = -i\sqrt{2}\sqrt{c} + \beta\left(\frac{2\gamma}{-\alpha + \sqrt{-\alpha^2 + 4\beta\gamma}\cot(0.5\sqrt{-\alpha^2 + 4\beta\gamma}\xi)}\right), \tag{24}$$

where $\xi = x + y - c\frac{t^k}{k}$.
Whenever, $\beta^2 - 4\alpha\gamma > 0$ and $\gamma \neq 0$,

$$P(\xi) = -i\sqrt{2}\sqrt{c} + \beta\left(\frac{2\gamma}{-\alpha + \sqrt{-\alpha^2 + 4\beta\gamma}\tanh(0.5\sqrt{-\alpha^2 + 4\beta\gamma}\xi)}\right), \tag{25}$$

or

$$P(\xi) = -i\sqrt{2}\sqrt{c} + \beta\left(\frac{2\gamma}{-\alpha + \sqrt{-\alpha^2 + 4\beta\gamma}\coth(0.5\sqrt{-\alpha^2 + 4\beta\gamma}\xi)}\right), \tag{26}$$

where $\xi = x + y - c\frac{t^k}{k}$.
Whenever, $\beta^2 - 4\alpha\gamma = 0$ and $\gamma \neq 0$,

$$P(\xi) = -i\sqrt{2}\sqrt{c} + \beta\left(\frac{2\gamma\xi}{2 - \alpha\xi}\right), \tag{27}$$

where $\xi = x + y - c\frac{t^k}{k}$.

**Family 3:**

$$a_0 \to 0,\ a_1 \to \frac{c}{2\beta},\ b_1 \to -\beta,\ \alpha \to 0,\ \gamma \to \frac{c}{2\beta},\ \beta = \beta. \tag{28}$$

As a consequence, the soliton solutions have the following forms:
Whenever $\beta^2 - 4\alpha\gamma < 0$ and $\gamma \neq 0$,

$$P(\xi) = \frac{c}{2\beta}\left(\frac{\sqrt{4\beta\gamma}\tan(0.5\sqrt{4\beta\gamma}\xi)}{2\gamma}\right) - \beta\left(\frac{2\gamma}{\sqrt{4\beta\gamma}\tan(0.5\sqrt{4\beta\gamma}\xi)}\right), \tag{29}$$

or

$$P(\xi) = \frac{c}{2\beta}\left(\frac{\sqrt{4\beta\gamma}\cot(0.5\sqrt{4\beta\gamma}\xi)}{2\gamma}\right) - \beta\left(\frac{2\gamma}{\sqrt{4\beta\gamma}\cot(0.5\sqrt{4\beta\gamma}\xi)}\right), \tag{30}$$

where $\xi = x + y - c\frac{t^k}{k}$.
Whenever, $\beta^2 - 4\alpha\gamma > 0$ and $\gamma \neq 0$,

$$P(\xi) = \frac{c}{2\beta}\left(\frac{\sqrt{4\beta\gamma}\tanh(0.5\sqrt{4\beta\gamma}\xi)}{2\gamma}\right) - \beta\left(\frac{2\gamma}{\sqrt{4\beta\gamma}\tanh(0.5\sqrt{4\beta\gamma}\xi)}\right), \tag{31}$$

or

$$P(\xi) = \frac{c}{2\beta}\left(\frac{\sqrt{4\beta\gamma}\coth(0.5\sqrt{4\beta\gamma}\xi)}{2\gamma}\right) - \beta\left(\frac{2\gamma}{\sqrt{4\beta\gamma}\coth(0.5\sqrt{4\beta\gamma}\xi)}\right), \tag{32}$$

where $\xi = x + y - c\frac{t^k}{k}$.
Whenever, $\beta^2 - 4\alpha\gamma = 0$ and $\gamma \neq 0$,

$$P(\xi) = \left(\frac{c}{2\beta}\right)\left(\frac{2 - \alpha\xi}{2\gamma\xi}\right) - \beta\left(\frac{2\gamma\xi}{2 - \alpha\xi}\right). \tag{33}$$

where $\xi = x + y - c\frac{t^k}{k}$.

**Family 4:**

$$a_0 \to 0,\ a_1 \to \frac{c}{\beta},\ b_1 \to \beta,\ \alpha \to 0,\ \gamma \to \frac{-c}{\beta},\ \beta = \beta. \tag{34}$$

As a consequence, the soliton solutions have the following forms:
Whenever $\beta^2 - 4\alpha\gamma < 0$ and $\gamma \neq 0$,

$$P(\xi) = \frac{c}{\beta}\left(\frac{\sqrt{4\beta\gamma}\tan(0.5\sqrt{4\beta\gamma}\xi)}{2\gamma}\right) + \beta\left(\frac{2\gamma}{\sqrt{4\beta\gamma}\tan(0.5\sqrt{4\beta\gamma}\xi)}\right), \tag{35}$$

or

$$P(\xi)=\frac{c}{\beta}\left(\frac{\sqrt{4\beta\gamma}\cot(0.5\sqrt{4\beta\gamma}\xi)}{2\gamma}\right)+\beta\left(\frac{2\gamma}{\sqrt{4\beta\gamma}\cot(0.5\sqrt{4\beta\gamma}\xi)}\right), \tag{36}$$

where $\xi\ =x+y-c\frac{t^k}{k}$.

Whenever, $\beta^2-4\alpha\gamma>0$ and $\gamma\neq 0$,

$$P(\xi)=\frac{c}{\beta}\left(\frac{\sqrt{4\beta\gamma}\tanh(0.5\sqrt{4\beta\gamma}\xi)}{2\gamma}\right)+\beta\left(\frac{2\gamma}{\sqrt{4\beta\gamma}\tanh(0.5\sqrt{4\beta\gamma}\xi)}\right), \tag{37}$$

or

$$P(\xi)=\frac{c}{\beta}\left(\frac{\sqrt{4\beta\gamma}\coth(0.5\sqrt{4\beta\gamma}\xi)}{2\gamma}\right)+\beta\left(\frac{2\gamma}{\sqrt{4\beta\gamma}\coth(0.5\sqrt{4\beta\gamma}\xi)}\right), \tag{38}$$

where $\xi\ =x+y-c\frac{t^k}{k}$.

Whenever, $\beta^2-4\alpha\gamma=0$ and $\gamma\neq 0$,

$$P(\xi)=\left(\frac{c}{\beta}\right)\left(\frac{2-\alpha\xi}{2\gamma\xi}\right)-\beta\left(\frac{2\gamma\xi}{2-\alpha\xi}\right), \tag{39}$$

where $\xi\ =x+y-c\frac{t^k}{k}$.

## 4.2 Analysis Of Solution Via The Sardar Sub-Equation Method

In this subsection, the Sardar sub-equation method is applied on to the nonlinear Bogoyavlenskii equations to develop various novel and distinct travelling wave solutions. By homogenous balancing principle, the value of $Z$ is obtained as $Z=1$. According to the mentioned method, Eq.(6) becomes

$$P(\xi)=\nu_0+\nu_1\psi^1(\xi), \tag{40}$$

by substituting Eq. (40) into Eq. (12) and then equating all coefficients of $\psi(\xi)$ to zero, an algebraic system of equations is gained as given below,

$$\psi^3:\ 2\nu_1-2\nu_1^3=0,$$
$$\psi^2:\ -6\nu_0\nu_1^2=0,$$

$$\begin{gathered}\psi^1:\ \nu_1 g-6\nu_0^2\nu_1-4c\nu_1=0,\\ Cnst.:\ -2\nu_0^3-4c\nu_0.\end{gathered} \tag{41}$$

Solve the above system with the aid of computer software maple or mathematica to get the following results

$$\nu_0\to i\sqrt{2}\sqrt{c},\quad \nu_1\to 1,\quad g\to 4c. \tag{42}$$

The formal solutions of Eq. (12) corresponding to the Eq. (40), along with solution Eq. (10) are

**Case 1:**

If $g=4c>0$ and $\alpha=0$, then

$$P_1^{\pm}=i\sqrt{2}\sqrt{c}\pm\sqrt{-4crs}(\mathrm{sech}_{rs}(\sqrt{4c}(x+y-c\frac{t^k}{k}))), \tag{43}$$

$$P_2^{\pm}=i\sqrt{2}\sqrt{c}\pm\sqrt{-4crs}(\mathrm{csch}_{rs}(\sqrt{4c}(x+y-c\frac{t^k}{k}))), \tag{44}$$

**Case 2:**
If $g = 4c < 0$ and $\alpha = 0$, then

$$P_3^{\pm} = i\sqrt{2}\sqrt{c} \pm \sqrt{-4crs}(\sec_{rs}(\sqrt{4c}(x + y - c\frac{t^k}{k}))), \tag{45}$$

$$P_4^{\pm} = i\sqrt{2}\sqrt{c} \pm \sqrt{-4crs}(\csc_{rs}(\sqrt{4c}(x + y - c\frac{t^k}{k}))), \tag{46}$$

**Case 3:**
If $g = 4c < 0$ and $\alpha = (\frac{g}{2})^2$, then

$$P_5^{\pm} = i\sqrt{2}\sqrt{c} \pm \sqrt{\frac{-4c}{2}} \tanh_{rs}\left(\sqrt{\frac{-4c}{2}}(x + y - c\frac{t^k}{k})\right), \tag{47}$$

$$P_6^{\pm} = i\sqrt{2}\sqrt{c} \pm \sqrt{\frac{-4c}{2}} \coth_{rs}\left(\sqrt{\frac{-4c}{2}}(x + y - c\frac{t^k}{k})\right), \tag{48}$$

$$P_7^{\pm} = i\sqrt{2}\sqrt{c}\pm\sqrt{\frac{-4c}{2}}\left( \tanh_{rs}\left(\sqrt{-2(4c)}(x+y-c\frac{t^k}{k})\right)\pm i\sqrt{rs}\,\mathrm{sech}_{rs}\left(\sqrt{-2(4c)}(x+y-c\frac{t^k}{k})\right)\right), \tag{49}$$

$$P_8^{\pm} = i\sqrt{2}\sqrt{c}\pm\sqrt{\frac{-4c}{2}}\left( \coth_{rs}\left(\sqrt{-2(4c)}(x+y-c\frac{t^k}{k})\right)\pm \sqrt{rs}\,\mathrm{csch}_{rs}\left(\sqrt{-2(4c)}(x+y-c\frac{t^k}{k})\right)\right), \tag{50}$$

$$P_9^{\pm} = i\sqrt{2}\sqrt{c} \pm \sqrt{\frac{-4c}{8}}\left( \tanh_{rs}\left(\sqrt{\frac{-4c}{8}}(x + y - c\frac{t^k}{k})\right) \pm \sqrt{rs}\coth_{rs}\left(\sqrt{\frac{-4c}{8}}(x + y - c\frac{t^k}{k})\right)\right). \tag{51}$$

**Case 4:**
If $g = 4c > 0$ and $\alpha = (\frac{g}{2})^2$, then

$$P_{10}^{\pm} = i\sqrt{2}\sqrt{c} \pm \sqrt{\frac{4c}{2}} \tan_{rs}\left(\sqrt{\frac{-4c}{2}}(x + y - c\frac{t^k}{k})\right), \tag{52}$$

$$P_{11}^{\pm} = i\sqrt{2}\sqrt{c} \pm \sqrt{\frac{4c}{2}} \cot_{rs}\left(\sqrt{\frac{-4c}{2}}(x + y - c\frac{t^k}{k})\right), \tag{53}$$

$$P_{12}^{\pm} = i\sqrt{2}\sqrt{c}\pm\sqrt{\frac{4c}{2}}\left( \tan_{rs}\left(\sqrt{2(4c)}(x+y-c\frac{t^k}{k})\right)\pm i\sqrt{rs}\sec_{rs}\left(\sqrt{2(4c)}(x+y-c\frac{t^k}{k})\right)\right), \tag{54}$$

$$P_{13}^{\pm} = i\sqrt{2}\sqrt{c}\pm\sqrt{\frac{4c}{2}}\left( \cot_{rs}\left(\sqrt{2(4c)}(x+y-c\frac{t^k}{k})\right)\pm \sqrt{rs}\csc_{rs}\left(\sqrt{2(4c)}(x+y-c\frac{t^k}{k})\right)\right), \tag{55}$$

$$P_{14}^{\pm} = i\sqrt{2}\sqrt{c} \pm \sqrt{\frac{4c}{8}}\left( \tan_{rs}\left(\sqrt{\frac{4c}{8}}(x + y - c\frac{t^k}{k})\right) \pm \sqrt{rs}\cot_{rs}\left(\sqrt{\frac{4c}{8}}(x + y - c\frac{t^k}{k})\right)\right). \tag{56}$$

## §5 Results and Discussions

In this section, the graphical descriptions of some retrieved wave solutions have been investigated. The graphical illustration is an important tool used for understanding the properties and mechanism of solutions physically. The numerical simulations of the extracted solutions have been presented using 3D-surface graphs and contour graphs. Moreover, the effects for different values of conformable order are graphical illustrated by 2D graphs for some of the obtained solutions. In each figure, (a) shows a 3D-surface graph, (b) shows contour graphs and (c) shows 2D-parametric graphs.

By using the computer software wolfram mathematica, the modified auxiliary equation method and the Sardar sub-equation method are implemented to establish the analytical travelling wave solutions of the nonlinear conformable Bogoyavlenskii equations model. A few of new kind of solutions, that aren't been added to literature previously, are successfully developed in this article.

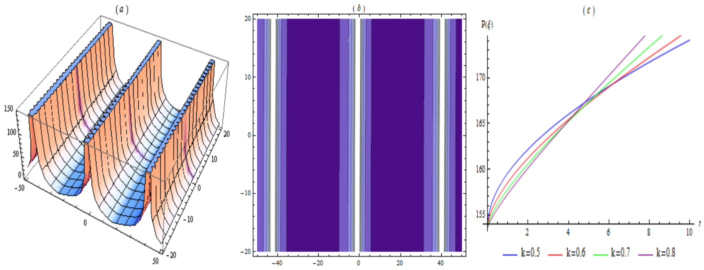


Figure 1. (a), (b), (c): 3D Plot, contour and 2D-parametric (for different values of $k$) representation of the singular periodic shape solution of Eq. (18) for the values of arbitrary constants taken as $a_0 = 1, \alpha = 0.03, \beta = 0.06, \gamma = 0.1, c = 0.02, k = 0.7$ and $y = 1$.

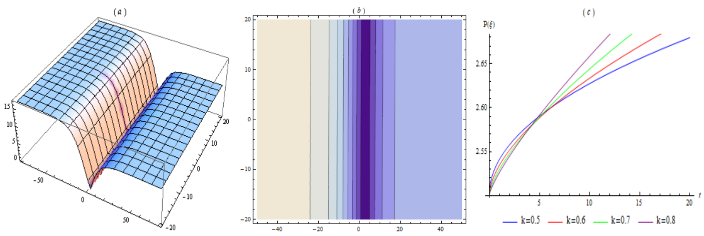


Figure 2. (a), (b), (c): 3D Plot, contour and 2D-parametric (for different values of $k$) representation of the anti-bell shaped bright-dark soliton solution of Eq. (19) for the values of arbitrary constants taken as $a_0 = 1, \alpha = 0.03, \beta = 0.06, \gamma = 0.1, c = 0.02, k = 0.7$ and $y = 1$.

In Fig. 1, 3-dimensional surface plot, contour and 2D-parametric (for different values of k) representation of the singular periodic shape solution of Eq. (18) for the values of arbitrary constants taken as $a_0 = 1, \alpha = 0.03, \beta = 0.06, \gamma = 0.1, c = 0.02, k = 0.7$ and $y = 1$. In Fig. 2, 3-dimensional surface plot, contour and 2D-parametric (for different values of k) representation of the anti-bell shaped soliton solution of Eq. (19) for the values of arbitrary constants taken as $a_0 = 1, \alpha = 0.03, \beta = 0.06, \gamma = 0.1, c = 0.02, k = 0.7$ and $y = 1$. In Fig. 3, 3-dimensional surface plot, contour and 2D-parametric (for different values of k) representation of the kink wave shape solution soliton of Eq. (33) for the values of arbitrary constants taken as $a_0 = 1, \alpha = 0.03, \beta = 0.06, \gamma = 0.1, c = 0.02, k = 0.7$ and $y = 1$. In Fig. 4, 3-dimensional surface plot, contour and 2D-parametric (for different values of k) representation of the peakon-shaped bright soliton of Eq. (43) for the values of arbitrary constants taken as $a_0 = 1, \alpha = 0.03, \beta = 0.06, \gamma = 0.1, c = 0.02, k = 0.7$ and $y = 1$. In Fig. 5, 3-dimensional surface plot, contour and 2D-parametric (for different values of k) representation of the cubic-quadratic soliton of Eq. (54) for the values of arbitrary constants taken as $a_0 = 1, \alpha = 0.03, \beta = 0.06, \gamma = 0.1, c = 0.02, k = 0.7$ and $y = 1$.

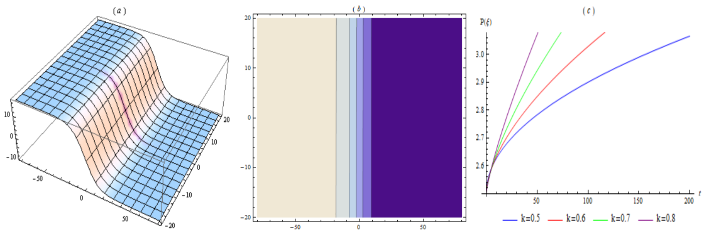


Figure 3. (a), (b), (c): 3D Plot, contour and 2D-parametric (for different values of $k$) representation of the kink wave-shaped solution soliton of Eq. (33) for the values of arbitrary constants taken as $a_0 = 1, \alpha = 0.03, \beta = 0.06, \gamma = 0.1, c = 0.02, k = 0.7$ and $y = 1$.

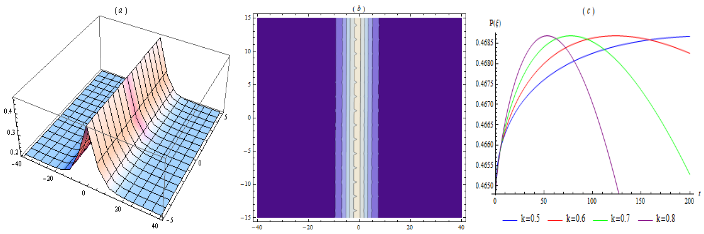


Figure 4. (a), (b), (c): 3D Plot, contour and 2D-parametric (for different values of $k$) representation of the peakon-shaped bright soliton of Eq. (43) for the values of arbitrary constants taken as $a_0 = 1, \alpha = 0.03, \beta = 0.06, \gamma = 0.1, c = 0.02, k = 0.7$ and $y = 1$.

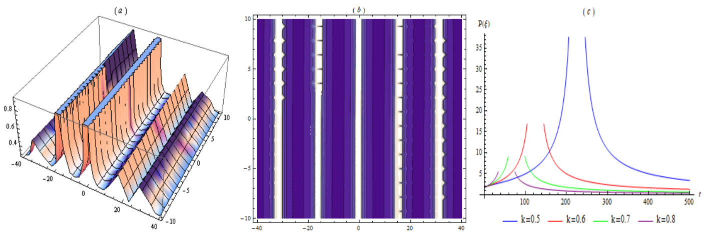


Figure 5. (a), (b), (c): 3D Plot, contour and 2D-parametric (for different values of $k$) representation of the cubic-quadratic soliton of Eq. (54) for the values of arbitrary constants taken as $a_0 = 1, \alpha = 0.03, \beta = 0.06, \gamma = 0.1, c = 0.02, k = 0.7$ and $y = 1$.

The variation in the extracted solutions for change in the value of $k$ is also observed in all Fig. 1-5 (c).

These results offer a sufficient foundation for physicists to redesign new experiments with appropriate initial conditions as recommended by the extracted solution expressions. The obtained wave solutions for the nonlinear conformable Bogoyavlenskii equations will be supportive to conducting laboratory experiments for a deeper insight into the possible physical changes.

## §6 Modulation stability of the nonlinear conformable Bogoyavlenskii equations model

Various nonlinear phenomena exhibit an instability that results the modulation of the steady state owing to the connection among nonlinear as well as the dispersive effects. Here, we investigate the modulation instability of the nonlinear conformable Bogoyavlenskii equations by using the concept of linear stability [54]. Consider the steady-state solutions of the nonlinear conformable Bogoyavlenskii equations is of the following form

$$\begin{gathered} p(x,y,t)=\sqrt{H}+\psi_1(x,y,t)\exp\phi(t),\\ q(x,y,t)=\sqrt{H}+\psi_2(x,y,t)\exp\phi(t),\\ where,\ \phi(t)=H(\alpha_1+H\alpha_2\epsilon)t, \end{gathered}\tag{57}$$

and $H$ represents the normalized optical power. Substituting Eq. (57) into Eq. (1) and linearizing, the following form is obtained:

$$\begin{aligned} 4\psi_1H\alpha+4\psi_2H^2\alpha_2\epsilon+4D_t^\alpha\psi_1+\frac{\partial^3\psi_1}{\partial y\partial x^2}-4H\frac{\partial\psi_1}{\partial y}-4\sqrt{H}\frac{\partial\psi_1}{\partial x}&=0,\\ \frac{\partial\psi_2}{\partial x}&=\sqrt{H}\frac{\partial\psi_1}{\partial y}. \end{aligned}\tag{58}$$

Suppose the solution of Eq. (58) in the form

$$\begin{aligned} \psi_1(x,y,t)&=\beta_1\exp i(\nu(x+y)+\frac{\omega ct^k}{k}),\\ \psi_2(x,y,t)&=\beta_2\exp i(\nu(x+y)+\frac{\omega ct^k}{k}), \end{aligned}\tag{59}$$

where $\omega$ and $\nu$ are the frequency of perturbation and normalized wave number. Putting Eq. (59) into Eq. (58), the following dispersion relation is obtained as

$$\begin{aligned} \beta_2&=\beta_1\sqrt{H},\\ \omega&=\frac{-\nu^3-4\nu(\sqrt{H}+H)+4H(\alpha+\alpha_2H^{3/2}\epsilon)}{4c}. \end{aligned}\tag{60}$$

The above dispersion relations signifies that steady state stability depends upon the normalized wave number and on the normalized optical power. The velocity dispersion $\omega$ is real for all of the wave numbers $\nu$ and if $c\neq 0$, $\sqrt{H}>0$. So the steady state is stable against perturbations.

## §7 Conclusions

In this study, the nonlinear Bogoyavlenskii equations with the conformable derivative are studied using the modified auxiliary equation method and Sardar sub-equation method for the first time. Some obtained solutions that are calculated for the first time shown that the proposed methods are most efficient, simple and more capable to implement. 3D-surface graphs, contour graphs as well as parametric plots of some obtained results are drawn to understand the physical behavior of the obtained solutions to some specific values for the arbitrary parameters. It can be observed that extracted new wave solutions includes some remarkable kind of solutions such as the singular periodic shape solution, anti-bell shaped soliton solution, kink wave shape soliton solution, the peakon shaped bright soliton and the cubic-quadratic soliton. They also agree with earlier observations reported in the literature, demonstrating the validity of the suggested method. The results will help in understanding the possible dynamical behaviors of the suggested problem, which is one of the cardinal focuses. The extracted soliton solutions will be valuable additions to the literature for understanding related physical systems. In the future, the space-time fractional nonlinear Bogoyavlenskii equations can also be examined by other nonlinearity laws and analytical methods.

## Declarations

**Conflict of interest** The authors declare no conflict of interest.

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