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Analytical solutions fractional order partial differential equations arising in fluid dynamics

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Abstract. This article describes the solution procedure of the fractional Pade-II equation and generalized Zakharov equation(GSEs) using the sine-cosine method. Pade-II is an important nonlinear wave equation modeling unidirectional propagation of long-wave in dispersive media and GSEs are used to model the interaction between one-dimensional high, and low-frequency waves. Classes of trigonometric and hyperbolic function solutions in fractional calculus are discussed. Graphical simulations of the numerical solutions are flaunted by MATLAB.

§1 Introduction

The theory related to fractional calculus are paramount importance to describe various phenomenas in the fields of applied physics, applied mathematics, and along with their engineering counter parts. Researchers around the globe are devoted to the interpretation, properties and applications of fractional calculus. [1–8]. Some important nonlinear physical phenomena successfully studied by fractional calculus such as: transmission of the impulses inside nerve [1,2], population growth dynamics model in biology [3], quantum field theory [4], reaction-diffusion equation in hydrology [5], applications of fractional calculus in physics [6,7], fluid mechanics [8]. Nonlinear fractional differential equations (NLFDEs) have been progressively concentrated by numerous scientists working in different fields of science [9–12]. The main target of the manuscript is to investigate the exact travelling wave solutions for the fractional Pade-II equation and fractional generalized Zakharov equation using sine-cosine method [13–15]. In last few decades, there has been remarkable progress in the field of NLFDEs and a large number of well-established methods such as the RB sub-ODE method [16], new stochastic robust solver method [17], new stochastic solutions [18], the jacobi elliptic functions method [19], $(\frac{G'}{G})$-expansion method [21–24, 33], modified $(\frac{G'}{G^2})$-expansion method [25], the Kudryashov

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method [26] , the F-expansion method [28], the direct algebraic method [29], have been used by various researchers.

It is fascinating to find physical solutions of NLFDEs and remains a open problem for scientist and mathematicians. Fractional calculus, hot topic for researchers but celebrated its triumph in pure mathematics without physical applications. Riemann-Liouville derivative and its other forms are used to find their applications. The intent of this article is implementing He's fractional derivative [27] as follows:

$$D_t^{\alpha} f = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_{t_0}^{t_n} (s-t)^{n-\alpha-1} [f_0(s) - f(s)] df, \tag{1.1}$$

It is notable that, $f(s)$ stands for solution in a discontinuous medium where as $f_0(s)$ is its counterpart for a continuous medium without violating the initial boundary conditions.

There are few studies in the literature that concentrate on the Pade-II equation. Richard and co-authors presented the main diagonal algorithm theory [30]. John P. Boyd [31] investigated pade approximation for nonlinear ordinary differential equation (NLODE) using the fact boundary value problem. The derivation of the Pade (2,2) approximation based on the phase velocity of the linear water waves approximated by Fetecau *et al.* having bounded dispersion relation [32]. Yazhou et al. conducted the wave variable method by implementing $(\frac{G'}{G})$-expansion method to obtain travelling wave solutions [33]. Therefore, to understand the long water wave dynamics in dispersive media Pade-II equation is very meaningful. The mathematical form of fractional order Pade-II equation

$$\frac{\partial^{\alpha} g}{\partial t^{\alpha}} + \frac{\partial^{\eta} g}{\partial x^{\eta}} + g\frac{\partial^{\eta} g}{\partial x^{\eta}} - (\frac{9}{10})\frac{\partial^{3\eta} u}{\partial x^{3\eta}} - (\frac{19}{10})\frac{\partial^{2\eta} u}{\partial x^{2\eta}}\frac{\partial^{\alpha} u}{\partial t^{\alpha}} = 0,\ t > 0,\ 0 < \alpha \leq 1, \tag{1.2}$$

with the initial condition $u(x, o) = u_0$, where α is the order of fractional derivatives for both time domain t and space domain x. When α has special values 1, then Eq.(1.2) becomes a classical Pade-II equation

$$g_t + g_x + gg_x - (\frac{9}{10})g_{xxx} - (\frac{19}{10})g_{xxt} = 0 \tag{1.3}$$

It is well known that the generalized Zakharov equation is used to study plasmonic waves. There is significant research work on a GZEs, Bao *et al.* [34] studied the GZEs using numerical methods. Zhang et al. [35] obtained solitary wave solutions using a variational iteration approach. Yasir Khana and others [36], investigated GZEs using he's variational approach and reported few new soliton solutions. Time-space fractional GZEs are studied by Zakia and Toufik [37] for some travelling wave solutions. Lu *et al.* [38] considered fractional order GZEs in their research work and recently Veeresha and Prakash [39] also considered fractional GZEs with importance to Mittag-Leffler functions. Some more authors have also studied this important model [40–42]. The mathematical form of fractional order generalized Zakharov equation

$$\begin{aligned} & i\frac{\partial^{\alpha} g}{\partial t^{\alpha}} + \frac{\partial^{2\eta} g}{\partial x^{2\eta}} + 2\gamma|g|^2 g + 2gh = 0, \\ & \frac{\partial^{2\alpha} h}{\partial t^{2\alpha}} - \frac{\partial^{2\eta} h}{\partial x^{2\eta}} + \frac{\partial^{2\eta}|g|^2}{\partial x^{2\eta}} = 0,\ t > 0,\ 0 < \alpha \leq 1. \end{aligned} \tag{1.4}$$

with the initial condition $u(x, o) = u_0$, where α is the order of fractional derivatives for both time domain t and space domain x. When α have special values 1, then Eq.(1.4) becomes generalized Zakharov equations (GZEs). The generalized Zakharov equations (GZEs) for the complex envelope $g(x,t)$ of the high-frequency wave and the real low-frequency field $h(x,t)$ in the form

$$\begin{aligned} &ig_t + g_{xx} + 2\gamma|g|^2 g + 2gh = 0, \\ &h_{tt} - h_{xx} + (|g|^2)_{xx} = 0. \end{aligned} \tag{1.5}$$

The structure of the manuscript is as follows. In Section 2, we introduce the concepts of the sine cosine method. In Section 3, the travelling wave transformation is presented. In Section 4, and 5, the travelling wave solutions of fractional Pade-II equation and fractional generalized Zakharov equations are discussed in detail. Finally, the conclusion is outlined in Section 6.

§2 Properties of fractional derivatives and mathematical formulation of the sine-cosine method

Some notable properties of space-time fractional derivatives are presented in the form of the following definitions: Definition [4]: Let $\alpha \in (0,1)$, where g(t) and f(t) are α-differentiable at t<0.

1. $D^\alpha(cf + dg) = cD^\alpha(f) + dD^\alpha(f)$.

2. $D^\alpha(t^p) = pt^{p-\alpha}$, for all p ϵR.

3. $D^\alpha(\lambda) = 0$, for all constant functions$f(t) = \lambda$.

4. $D^\alpha(fg) = fD^\alpha(g) + gD^\alpha(f)$.

5. $D^\alpha(fg)(t) = \frac{gD^\alpha(f) - fD^\alpha(g)}{g^2}$.

6. $D^\alpha f(t) = t^{1-\alpha}\frac{\partial f(t)}{\partial t}$, if f is differentiable.

A NLFDE is presented as a combination of some dependent and independent terms and some of their fractional order partial derivatives.

$$S(g, D^\alpha_{\mathrm{t}} g, D^\alpha_{\mathrm{x}} g, D^{2\alpha}_{\mathrm{tt}} g, D^{2\alpha}_{\mathrm{xx}} g, D^{2\alpha}_{\mathrm{xt}} g...) = 0, \quad 0 < \alpha \leq 1 \tag{2.1}$$

Left-hand side of Eq. (2.1) is the polynomial of $g(x,t)$. The travelling wave solution to Eq. (2.1) by the use of sine-cosine method is presented as the following 3 steps:

Step 1:Analytical traveling wave solutions of Eq.(2.1) can be obtained considering the complex wave variable [37] as below;

$$g(x,t) = G(\phi), \phi = \frac{kx^\alpha}{\Gamma(\alpha+1)} - \frac{ct^\alpha}{\Gamma(\alpha+1)} \tag{2.2}$$

The following change for higher orders can be realized as:

$$\begin{cases} \frac{\partial}{\partial t} = -c\frac{\partial}{\partial \phi}, \\ \frac{\partial^2}{\partial t^2} = c^2\frac{\partial^2}{\partial \phi^2}, \\ \frac{\partial}{\partial x} = \frac{\partial}{\partial \phi}, \\ \frac{\partial^2}{\partial x^2} = \frac{\partial^2}{\partial \phi^2}, \\ \ldots\ldots\ldots\ldots, \end{cases} \tag{2.3}$$

Now, Eq.(2.1) reduced to an ordinary differential equation(ODE) by the use of Eq. (2.2),

$$S(G, G_\phi, G_{\phi\phi}, G_{\phi\phi\phi}, ...) = 0. \tag{2.4}$$

Integrating the reduced ODE (2.4) to a comparatively simpler form in such a way that it contains all G and its derivatives where G_ϕ stands for $\frac{dG}{d\phi}$, for simplicity, assume the constant of integration as zero.

Step 2: The general solutions of the NLFDE (2.4) by sine-cosine method in the (cos) form expressed as

$$G(x,t) = \lambda \cos^\beta(\mu\phi), [\phi] < \frac{\pi}{2\mu}, \tag{2.5}$$

similarly in the (sin) form expressed as

$$G(x,t) = \lambda \sin^\beta(\mu\phi), [\phi] < \frac{\pi}{2\mu}, \tag{2.6}$$

Where λ, β and μ are mere constants, μ is the wave number and c is the wave speed. Eq. (2.5) can be generalized

$$\begin{cases} G(\phi) = \lambda \cos^\beta(\mu\phi), \\ G^n(\phi) = \lambda^n \cos^{n\beta}(\mu\phi), \\ (G^n)_\phi = n\mu\beta\lambda^n \cos(\mu\phi)\sin^{n\beta-1}(\mu\phi) \\ (G^n)_{\phi\phi} = -n^2\mu^2\beta^2\lambda^n \cos^n \beta(\mu\phi) + n\mu^2\lambda^n\beta(n\beta-1)\cos^{n\beta-2}(\mu\phi). \end{cases} \tag{2.7}$$

Similarly Eq. (2.6) can be generalized

$$\begin{cases} G(\phi) = \lambda \sin^\beta(\mu\phi), \\ G^n(\phi) = \lambda^n \sin^{n\beta}(\mu\phi), \\ (G^n)_\phi = n\mu\beta\lambda^n \sin(\mu\phi)\cos^{n\beta-1}(\mu\phi) \\ (G^n)_{\phi\phi} = -n^2\mu^2\beta^2\lambda^n \sin^n \beta(\mu\phi) + n\mu^2\lambda^n\beta(n\beta-1)\sin^{n\beta-2}(\mu\phi). \end{cases} \tag{2.8}$$

Step 3: By proper substation of Eq. (2.7) or Eq. (2.8) into Eq. (2.2) gives trigonometric equations in terms of $\cos^\beta(\mu\phi)$ or $\sin^\beta(\mu\phi)$. After algebraic simplification of obtained trigonometric equation α, β and μ can be found out.

§3 Application of The sine-cosine method

In this section, sine-cosine method is considered to simplify two NLFDEs: fractional order Pade-II equation and fractional order generalised Zakharov equation.

3.1 The fractional order Pade-II equation

By the use of complex wave transformation $g(x,t) = G(\phi)$ where the wave variable $\phi = \frac{kx^\alpha}{\Gamma(\alpha+1)} - \frac{Vt^\alpha}{\Gamma(\alpha+1)}$, and V is the velocity of the wave, Eq.(1.2) is reduced to an ODE.

$$(1-V)G + \frac{1}{2}G^2 + (\frac{19}{10}V - \frac{9}{10})G_{\phi\phi} = 0. \tag{3.1}$$

By considering the trigonometric solution in (cos) form the following changes can be realized

$$G(\phi) = \lambda \cos^\beta(\mu\phi), \tag{3.2}$$

$$(G)_\phi = -\mu\beta\lambda \sin^\beta(\mu\phi)\cos^{\beta-1}(\mu\phi), \tag{3.3}$$

$$(G)_{\phi\phi} = -\mu^2\beta^2\lambda\cos^\beta(\mu\phi) + \mu^2\lambda\beta(\beta-1)\cos^{\beta-2}(\mu\phi). \tag{3.4}$$

After suitable substitution, Eq. (3.2) and Eq. (3.4) in Eq. (3.1), we have

$$(1-V)\lambda\cos^\beta\mu\phi - \frac{1}{2}\lambda^2\cos^{2\beta}\mu\phi + (\frac{19}{10}V - \frac{9}{10})\left(\lambda\mu^2\beta(\beta-1)\cos^{\beta-2}(\mu\phi) - \lambda\mu^2\beta^2\cos^\beta(\mu\phi)\right) = 0. \tag{3.5}$$

Now collecting the coefficients by homogeneous balance rule and equating them to zero separately, the following algebraic systems come into picture:

$$(\beta - 1) \neq 0, \tag{3.6}$$

$$(\beta - 2) = 2\beta, \tag{3.7}$$

$$(\frac{19}{10}V - \frac{9}{10})\mu^2\beta(\beta-1) = \frac{1}{2}\lambda, \tag{3.8}$$

$$-(\frac{19}{10}V - \frac{9}{10})\mu^2\beta^2 = (1-V), \tag{3.9}$$

Solving these systems, we can easily obtain the following values:

$$\beta = -2, \mu = \sqrt{\frac{1-V}{-4(\frac{19}{10}V - \frac{9}{10})}}, \lambda = 3(V-1). \tag{3.10}$$

The traveling wave solutions can be successfully written as,
for $V < 0$

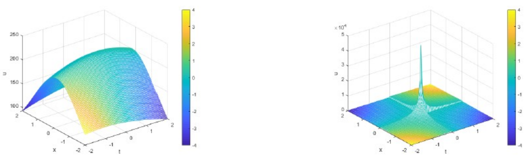
$$G_1(x,t) = 3(V-1)\sec^2\left[\sqrt{\frac{1-V}{-4(\frac{19}{10}V - \frac{9}{10})}}(x - Vt)\right], V < 0, \tag{3.11}$$

$$G_2(x,t) = 3(V-1)\csc^2\left[\sqrt{\frac{1-V}{-4(\frac{19}{10}c - \frac{9}{10})}}(x - Vt)\right], V < 0, \tag{3.12}$$

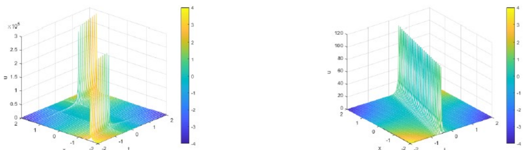
Similarly for $V > 0$

$$G_3(x,t) = 3(V-1)\,\mathrm{sech}^2\left[\sqrt{\frac{1-V}{-4(\frac{19}{10}V - \frac{9}{10})}}(x - Vt)\right], V > 0, \tag{3.13}$$

$$G_4(x,t) = 3(V-1)\,\mathrm{csch}^2\left[\sqrt{\frac{1-V}{-4(\frac{19}{10}V - \frac{9}{10})}}(x - Vt)\right], V > 0. \tag{3.14}$$



(a) Plot of $u_6(left)$ and plot of $u_7(right)$ when, $c = -0.5, r = 1, p = 2$.



(b) Plot of $u_8(left)$ and plot of $u_9(right)$ when, $c = -0.5, r = 1, p = 2$.

Figure 1. Traveling wave solution of fractional Pade-II equation.

3.2 The fractional order Generalized Zakharov equation

By the use of complex wave transformation $g(x,t) = e^{i\theta}G(\phi), h(x,t) = H(\phi)$ where the wave variable $\theta = px + rt$, $\phi = \frac{kx^\alpha}{\Gamma(\alpha+1)} - \frac{2kct^\alpha}{\Gamma(\alpha+1)}$, and V is the velocity wave, Eq.(1.4) reduced to an ODE.

$$\begin{aligned} &k^2G'' + 2GH - (p^2 + r)G - 2\gamma G^3 = 0, \\ &k^2(4p^2 - 1)H'' + k^2(G^2)''. \end{aligned} \tag{3.15}$$

Integrating the second equation of the system (3.15) twice with respect to ϕ, we can find out the value of $'H'$,

$$H(\phi) = \frac{G^2}{1 - 4p^2} + C, \text{if } p^2 \neq \frac{1}{4} \tag{3.16}$$

Where the integration constant, C is a mere constant, now after the substitution of Eq.(3.16) into the system (3.15), we obtain

$$k^2G'' + (2C - p^2 - r)G + 2(\frac{1}{1 - 4p^2} - \gamma)G^3 = 0. \tag{3.17}$$

By considering the trigonometric solution in (cos) form the following changes can be realized

$$G(\phi) = \lambda \cos^\beta(\mu\phi), \tag{3.18}$$

$$(G)_\phi = -\mu\beta\lambda \sin^\beta(\mu\phi) \cos^{\beta-1}(\mu\phi), \tag{3.19}$$

$$(G)_{\phi\phi} = -\mu^2\beta^2\lambda \cos^\beta(\mu\phi) + \mu^2\lambda\beta(\beta - 1) \cos^{\beta-2}(\mu\phi). \tag{3.20}$$

After suitable substitution, Eq. (3.18) and Eq. (3.20) in Eq. (3.17), we have

$$\begin{aligned} k^2\mu^2\beta^2\lambda \cos^\beta \mu\phi + 2(\frac{1}{1 - 4p^2} - \gamma)\lambda^3 \cos^{3\beta} \mu\phi - k^2\lambda\mu^2\beta(\beta - 1) \cos^{\beta-2}(\mu\phi) \\ + (2C - p^2 - r)\lambda \cos^\beta(\mu\phi) = 0. \end{aligned} \tag{3.21}$$

Now collecting the coefficients by homogeneous balance rule and equating them to zero separately, the following algebraic systems come into picture:

$$(\beta - 1) \neq 0, \tag{3.22}$$

$$(\beta-2)=3\beta, \tag{3.23}$$

$$(2C-p^2-r)=k^2\mu^2\beta^2, \tag{3.24}$$

$$2(\frac{1}{1-4p^2}-\gamma)\lambda^2=-k^2\beta(\beta-1), \tag{3.25}$$

Solving these systems, we can easily obtain the following values:

$$\beta=-1, \mu=\sqrt{\frac{(2C-p^2-r)}{k^2}}, \lambda=\sqrt{\frac{-(2C-p^2-r)}{(\frac{1}{1-4p^2}-\gamma)}}, \tag{3.26}$$

The following traveling wave solutions can be constructed,
for $C^2>0$

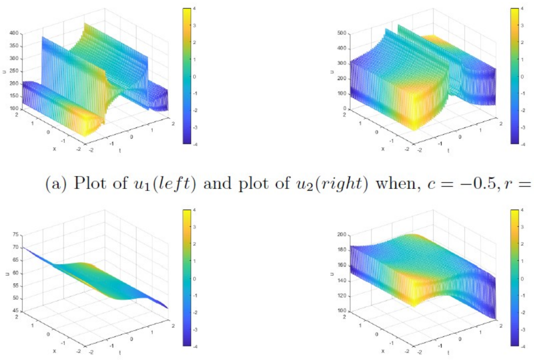
$$g_5(x,t)=e^{i(px+rt)}\left[\sqrt{\frac{-(2C-p^2-r)}{(\frac{1}{1-4p^2}-\gamma)}}\sec\left[\sqrt{\frac{(2C-p^2-r)}{k^2}}k(x-2pt)\right]\right], C^2>0 \tag{3.27}$$

$$g_6(x,t)=e^{i(px+rt)}\left[\sqrt{\frac{-(2C-p^2-r)}{(\frac{1}{1-4p^2}-\gamma)}}\csc\left[\sqrt{\frac{(2C-p^2-r)}{k^2}}k(x-2pt)\right]\right], C^2>0, \tag{3.28}$$

Similarly for $C^2<0$

$$g_7(x,t)=e^{i(px+rt)}\left[\sqrt{\frac{-(2C-p^2-r)}{(\frac{1}{1-4p^2}-\gamma)}}\operatorname{sech}\left[\sqrt{\frac{(2C-p^2-r)}{k^2}}k(x-2pt)\right]\right], C^2<0, \tag{3.29}$$

$$g_8(x,t)=e^{i(px+rt)}\left[\sqrt{\frac{-(2C-p^2-r)}{(\frac{1}{1-4p^2}-\gamma)}}\operatorname{csch}\left[\sqrt{\frac{(2C-p^2-r)}{k^2}}k(x-2pt)\right]\right], C^2<0, \tag{3.30}$$



(a) Plot of $u_1(left)$ and plot of $u_2(right)$ when, $c=-0.5, r=1, p=2$.

(b) Plot of $u_3(left)$ and plot of $u_4(right)$, when $c=-0.5, r=1, p=2$.

Figure 2. Traveling wave solution of fractional Generalized Zakharov equation.

§4 Results and discussion

The Pade-II equation and the generalized Zakharov equation have been focused by many researchers in the last few decades. In this work, the authors have explored, for the first time, the Pade-II with fractional form along with the fractional generalized Zakharov equation. The implementation of the sine-cosine method gives some novel solutions which includ bell type bright, peakon, periodic solitons, symmetric solitons and kink solutions. There are ample examples of such solitary waves having finite (compact) support, and peakons having peaks of discontinuous for the first derivative, in terms of hyperbolic secant solutions the dynamics of laminar jet problem can be studied, hyperbolic tangent solutions are sometimes used to study the rapidity in special relativity, and the hyperbolic cotangent solutions are needed to study Langevin function for magnetic polarization [43].

§5 Conclusion

In this study, the sine-cosine method is successfully implemented on fractional order Pade-II equation and generalized Zakharov equation. New solitary wave profiles of aforementioned equations are obtained with the help of MATLAB, in the form of trigonometric functions and hyperbolic functions. These new solutions of the two families indicate the effectiveness, simplicity, power, capability and realizabilities in terms of less computation, and fruitfulness of the method. The obtained solutions have many potential applications in the field of science and engineering. This simple yet effective method can also be used to solve a wide class of NLFDEs that appeared in various branches of science.

Declarations

Conflict of interest The authors declare no conflict of interest.

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