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Solution approximations for a mathematical model of relativistic electrons with beta derivative

Ibrahim Yalcinkaya[1] Orkun Tasbozan[2]
Ali Kurt[3] Hijaz Ahmad[4,5]

Abstract. The main aim of this paper is to obtain the exact and semi-analytical solutions of the nonlinear Klein-Fock-Gordon (KFG)equation which is a model of relativistic electrons arising in the laser thermonuclear fusion with beta derivative. For this purpose, both the modified extended tanh-function (mETF) method and the homotopy analysis method (HAM) are used. While applying the mETF the chain rule for beta derivative and complex wave transform are used for obtaining the exact solution. The advantage of this procedure is that discretization or normalization is not required. By applying the mETF, the exact solutions are obtained. Also, by applying the HAM semi-analytical results for the considered equation are acquired. In HAM $\hbar$ curve gives us a chance to find the suitable value of the $\hbar$ for the convergence of the solution series. Also, comparative graphical representations are given to show the effectiveness, reliability of the methods. The results show that the mETF and HAM are reliable and applicable tools for obtaining the solutions of non-linear fractional partial differential equations that involve beta derivative. This study can bring a new perspective for studies on fractional differential equations. On the other hand, it can be said that scientists can apply the considered methods for different mathematical models arising in physics, chemistry, engineering, social sciences and etc. which involves fractional differentiation. Briefly the results may cause a new insight who studies on relativistic electron modelling.

§1 Introduction

Humankind has been in an effort to understand and interpret nature since its existence. The reason for this effort is the attraction of discovering the unknown as well as changing nature to suit his own comfort and desires. As a result of this effort, it is understood that humankind lives in a nonlinear world and to interpret this nonlinearity a toll is needed. After long research,

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scientists realized that the nonlinear phenomena that arise in the world can be modelled mathematically using experimental or theoretical results [1,2,3, 19-29].But they deduced that not only modelling but also obtaining the solution of these nonlinear phenomena is very important.It is seen that using nonlinear equations seems a very good alternative to modelling natural events [45-58]. Especially differential equations are frequently used for this modelling process. In these differential equations, Newtonian concept differentiation is used. But some deficiencies of the well-known Newtonian concept are determined. For instance, integer calculus can't express the historical dependence of the evolution of system analysis by taking global correlation into consideration. Integer calculus fails to coincide with the experimental results. Also, some analytical and numerical methods were improved to obtain the expected results [30-45]. Besides integer calculus does not have a clearer physical significance while describing complicated physical mechanics problems. In the last decades, scientists worked to eliminate these deficiencies of integer order calculus. So, the story of fractional calculus started with these studies. A lot of definitions of fractional derivative and integration are expressed. Each definition that was introduced to the literature over time had an advantage over another definition. For instance, while modelling the real-world applications with Riemann-Liouville definition, the boundary/initial conditions must be given in a fractional type. This difficulty of Riemann-Liouville definition is eliminated by Caputo definition [4,5,6,16,17,18]. Both Riemann-Liouville and Caputo differentiation involve integral form definitions. So the calculations become more complicated and difficult. To apply some numerical methods, discretization or some modifications are needed. Moreover, analytical solutions of mathematical models that involve these definitions can't be obtained. Atangana who works on fractional calculus expressed a new type of fractional differentiation operator called "Beta derivative" [7]. Atangana claimed that this new fractional derivative operator provides basic criteria that need to be satisfied for the operator to be called fractional derivative [7].

Definition 1.[7] Let $a \in \mathbb{R}$ and $g : [a, \infty] \to \mathbb{R}$ be a function. Then β-derivative of g is defined as:

$$ {}_0^A D_t^\beta = \begin{cases} \lim\limits_{\epsilon \to \infty} \frac{g\left(t+\epsilon\left(t+\frac{1}{\Gamma(\beta)}\right)^{1-\beta}\right)-g(t)}{\epsilon}, & \text{for all } t \geq 0, \beta = 0, \\ g(t), & \text{for all } t \geq 0, 0 < \beta \leq 1. \end{cases} $$

where Γ the gamma-function

$$\Gamma(v) = \int_0^\infty t^{v-1} e^{-t} dt.$$

Especially, nowadays great attention is paid to the mathematical models involving beta derivative. For instance, Yusuf et al. [8] employed Ricatti-Bernoulli(RB) sub-ODE, generalized Bernoulli (GB) sub-ODE and generalized tanh (GT) methods to get the exact solutions of Chen-Lee-Liu equation with beta derivative. Zafar et al. [9] used three distinctive methods to get the analytical results for the space-time fractional Peyrard-Bishop model. Gurefe applied the generalized Kudryashov method for the nonlinear fractional partial differential equations with the beta-derivative [10].

In this study, the numerical and exact solutions of the fractional KFG equation [11]

$$D_t^\beta u - D_{xx} u - au - bu^n = 0$$

which is a mathematical model of relativistic electrons arising in laser thermonuclear fusion where $u(x,t)$ describes the dynamical behavior of the relativistic electrons and a, b, n are arbitrary constants. By the way, a new solution series can be expressed by implementing considered methods. In addition, a new insight can be shown who studies relativistic electrons and can compare their experimental results with theoretical results.

To the best of our knowledge, both numerical and analytical solutions of fractional KFG equation with beta derivative appear for the first time in the literature.

§2 Mathematical Methods

In this section, brief descriptions of the mETF [12] and HAM [13,14,15] methods are given.

2.1 Fundamentals of mETF

Consider the following nonlinear partial differential equation including beta derivative with fractional term

$$P\left(u, D_t^{\beta} u, D_x u, D_t^{(2\beta)} u, D_{xx} u, ...\right) = 0 \tag{1}$$

where $D_t^{(2\beta)}$ means two times the sequential beta derivative of function $u(x,t)$. By using the wave transformation

$$\chi = x + \frac{\lambda(\frac{1}{\Gamma(\beta)} + t)^{\beta}}{\beta},$$

Eq. (1) turns into a nonlinear ordinary differential equation (NODE) as

$$P(u, u', u'', ...) = 0 \tag{2}$$

where the prime indicates the Newtonian concept differentiation of function u with respect to the new independent variable χ. The new step is that the solution we are expecting in the form

$$u(\phi) = A_0 + \sum_{j=1}^{m} \phi^j (A_j + B_j \phi^{-2j}) \tag{3}$$

where

$$\phi' = \eta + \phi^2 \tag{4}$$

the parameter η can be evaluated later and $\phi = \phi(\chi)$, $\phi' = \frac{d\phi}{d\chi}$. The parameter m can be determined by using a balancing procedure that depends on matching the highest order nonlinear term with the highest order linear term. Inserting Eq. (3) and Eq. (4) into Eq. (2) will produce a nonlinear equation system with respect to $A_j, B_j, \eta, \lambda (j = 1, 2, ..., n)$ by vanishing all the coefficients. By using computer software,the values of variables $A_j, B_j, \eta, \lambda (j = 1, 2, ..., n)$ can be evaluated. By using these values and the following solution cases for Eq. (4)

Case 1. When $b < 0$ Hyperbolic function solutions

$$\phi(\chi) = -\sqrt{-\eta}\tanh(\sqrt{-\eta}\chi),$$

$$\phi(\chi) = -\sqrt{-\eta}\coth(\sqrt{-\eta}\chi).$$

Case 2. When $b = 0$ Rational function solution

$$\phi(\chi) = \frac{1}{\eta}.$$

Case 3. When $b > 0$ Trigonometric function solutions

$$\phi(\chi) = \sqrt{\eta}\tan(\sqrt{\eta}\chi),$$
$$\phi(\chi) = -\sqrt{\eta}\cot(\sqrt{\eta}\chi)$$

the exact solutions for time fractional nonlinear partial differential equation can be governed.

2.2 Fundamentals of HAM

To explain the process of HAM, let us consider the following differential equation

$$\mathcal{N}\left[u(x,t)\right] = 0 \tag{5}$$

where $\mathcal{N}$ corresponds to the nonlinear operator and x, t are the independent variables of unknown function $u(x,t)$. Thinking of the generalization of HAM, the zeroth order deformation equation [14] has been organized as

$$(1-p)\mathcal{L}\left[\Omega(x,t;p) - u_0(x,t)\right] = p\hbar\mathcal{N}\left[\Omega(x,t;p)\right] \tag{6}$$

which includes the embedding parameter $p \in [0,1]$, auxiliary parameter $\hbar \neq 0$, auxiliary linear operator $\mathcal{L}$, initial condition of $u(x,t)$, and unknown function $\Omega(x,t;p)$. Thus, choosing auxiliary parameters and operators in HAM can be acceptable. For instance, when p is chosen as $p = 0$ and $p = 1$, then the following results can be obtained respectively.

$$\Omega(x,t;0) = u_0(x,t), \Omega(x,t;1) = u(x,t).$$

It means that when the embedding parameter changes from 0 to 1, the solution $\Omega(x,t;p)$ differs from the initial value $u_0(x,t)$ to solution $u(x,t)$. When the $\Omega(x,t;p)$ is expanded in Taylor series due to embedding parameter p, we acquire

$$\Omega(x,t;p) = u_0(x,t) + \sum_{m=1}^{\infty} u_m(x,t)p^m$$

where

$$u_m(x,t) = \left.\frac{1}{m!}\frac{\partial^m\Omega(x,t;p)}{\partial p^m}\right|_{p=0}. \tag{7}$$

For suitable choices of auxiliary linear operator, the auxiliary parameter $\hbar$ and the initial guess above mentioned series converges at $p = 1$ and

$$u(x,t) = u_0(x,t) + \sum_{m=1}^{\infty} u_m(x,t).$$

As shown by Liao [13,14,15], this must be one of the solutions of the nonlinear equation. Due to Eq.(7), a governing equation can be reduced from Eq.(6). Establish the vector:

$$\mathbf{u}_n = \left\{u_0(x,t), u_1(x,t), ..., u_n(x,t)\right\}.$$

The mth-order deformation equation can be procured by differentiating Eq.(6) m times with respect to p and arranging $p = 0$ and dividing by $m!$

$$\mathcal{L}\left[u_m(x,t) - \chi_m u_{m-1}(x,t)\right] = \hbar R_m\left(\mathbf{u}_{m-1}\right) \tag{8}$$

where

$$R_m\left(\mathbf{u}_{m-1}\right) = \left.\frac{1}{(m-1)!}\frac{\partial^{m-1}\mathcal{N}\left[\Omega(x,t;p)\right]}{\partial p^{m-1}}\right|_{p=0}$$

and

$$\chi_m = \begin{cases} 0, & m \le 1, \\ 1, & m > 1. \end{cases}$$

So $u_m(x,t)$ can be acquired from Eq.(8) by the boundary condition for $m \ge 1$.

§3 Applications of Considered Methods

This section will discuss applications of the considered methods for the KFG equation

$$D_t^{\beta} u - D_{xx} u - au - bu^n = 0 \tag{9}$$

where $D_t^{\beta} u$ means the beta derivative of an unknown function $u(x,t)$ with a fractional term.

3.1 Application of mETF

Firstly, using the wave transformation

$$\chi = x + \frac{\lambda(\frac{1}{\Gamma(\beta)} + t)^{\beta}}{\beta}$$

and the chain rule for beta derivative Eq.(9) converts into NODE as follows

$$\lambda u' - u'' - au - bu^n = 0 \tag{10}$$

where the prime indicates the derivative of the function u with respect to χ. Using the balancing procedure in Eq.(10) leads to $m = \frac{2}{n-1}$. Thus, using the transformation $u = \psi^{\frac{2}{n-1}}$ in Eq.(10) and multiplying the obtained equation by $\psi^{\frac{2n-4}{n-1}}$, we acquire the following NODE

$$\frac{2\lambda}{n-1}\psi\psi' - \frac{6-2n}{(n-1)^2}(\psi')^2 - \frac{2}{n-1}\psi\psi'' - a\psi^2 - b\psi^4 = 0. \tag{11}$$

Again using the balancing procedure yields $m = 1$. So by the suggested method, the auxiliary solution of the equation can be regarded as

$$\psi(\chi) = A_0 + A_1\phi(\chi) + B_1(\phi(\chi))^{-1}. \tag{12}$$

Putting Eq.(12) into Eq.(11) and using Eq. (4) an equation arises that includes powers of the function $\phi(\chi)$. By equating all the coefficients of the powers to zero an algebraic equation system comes into existence involving the variables $\eta, \lambda, A_0, A_1, B_1, a, b$. Solving this system with the aid of symbolic computer software, the following solution sets yield.

Set 1:

$$A_1 = -\frac{2A_0(3+n)}{(n-1)\lambda}, a = \frac{2\lambda^2\,(1+n)}{(3+n)^2}, B_1 = \frac{A_0\lambda(1-n)}{8(3+n)}, \eta = -\frac{(-1+n)^2\lambda^2}{16(3+n)^2}, b = -\frac{(1+n)\lambda^2}{2A_0^2(3+n)^2},$$

Set 2:

$$A_1 = -\frac{2A_0(3+n)}{(n-1)\lambda}, a = \frac{2\lambda^2\,(1+n)}{(3+n)^2}, B_1 = 0, \eta = -\frac{(-1+n)^2\lambda^2}{4(3+n)^2}, b = -\frac{(1+n)\lambda^2}{2A_0^2(3+n)^2},$$

Set 3:

$$A_1 = 0, a = \frac{2\lambda^2\,(1+n)}{(3+n)^2}, B_1 = \frac{A_0\lambda(1-n)}{2(3+n)}, \eta = -\frac{(-1+n)^2\lambda^2}{4(3+n)^2}, b = -\frac{(1+n)\lambda^2}{2A_0^2(3+n)^2}.$$

Using the solution sets Eq.(12), the solution cases for Eq. (4) and transformation $u = \psi^{\frac{2}{n-1}}$ the analytical solutions for Eq.(9) can be obtained as

Solutions for Set 1.

$$u_1(x,t) = \left(A_0 - \frac{A_0\lambda(1-n)\coth\left(\frac{1}{4}\sqrt{\frac{(-1+n)^2\lambda^2}{(3+n)^2}}\left(x+\frac{\lambda\left(t+\frac{1}{\Gamma(\beta)}\right)^\beta}{\beta}\right)\right)}{2(3+n)\sqrt{\frac{(-1+n)^2\lambda^2}{(3+n)^2}}} + \frac{(3A_0+A_0n)\sqrt{\frac{(-1+n)^2\lambda^2}{(3+n)^2}}\tanh\left(\frac{1}{4}\sqrt{\frac{(-1+n)^2\lambda^2}{(3+n)^2}}\left(x+\frac{\lambda\left(t+\frac{1}{\Gamma(\beta)}\right)^\beta}{\beta}\right)\right)}{2(-1+n)\lambda}\right)^{\frac{2}{-1+n}},$$

$$u_2(x,t) = \left(A_0 + \frac{(3A_0+A_0n)\sqrt{\frac{(-1+n)^2\lambda^2}{(3+n)^2}}\coth\left(\frac{1}{4}\sqrt{\frac{(-1+n)^2\lambda^2}{(3+n)^2}}\left(x+\frac{\lambda\left(t+\frac{1}{\Gamma(\beta)}\right)^\beta}{\beta}\right)\right)}{2(-1+n)\lambda} - \frac{(A_0\lambda-A_0n\lambda)\tanh\left(\frac{1}{4}\sqrt{\frac{(-1+n)^2\lambda^2}{(3+n)^2}}\left(x+\frac{\lambda\left(t+\frac{1}{\Gamma(\beta)}\right)^\beta}{\beta}\right)\right)}{2(3+n)\sqrt{\frac{(-1+n)^2\lambda^2}{(3+n)^2}}}\right)^{\frac{2}{-1+n}},$$

$$u_3(x,t) = \left(A_0 + \frac{(A_0\lambda-A_0n\lambda)\cot\left(\frac{1}{4}\sqrt{-\frac{(-1+n)^2\lambda^2}{(3+n)^2}}\left(x+\frac{\lambda\left(t+\frac{1}{\Gamma(\beta)}\right)^\beta}{\beta}\right)\right)}{2(3+n)\sqrt{-\frac{(-1+n)^2\lambda^2}{(3+n)^2}}} - \frac{(3A_0+A_0n)\sqrt{-\frac{(-1+n)^2\lambda^2}{(3+n)^2}}\tan\left(\frac{1}{4}\sqrt{-\frac{(-1+n)^2\lambda^2}{(3+n)^2}}\left(x+\frac{\lambda\left(t+\frac{1}{\Gamma(\beta)}\right)^\beta}{\beta}\right)\right)}{2(-1+n)\lambda}\right)^{\frac{2}{-1+n}},$$

$$u_4(x,t) = \left(A_0 + \frac{(3A_0+A_0n)\sqrt{-\frac{(-1+n)^2\lambda^2}{(3+n)^2}}\cot\left(\frac{1}{4}\sqrt{-\frac{(-1+n)^2\lambda^2}{(3+n)^2}}\left(x+\frac{\lambda\left(t+\frac{1}{\Gamma(\beta)}\right)^\beta}{\beta}\right)\right)}{2(-1+n)\lambda} - \frac{(A_0\lambda-A_0n\lambda)\tan\left(\frac{1}{4}\sqrt{-\frac{(-1+n)^2\lambda^2}{(3+n)^2}}\left(x+\frac{\lambda\left(t+\frac{1}{\Gamma(\beta)}\right)^\beta}{\beta}\right)\right)}{2(3+n)\sqrt{-\frac{(-1+n)^2\lambda^2}{(3+n)^2}}}\right)^{\frac{2}{-1+n}}.$$

Solutions for Set 2.

$$u_5(x,t) = \left(A_0 + \frac{(3A_0+A_0n)\sqrt{\frac{(-1+n)^2\lambda^2}{(3+n)^2}}\tanh\left(\frac{1}{2}\sqrt{\frac{(-1+n)^2\lambda^2}{(3+n)^2}}\left(x+\frac{\lambda\left(t+\frac{1}{\mathrm{Gamma}[\beta]}\right)^\beta}{\beta}\right)\right)}{(-1+n)\lambda}\right)^{\frac{2}{-1+n}},$$

$$u_6(x,t) =$$

$$\left(A_0+\frac{(3A_0+A_0n)\sqrt{\frac{(-1+n)^2\lambda^2}{(3+n)^2}}\coth\left(\frac{1}{2}\sqrt{\frac{(-1+n)^2\lambda^2}{(3+n)^2}}\left(x+\frac{\lambda\left(t+\frac{1}{\mathrm{Gamma}[\beta]}\right)^\beta}{\beta}\right)\right)}{(-1+n)\lambda}\right)^{\frac{2}{-1+n}},$$

$$u_7(x,t)=$$

$$\left(A_0-\frac{(3A_0+A_0n)\sqrt{-\frac{(-1+n)^2\lambda^2}{(3+n)^2}}\tan\left(\frac{1}{2}\sqrt{-\frac{(-1+n)^2\lambda^2}{(3+n)^2}}\left(x+\frac{\lambda\left(t+\frac{1}{\mathrm{Gamma}[\beta]}\right)^\beta}{\beta}\right)\right)}{(-1+n)\lambda}\right)^{\frac{2}{-1+n}},$$

$$u_8(x,t)=\left(A_0+\frac{(3A_0+A_0n)\sqrt{-\frac{(-1+n)^2\lambda^2}{(3+n)^2}}\cot\left(\frac{1}{2}\sqrt{-\frac{(-1+n)^2\lambda^2}{(3+n)^2}}\left(x+\frac{\lambda\left(t+\frac{1}{\Gamma(\beta)}\right)^\beta}{\beta}\right)\right)}{(-1+n)\lambda}\right)^{\frac{2}{-1+n}}.$$

Solutions for Set 3.

$$u_9(x,t)=$$

$$\left(A_0-\frac{(A_0\lambda-A_0n\lambda)\coth\left(\sqrt{-\frac{\frac{(1+n)\lambda^2}{2(3+n)^2}-\frac{n(1+n)\lambda^2}{(3+n)^2}+\frac{n^2(1+n)\lambda^2}{2(3+n)^2}}{-2-2n}}\left(x+\frac{\lambda\left(t+\frac{1}{\Gamma(\beta)}\right)^\beta}{\beta}\right)\right)}{2(3+n)\sqrt{-\frac{\frac{(1+n)\lambda^2}{2(3+n)^2}-\frac{n(1+n)\lambda^2}{(3+n)^2}+\frac{n^2(1+n)\lambda^2}{2(3+n)^2}}{-2-2n}}}\right)^{\frac{2}{-1+n}},$$

$$u_{10}(x,t)=$$

$$\left(A_0-\frac{(A_0\lambda-A_0n\lambda)\tanh\left(\sqrt{-\frac{\frac{(1+n)\lambda^2}{2(3+n)^2}-\frac{n(1+n)\lambda^2}{(3+n)^2}+\frac{n^2(1+n)\lambda^2}{2(3+n)^2}}{-2-2n}}\left(x+\frac{\lambda\left(t+\frac{1}{\Gamma(\beta)}\right)^\beta}{\beta}\right)\right)}{2(3+n)\sqrt{-\frac{\frac{(1+n)\lambda^2}{2(3+n)^2}-\frac{n(1+n)\lambda^2}{(3+n)^2}+\frac{n^2(1+n)\lambda^2}{2(3+n)^2}}{-2-2n}}}\right)^{\frac{2}{-1+n}},$$

$$u_{11}(x,t)=$$

$$\left(A_0+\frac{(A_0\lambda-A_0n\lambda)\cot\left(\sqrt{\frac{\frac{(1+n)\lambda^2}{2(3+n)^2}-\frac{n(1+n)\lambda^2}{(3+n)^2}+\frac{n^2(1+n)\lambda^2}{2(3+n)^2}}{-2-2n}}\left(x+\frac{\lambda\left(t+\frac{1}{\Gamma(\beta)}\right)^\beta}{\beta}\right)\right)}{2(3+n)\sqrt{\frac{\frac{(1+n)\lambda^2}{2(3+n)^2}-\frac{n(1+n)\lambda^2}{(3+n)^2}+\frac{n^2(1+n)\lambda^2}{2(3+n)^2}}{-2-2n}}}\right)^{\frac{2}{-1+n}},$$

$$u_{12}(x,t)=$$

$$\left(A_0-\frac{(A_0\lambda-A_0n\lambda)\tan\left(\sqrt{\frac{\frac{(1+n)\lambda^2}{2(3+n)^2}-\frac{n(1+n)\lambda^2}{(3+n)^2}+\frac{n^2(1+n)\lambda^2}{2(3+n)^2}}{-2-2n}}\left(x+\frac{\lambda\left(t+\frac{1}{\Gamma(\beta)}\right)^\beta}{\beta}\right)\right)}{2(3+n)\sqrt{\frac{\frac{(1+n)\lambda^2}{2(3+n)^2}-\frac{n(1+n)\lambda^2}{(3+n)^2}+\frac{n^2(1+n)\lambda^2}{2(3+n)^2}}{-2-2n}}}\right)^{\frac{2}{-1+n}}.$$

As it is seen from the obtained results we get both trigonometric and hyperbolic function solutions. These results have never been seen in the literature so far. All of them are new and original results. Also, solutions can show a new perspective to the scientists who study relativistic electron models.

3.2 Application of HAM

Consider the time fractional KFG equation (9) for $n = 3$ with the initial condition

$$u(x,0) = A_0 + \frac{A_0\coth\left(\frac{1}{12}\lambda\left(x + \frac{\lambda\left(\frac{1}{\Gamma(\beta)}\right)^{\beta}}{\beta}\right)\right)}{2} + \frac{A_0\tanh\left(\frac{1}{12}\lambda\left(x + \frac{\lambda\left(\frac{1}{\Gamma(\beta)}\right)^{\beta}}{\beta}\right)\right)}{2} \tag{13}$$

where $0 < \beta < 1$ and the fractional derivative is in the beta derivative sense. To get the series solution of Eq. (9) with initial condition (13) we can choose the linear operator as

$$\mathcal{L}\left[\Omega(x,t;p)\right] = D_t^{\beta}\Omega(x,t;p)$$

which satisfies the following property for a constant s

$$\mathcal{L}\left[s\right] = 0.$$

From Eq. (9) choice of nonlinear operator can be shown as

$$\mathcal{N}\left[\Omega(x,t;p)\right] = \frac{\partial^{\beta}\Omega(x,t;p)}{\partial t^{\beta}} - \frac{\partial^{2}\Omega(x,t;p)}{\partial x^{2}} - a\Omega(x,t;p) - b\Omega(x,t;p)^{3}. \tag{14}$$

So the deformation equation for zeroth order can be concluded as

$$(1-p)\mathcal{L}\left[\Omega(x,t;p) - u_0(x,t)\right] = p\hbar\mathcal{N}\left[\Omega(x,t;p)\right].$$

For the choices $p = 0$ and $p = 1$ we acquire

$$\Omega(x,t;0) = u_0(x,t), \Omega(x,t;1) = u(x,t).$$

As we mentioned before while the embedding parameter p differs from 0 to 1, $\Omega(x,t;p)$ changes from the initial value $u_0(x,t)$ to $u(x,t)$. Expanding $\Omega(x,t;p)$ in Taylor series due to parameter p

$$\Omega(x,t;p) = u_0(x,t) + \sum_{m=1}^{\infty} u_m(x,t)p^m \tag{15}$$

where

$$u_m(x,t) = \left.\frac{1}{m!}\frac{\partial^m\Omega(x,t;p)}{\partial p^m}\right|_{p=0}. \tag{16}$$

Eq. (15) converges at $p = 1$ for the appropriate choices of the initial guess and the auxiliary parameter $\hbar$ and

$$u(x,t) = u_0(x,t) + \sum_{m=1}^{\infty} u_m(x,t).$$

must be one of the solutions of the nonlinear equation. Equations in display format are separated from the paragraphs of the text. Differentiating Eq. (15) with respect to p yields mth order deformation equation

$$\mathcal{L}\left[u_m(x,t) - \chi_m u_{m-1}(x,t)\right] = \hbar R_m\left(\mathbf{u}_{m-1}\right) \tag{17}$$

where

$$\begin{aligned} R_m\left(\mathbf{u}_{m-1}\right) \quad = \quad & \frac{\partial^{\beta}u_{m-1}(x,t)}{\partial t^{\beta}} - \frac{\partial^{2}u_{m-1}(x,t)}{\partial x^{2}} - au_{m-1}(x,t) \\ & -b\sum_{n=0}^{m-1}\left(\sum_{k=0}^{n} u_k(x,t)u_{n-k}(x,t)\right)u_{m-1-n}(x,t) \end{aligned}$$

and

$$\chi_m = \begin{cases} 0, & m \leq 1, \\ 1, & m > 1. \end{cases}$$

The solution of Eq. (17) for $m \geq 1$ arises in

$$u_m(x,t) = \chi_m u_{m-1}(x,t) + \hbar \mathcal{L}^{-1}[R_m(\mathbf{u}_{m-1})]. \tag{18}$$

Considering Eq. (18) with the initial condition Eq. (15) we have

$$\begin{aligned} u_0(x,t) &= A_0 + \frac{A_0 \coth\left(\frac{1}{12}\lambda\left(x + \frac{\lambda\left(\frac{1}{\Gamma(\beta)}\right)^\beta}{\beta}\right)\right)}{2} + \frac{A_0 \tanh\left(\frac{1}{12}\lambda\left(x + \frac{\lambda\left(\frac{1}{\Gamma(\beta)}\right)^\beta}{\beta}\right)\right)}{2} \\ u_1(x,t) &= -\frac{1}{36\beta} A_0 \hbar \left(1 + \coth\left(\frac{1}{6}\lambda\left(x + \frac{\lambda\Gamma(\beta)^{-\beta}}{\beta}\right)\right)\right) \operatorname{csch}\left(\frac{1}{6}\lambda\left(x + \frac{\lambda\Gamma(\beta)^{-\beta}}{\beta}\right)\right)^2 \\ &\times \; \Gamma(\beta)^{-\beta}\left(-1 + (1 + t\Gamma(\beta))^\beta\right)\left(-18a + \lambda^2 + \left(18a + 36A_0^2 b + \lambda^2\right)\right. \\ &\times \; \left.\cosh\left(\frac{1}{3}\lambda\left(x + \frac{\lambda\Gamma(\beta)^{-\beta}}{\beta}\right)\right) + \left(36A_0^2 b - \lambda^2\right)\sinh\left(\frac{1}{3}\lambda\left(x + \frac{\lambda\Gamma(\beta)^{-\beta}}{\beta}\right)\right)\right) \\ &\vdots \end{aligned}$$

So, the series of solutions found out by HAM can be written in the form

$$u(x,t) = u_0(x,t) + u_1(x,t) + u_2(x,t) + \ldots. \tag{19}$$

To show the convergence of the above-mentioned series, it is very important to obtain the suitable range for the auxiliary parameter $\hbar$. For this aim, we use $\hbar$-curve given in Fig.1, Fig.4 and Fig.7 to choose a suitable value of $\hbar$ for the convergence of the solution series. This suitable value for $\hbar$ can be determined by finding the valid region of $\hbar$ which corresponds to the line segments nearly parallel to the horizontal axis. To show the convergence of the series solution, the graphical representations can be given for different values of β. As it is seen from the comparative Fig.2, 3, 5, 6, 8 and 9, the graphics of the approximate solutions are similar to the graphics of analytical solutions. This situation is provided by choosing the most appropriate value of $\hbar$ that obtained from the novel $\hbar$-curves. Also, this situation shows the efficiency, reliability and applicability of the so-called HAM method.

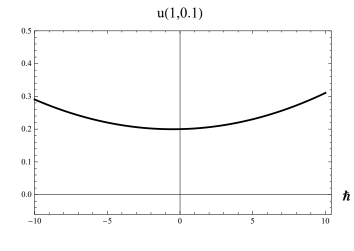

Figure 1. The $\hbar$-curve for 3rd order approximate solution obtained by HAM for $\beta = 0.9$.

The figures indicate that the approximate solutions are very close to the exact solutions. This means that HAM is an efficient, reliable and applicable technique for the solutions of fractional mathematical models which involve the beta fractional derivative.

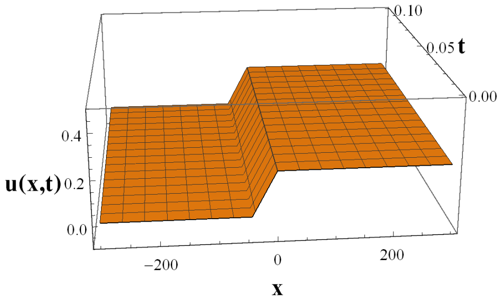

Figure 2. The graphical representation of 3rd order approximate solution obtained by HAM for $A_0 = 0.1, a = -0.05, b = -0.001, \beta = 0.9, \lambda = 10, \hbar = -1$ and $-300 \le x \le 300, 0 \le t \le 0.01$.

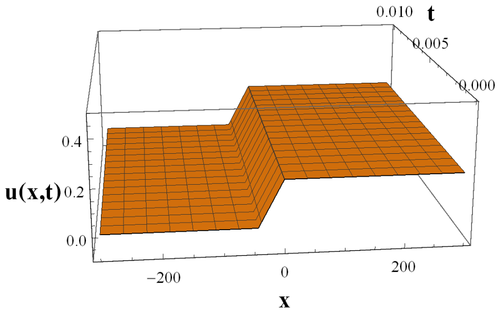

Figure 3. The graphical representation $u_1(x,t)$ for $A_0 = 0.1, a = -0.05, b = -0.001, \beta = 0.9, \lambda = 10$ and $-300 \le x \le 300, 0 \le t \le 0.01$.

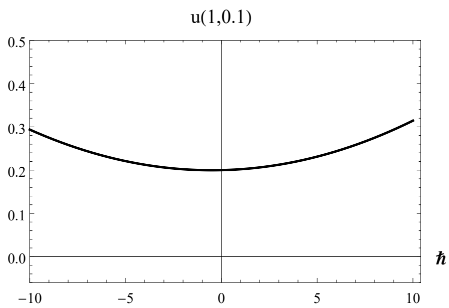

Figure 4. The $\hbar$-curve for 3rd order approximate solution obtained by HAM for $\beta = 0.75$.

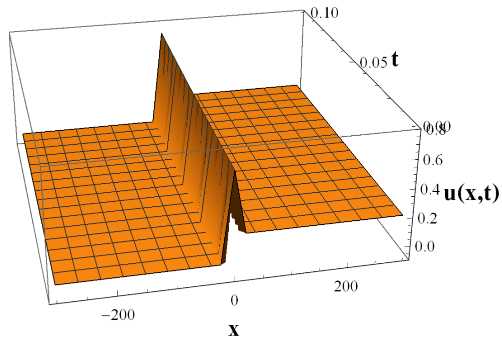

Figure 5. The graphical representation of 3rd order approximate solution obtained by HAM for $A_0 = 0.1, a = -0.05, b = -0.001, \beta = 0.75, \lambda = 1, \hbar = -1$ and $-300 \leq x \leq 300, 0 \leq t \leq 0.01$.

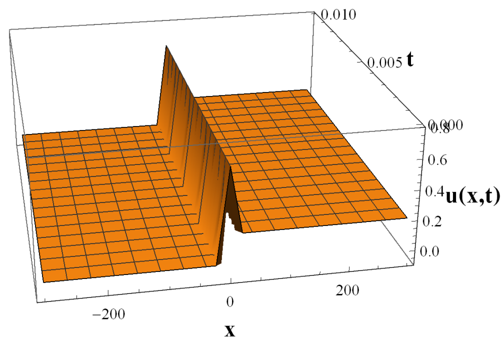

Figure 6. The graphical representation $u_1(x,t)$ for $A_0 = 0.1, a = -0.05, b = -0.001, \beta = 0.75, \lambda = 1$ and $-300 \leq x \leq 300, 0 \leq t \leq 0.01$.

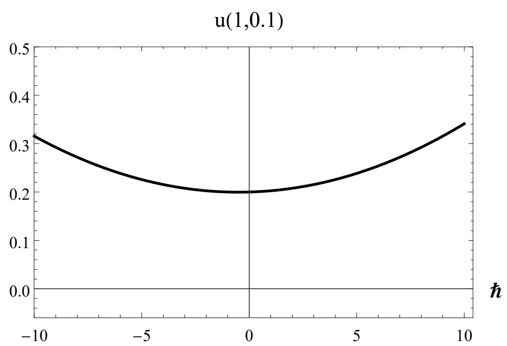

Figure 7. The $\hbar$-curve for 3-rd order approximate solution obtained by HAM for $\beta = 0.5$.

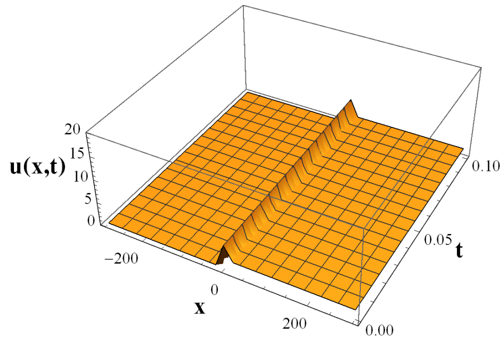

Figure 8. The graphical representation of 3rd order approximate solution obtained by HAM for $A_0 = 1, a = -0.05, b = -0.1, \beta = 0.5, \lambda = 1, \hbar = -1$ and $-300 \leq x \leq 300, 0 \leq t \leq 0.1$.

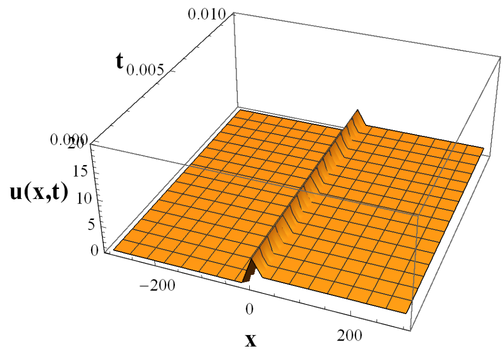

Figure 9. The graphical representation $u_1(x, t)$ for $A_0 = 1, a = -0.05, b = -0.1, \beta = 0.5, \lambda = 1$ and $-300 \leq x \leq 300, 0 \leq t \leq 0.1$.

§4 Conclusion

Authors obtained the analytical and semi-analytical solutions of KFG equations by using mETF and HAM respectively where the fractional term using Atangana's beta derivative. To show the effectiveness, reliability and applicability of the considered methods, comparative graphical representations of some solutions for different values of the fractional term β and other parameters. To the best of our knowledge, both of the mETF and HAM methods are applied firstly to a mathematical model that involves fractional terms in the sense of beta derivative. All the results and graphics show both of these methods can attract the attention of the scientists that studying on mathematical models involving fractional terms. This study may open a new path and a new sight who study and want to compare the experimental results with the theoretical findings especially on relativistic electron models.

Declarations

Conflict of interest The authors declare no conflict of interest.

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[1]Department of Mathematics and Computer Sciences, Faculty of Science, Necmettin Erbakan University, Konya, Turkey.

Email: iyalcinkaya@erbakan.edu.tr

[2]Department of Mathematics, Faculty of Science and Literature, Hatay Mustafa Kemal University, Hatay, Turkey.

Email: otasbozan@mku.edu.tr

[3]Department of Mathematics, Faculty of Science, Pamukkale University, Denizli, Turkey.

Email: akurt@pau.edu.tr

[4]Near East University, Operational Research Center in Healthcare, TRNC Mersin 10, Nicosia, 99138, Turkey.

[5]Department of Mathematics, Faculty of Science, Islamic University of Madinah, Madinah, Saudi Arabia.

Email: hijaz555@gmail.com