Appl. Math. J. Chinese Univ.
2024, 39(3): 448-457

A nonlocal dispersal and time delayed HIV infection model with general incidences

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Abstract. Biologically, because of the impact of reproduction period and nonlocal dispersal of HIV-infected cells, time delay and spatial heterogeneity should be considered. In this paper, we establish an HIV infection model with nonlocal dispersal and infection age. Moreover, applying the theory of Fourier transformation and von Foerster rule, we transform the model to an integro-differential equation with nonlocal time delay and dispersal. The well-posedness, positivity, and boundedness of the solution for the model are studied.

§1 Introduction

During the HIV infection stage, the diffusion of the virus within the host plays an important role in understanding the persistence of the infection and how it will affect the HIV infection within the host. In other words, the diffusion of the virus in heterogeneous space should be considered when applying mathematical models to investigate HIV infection. In view of this, there are many reaction-diffusion models with Neumann boundary condition (this implies that no virus can across the boundary of bounded domain) have been established to study the dynamics of HIV infection. For the corresponding reaction-diffusion HIV models and numerical simulation of the HIV infection model, the reader is referred to [1–6].

However, on the one hand, clinical therapy shows that HIV not only invades the lymphatic tissues of the host, but directly and indirectly invades many tissues within the host, such as the hematopoietic system, central nervous system and gastrointestinal system [7]. Thus, the mentioned reaction-diffusion model with Neumann boundary conditions are not able to capture the virus diffusive among different tissues aspect of HIV infection, and the nonlocal dispersal model may be preferred to the reaction-diffusion model [8], where the nonlocal operator $D\int_\Omega J(x-y)wdy - Dw$ is defined as the probability distribution of w jumps from position y to position x, i.e., the convolution $D\int_\Omega J(x-y)wdy$ is the rate at which w is arriving at position x from other places, and Dw is the rate at which they are leaving position x to move to other

Received: 2021-06-21.   Revised: 2021-08-12.
MR Subject Classification: 34K04, 34K30, 35F50.
Keywords: HIV model, nonlocal dispersal, time delay, general incidences.
Digital Object Identifier(DOI): https://doi.org/10.1007/s11766-024-4490-0.
Supported by Funding for the National Natural Science Foundation of China(12201557, 12001483, 61807006).
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positions [9]. On the other hand, time delay plays a crucial role in the HIV infection, which has been verified by many works [10–12]. Although the nonlocal dispersal HIV models have been developed in [13–17]. To our knowledge, research addressing the joint effects of three important factors: heterogeneous environment, nonlocal dispersal and time delay on HIV infection, is still at the preliminary stage. Inspired by the above discussions, we present a novel nonlocal dispersal and time delayed HIV infection model in a heterogeneous environment.

§2 Model formulation

To consider the joint effect of infection age, the nonlocal dispersal and the heterogeneous environment on the process of HIV infection. In the present paper, we formulate the following model

$$\begin{cases} \dfrac{\partial T(x,t)}{\partial t} = D_1 \displaystyle\int_\Omega J(x-y)T(y,t)dy - D_1 T(x,t) + h(T(x,t)) - \mu_1(x)T(x,t) \\ \qquad\qquad - f\left(T(x,t),V(x,t)\right) - g\left(T(x,t), \displaystyle\int_0^\infty i(x,a,t)da\right), \\ \dfrac{\partial i(x,a,t)}{\partial t} + \dfrac{\partial i(x,a,t)}{\partial a} = D_2 \displaystyle\int_\Omega J(x-y)i(y,a,t)dy - D_2 i(x,a,t) - \mu_2(a)i(x,a,t), \\ \dfrac{\partial V(x,t)}{\partial t} = D_3 \displaystyle\int_\Omega J(x-y)V(y,t)dy - D_3 V(x,t) - \mu_3(x)V(x,t) + \displaystyle\int_0^\infty p(a,x)i(x,a,t)da, \\ i(x,0,t) = f\left(T(x,t),V(x,t)\right) + g\left(T(x,t), \displaystyle\int_0^\infty i(x,a,t)da\right), \end{cases} \tag{1}$$

for $(t,x) \in \mathbb{R}^+ \times \bar{\Omega}$, where $\bar{\Omega}$ represents the set which includes itself and the boundary of Ω. Here, we denote the concentration of the susceptible cells at position x and at time t as $T(t,x)$, $i(t,a,x)$ represents the infection age (a)-dependent concentration of HIV-infected cells at position x and at time t, the concentration of free virus at position x and at time t denoted by $V(t,x)$. Variables $h(T(x,t))$ and $\mu_1(x)$ denote the reproduction rate and the removal rate of the susceptible cells at position x. Variables $\mu_2(x)$ and $\mu_3(x)$ represent the death rate of HIV-infected cells and free viruses, respectively. The viral production rate of free viruses at position x and with infection age a is $p(a,x)$. Infection incidence of susceptible cells by infected cells and viruses are expressed as $g\left(T(x,t), \int_0^\infty i(x,a,t)da\right)$ and $f\left(T(x,t),V(x,t)\right)$, respectively. Parameters $D_1 > 0, D_3 > 0$ are constants that stand for the nonlocal diffusion coefficients of the susceptible cells and the free viruses. Similarly, $D_2 > 0$ is defined as the diffusion rate of the HIV-infected cells. $D = (D_1, D_2, D_3)$ and $w = (T(x,t), i(x,a,t), V(x,t))$.

For the $i(x,a,t)$ equation in model (1), we introduce the average latency period denoted by r, resulting in the division of the HIV-infected cells into two epidemiology categories: latently infected cells and actively infected cells, denoted by $I_1(x,t)$ and $I_2(x,t)$, respectively. Then we immediately have

$$I_1(x,t) = \int_0^r i(x,a,t)da, \ I_2(x,t) = \int_r^\infty i(x,a,t)da. \tag{2}$$

Calculating the derivative of (2) associated with t and making use of model (1), yields

$$\begin{aligned}\frac{\partial}{\partial t}I_1(x,t)&=D_2\int_\Omega J(x-y)(I_1(y,t)-I_1(x,t))dy-\mu_2(a)I_1(x,t)-i(x,r,t)+i(x,0,t),\\ \frac{\partial}{\partial t}I_2(x,t)&=D_2\int_\Omega J(x-y)(I_2(y,t)-I_2(x,t))dy-\mu_2(a)I_2(x,t)-i(x,\infty,t)+i(x,r,t),\end{aligned}\tag{3}$$

Biologically, there is no cell can survive forever. Hence, we assume that $i(x,\infty,t)=0$. Note that $i(x,0,t)=f(T(x,t),V(x,t))+g(T(x,t),I_2(x,t))$. Now we derive the explicit expression of $i(x,r,t)$. To this goal, we set $\omega^\tau(x,t)=i(x,t-\tau,t),\ t\in[\tau,\tau+r]$, it follows from $i(x,a,t)$ equation of model (1) and the epidemical meaning that

$$\begin{aligned}\frac{\partial}{\partial a}\omega^\tau(x,t)&=\left[\frac{\partial i(x,a,t)}{\partial t}+\frac{\partial i(x,a,t)}{\partial a}\right]_{a=t-\tau}\\ &=D_2\int_\Omega J(x-y)[i(y,t-\tau,t)-i(x,t-\tau,t)]dy-\mu_2(t-\tau)\omega^\tau(x,t)\\ &=D_2\int_\Omega J(x-y)[\omega^\tau(y,t)-\omega^\tau(x,t)]dy-\mu_2(t-\tau)\omega^\tau(x,t)\end{aligned}$$

Note that $\omega^\tau(x,t)=i(x,t-\tau,t)$, one has $\omega^\tau(x,\tau)=i(x,0,\tau)=f(T(x,\tau),V(x,\tau))+g(T(x,\tau), I_2(x,\tau))$.

In the next, we introduce the following Fourier transform [17] $\mathcal{F}(h)$ and inverse Fourier transform $\mathcal{F}^{-1}(\hat{h})$

$$\hat{h}(\sigma)=\int_\Omega e^{i\sigma x}h(x)dx,\ h(x)=\frac{1}{2\pi}\int_\Omega e^{-i\sigma x}\hat{h}(\sigma)d\sigma.$$

Regarding τ as a parameter and letting $\hat{\omega}^\tau(\sigma,t)$ and $\hat{J}(\sigma)$ express the Fourier transform of $\omega^\tau(x,t)$ and $J(x)$, we obtain that

$$\begin{aligned}\frac{\partial}{\partial a}\hat{\omega}^\tau(\sigma,t)&=\int_\Omega e^{i\sigma x}\left(D_2\int_\Omega J(x-y)[\omega^\tau(y,t)-\omega^\tau(x,t)]dy-\mu_2(t-\tau)\omega^\tau(x,t)\right)dx\\ &=\left[D_2\hat{J}(\sigma)-D_2-\mu_2(t-\tau)\right]\hat{\omega}^\tau(\sigma,t).\end{aligned}$$

By integral from τ to t with respect to a, we obtain that

$$\hat{\omega}^\tau(\sigma,t)=\hat{\omega}^\tau(\sigma,\tau)\exp\left\{\int_\tau^t[D_2\hat{J}(\sigma)-D_2-\mu_2(s-\tau)]ds\right\}.$$

Note that $\hat{\omega}^\tau(\sigma,\tau)=\int_\Omega e^{i\sigma y}i(y,0,\tau)dy$. Therefore, letting $\delta(t,\tau):=\exp\left\{-\int_\tau^t\mu_2(s-\tau)da\right\}$, it follows from the inverse Fourier transform that

$$\omega^\tau(x,t)=\frac{\delta(t,\tau)}{2\pi}\int_\Omega e^{-i\sigma x}\left\{\exp\left\{\int_\tau^t D_2(\hat{J}(\sigma)-1)ds\right\}\int_\Omega e^{i\sigma y}i(y,0,\tau)dy\right\}d\sigma.$$

Let $\tau=t-r$, $\Pi(r)=\exp\left\{-\int_0^r\mu_2(a)da\right\}$, $\alpha=\int_0^r D_2da$, then we can rewrite $\omega^\tau(x,t)$ as

$$\begin{aligned}\omega^\tau(x,t)=&i(x,r,t)=\frac{\Pi(r)}{2\pi}\int_\Omega e^{-i\sigma x}\left\{e^{\alpha(\hat{J}(\sigma)-1)}\int_\Omega e^{i\sigma y}i(y,0,t-r)dy\right\}d\sigma\\ =&\Pi(r)\int_\Omega H_\alpha(x-y)[f(T(x,t-r),V(x,t-r))+g(T(x,t-r),I_2(t-r)]dy,\qquad(4)\\ H_\alpha(x):=&\frac{1}{2\pi}\int_\Omega e^{\alpha(\hat{J}(\sigma)-1)}e^{-i\sigma x}d\sigma.\end{aligned}$$

For $J(x)$, we assume that it is a nonnegative Lebesgue measurable function, and $\int_\Omega J(x)dx=\tilde{J}>0$. Then similar arguments as those in Lemma 3.1 in [17], we give the following assertions for $H_\alpha(x)$.

Proposition 2.1. *For $H_\alpha(x)$, we have*

(1) $\int_\Omega H_\alpha(x)dx = 1$ if $\alpha = 0$; $\int_\Omega H_\alpha(x)dx = e^{-\alpha(\tilde{J}-1)} < \infty$ when $\alpha > 0$;

(2) If $\alpha = 0$, $H_\alpha(x) = \delta(x)$, where $\delta(x)$ is the Dirac-delta function [18], and $H_\alpha(x) > 0$ for $x \in \Omega$ if $\alpha > 0$;

(3) $H_\alpha(-x) = H_\alpha(x)$ holds for $J(-x) = j(x)$, $x \in \Omega$;

(4) if $\int_\Omega J(x)e^{\mu x}dx < \infty$ for any $\mu \geqslant 0$, then $\int_\Omega H_\alpha(x)e^{\mu x}dx < \infty$ for any $\mu \geqslant 0$;

(5)$\int_\Omega H_\alpha(x-y)H_\gamma(y-z)dy = H_{\alpha+\gamma}(x-z)$ for $x, y, z \in \Omega$ and $\alpha, \gamma > 0$.

Remark 2.2. *[17] For local diffusion operator $D\Delta w$ is applied in some reaction-diffusion epidemic models [19–21], the corresponding normal expression of kernel function $\Gamma_\alpha(x)$ is derived as*

$$\gamma_\alpha(x) = \frac{\Pi(r)}{\sqrt{4\pi\alpha}} e^{-\frac{x^2}{4\alpha}} = \frac{\Pi(r)}{2\pi}\int_\Omega e^{-\alpha\sigma^2} e^{-i\sigma x} d\sigma,$$

which is a Green function associated with the linear diffusion equation $\partial\Gamma/\partial t = D\Delta\Gamma$ with Neumann boundary condition $\partial\Gamma/\partial\varphi = 0$, $x \in \partial\Omega$. For nonlocal diffusion operator $D\int_\Omega J(x-y)(w(y,a,t) - w(x,a,t))dy$, one has the kernel function

$$\Pi(r)H_\alpha(x) = \frac{\Pi(r)}{2\pi}\int_\Omega e^{\alpha(\hat{J}(\sigma)-1)} e^{-i\sigma x} d\sigma.$$

Let $J(x) = \delta - \delta^{(2)}(x)$, where $\delta^{(2)}(x)$ represents the second-order derivative of $\delta(x)$ with respect to x. Then in view of the basic property of Dirac-δ function, we have $\hat{J}(\sigma) = 1 - \sigma^2$, it immediately follows that $\Pi(r)H_\alpha(x) = \Gamma_\alpha(x)$ holds for $J(x) = \delta - \delta^{(2)}(x)$. That is, the Laplacian operator $D\Delta w$ is a particular case of nonlocal diffusion operator $D\int_\Omega J(x-y)(w(y,\cdot,t) - w(x,\cdot,t))dy$ as $J(x) = \delta - \delta^{(2)}(x)$.

Up to now, we have the following equations from (4)

$$\begin{aligned}
\frac{\partial}{\partial t} I_1(x,t) =& D_2 \int_\Omega J(x-y)(I_1(y,t) - I_1(x,t))dy - \mu_2(a) I_1(x,t) \\
&+ i(x,0,t) - \Pi(r)\int_\Omega H_\alpha(x-y) i(y,0,t-r)dy, \\
\frac{\partial}{\partial t} I_2(x,t) =& D_2 \int_\Omega J(x-y)(I_2(y,t) - I_2(x,t))dy - \mu_2(a) I_2(x,t) \\
&+ \Pi(r)\int_\Omega H_\alpha(x-y)[f(T(y,t-r), V(y,t-r)) + g(T(y,t-r), I_2(y,t-r))]dy.
\end{aligned}$$

For the $V(x,t)$ equation of model (1), we assume that HIV-free viruses $V(t,x)$ are produced only by actively infected cells due to the budding. Hence, we set $q(x,a) = q(x)$ for $a \geqslant r$ and $q(x,a) = 0$ for $a < r$. Then, we have

$$\frac{\partial V(x,t)}{\partial t} = D_3 \int_\Omega J(x-y)(V(y,t) - V(x,t))dy + q(x)I_2(x,t) - \mu_3(x)V(x,t).$$

Note that $I_1(x,t)$ can be determined by $T(x,t), I_2(x,t), V(x,t)$, we replace $(T(x,t), I_2(x,t), V(x,t))$ with $(w_1(x,t), w_2(x,t), w_3(x,t))$ and consider the following system with more general kernel function form $H(x-y)$ as follows:

$$\begin{cases} \dfrac{\partial w_1(x,t)}{\partial t} = D_1 \displaystyle\int_\Omega J(x-y)(w_1(y,t)-w_1(x,t))dy - f(w_1(x,t),w_3(x,t)) \\ \qquad - \mu_1(x)w_1(x,t) - g(w_1(x,t),w_2(x,t)) + h(w_1(x,t)), \\ \dfrac{\partial}{\partial t}w_2(x,t) = D_2 \displaystyle\int_\Omega J(x-y)(w_2(y,t)-w_2(x,t))dy - \mu_2(a)w_2(x,t) \\ \qquad + \Pi(r)\displaystyle\int_\Omega H(x-y)[f(w_1(y,t-r),w_3(y,t-r)) + g(w_1(y,t-r),w_2(y,t-r))]dy, \\ \dfrac{\partial w_3(x,t)}{\partial t} = D_3 \displaystyle\int_\Omega J(x-y)(w_3(y,t)-w_3(x,t))dy + q(x)w_2(x,t) - \mu_3(x)w_3(x,t), \end{cases} \tag{5}$$

for$(x,t) \in (\bar{\Omega} \times \mathbb{R}^+)$ and subject to the following initial value

$$w_{10}(x) = \psi_1(x,s) \geqslant 0, w_{20}(x) = \psi_2(x,s) \geqslant 0, w_{30}(x) = \psi_3(x,s) \geqslant 0, x \in \bar{\Omega},\ s \in [-r,0]. \tag{6}$$

where $\psi_j(x,s) \in \mathbb{C} := C(\bar{\Omega} \times [-r,0], \mathbb{R})$ $(j = 1,2,3)$ with $\psi = (\psi_1, \psi_2, \psi_3)^T$. Here $\psi(\cdot,0) > 0$ implies $\psi(x,0) \geqslant 0, \psi(x,0) \not\equiv 0$. We define $w_t \in \mathbb{C}$ by $w_t(\theta) = w(t+\theta), \theta \in [-r,0]$.

Throughout the paper, we always make the following basic assumptions:

Assumption 2.3. *For $J(x)$, kernel function H and infection incidences $f(w_1,w_3)$, $g(w_1,w_2)$, we always make the following basic assumptions:*

(1) $J(x)$ is Lebesgue measurable function, $\int_\Omega J(x)dx = \tilde{J} = 1$, and $J(x) = J(-x)$ for $x \in \overline{\Omega}$;

(2) $H(x) \geqslant 0, H(x) = H(-x)$ and Lebesgue measurable for $x \in \Omega$, $\int_\Omega H(x)dx = 1$;

(3) For $f(w_1,w_3)$, we assume that: (i) $f(w_1,w_3) = 0$, $x \in \bar{\Omega}$ if and only if $w_1(x,t) = 0$(or $w_3(x,t) = 0$); (ii) There exists a positive constant $\bar{\beta} > 0$ such that $f(w_1,w_3) \leqslant \bar{\beta} w_1(x,t)w_3(x,t)/(1 + w_1(x,t) + w_3(x,t))$, $w_1(x,t), w_3(x,t) \in \mathbb{R}^+$, $x \in \bar{\Omega}$. Moreover, $g(w_1,w_3)$ also satisfies the above conditions;

(4) $h(w_1(x,t))$ is nondecreasing on $[0,+\infty)$. There exists a positive constant $\tilde{h} > 0$ and $\Lambda(x)$ such that $h(w_1(x,t)) \leqslant \tilde{h}w_1(x,t)$ and $h(w_1(x,t)) \leqslant \Lambda(x)$ for all $x \in \overline{\Omega}$.

§3 Well-posedness, positivity, and boundedness of the solution for system (5)

For convenience, we define the following function spaces and positive cones

$\mathbb{X} := \{\psi : \mathbb{R} \to \mathbb{R} | \psi = \{\psi(x)\}_{x\in\Omega}$ is bounded and uniformly continuous$\}$,

$\mathbb{X}^+ = \{\psi \in \mathbb{X} | \psi(x) \geqslant 0$ for $x \in \Omega\}$,

$\mathbf{X} := C(\overline{\Omega} \times [-r,0], \mathbb{R})$, $\mathbf{Y} := \mathbf{X}^3$ with the norm $||\psi||_{\mathbf{X}} = \sup\limits_{x\in\Omega} |\psi(x)|$, $\psi \in \mathbf{X}$,

$\mathbf{X}_+ := C(\overline{\Omega} \times [-r,0], \mathbb{R}_+)$, $\mathbf{Y}_+ = \mathbf{X}_+^3$, $\psi^+ = \sup\limits_{x\in\Omega} \psi(x)$, $\psi^- = \inf\limits_{x\in\Omega} \psi(x)$, $\psi \in \mathbf{X}$.

Define a linear operator $\mathcal{A}w = (\mathcal{A}_1 w_1, \mathcal{A}_2 w_2, \mathcal{A}_3 w_3)^T$, where $\mathcal{A}_j w_j = D_j \int_\Omega J(x-y)(w_j(y,t) - w_j(\cdot,t))dy - \mu_j w_j(\cdot,t)$, $j = 1,2,3$, and a nonlinear operator $\mathcal{F} = (\mathcal{F}_1, \mathcal{F}_2, \mathcal{F}_3)$ as follows:

$$\mathcal{F}_1 w = h(w_1(\cdot,t)) - f(w_1(\cdot,t), w_3(\cdot,t)) - g(w_1(\cdot,t), w_2(\cdot,t)),$$

$$\mathcal{F}_2 w = \Pi(r)\int_\Omega H(x-y)(f(w_1(\cdot,t-r),w_3(\cdot,t-r))+g(w_1(\cdot,t-r),w_2(\cdot,t-r)))dy,$$

$$\mathcal{F}_3 w = q(\cdot)w_2(\cdot,t).$$

Under the Assumption 2.3, $\mathcal{A}_j, j=1,2,3$ are bounded linear operators and the generators of uniformly continuous positive C_0-semigroups $\{\mathcal{T}_j(t)\}_{t\geqslant 0}$ on $\mathbf{X}$. Let $\mathcal{T}(t)=(\mathcal{T}_1,\mathcal{T}_2,\mathcal{T}_3)$, then system (5) can be rewritten as

$$\begin{cases} w(\cdot,t)=\mathcal{T}(t)\phi(\cdot,0)+\displaystyle\int_o^t \mathcal{T}(t-s)\mathcal{F}(w_s)(\cdot)ds, t>0, \\ w(\cdot,\theta)=\phi(\cdot,\theta),\ \theta\in[-r,0]. \end{cases} \tag{7}$$

On the existence and uniqueness of the positive solution of system (7), we have the following theorem:

Theorem 3.1. *If assumption 2.3 holds and $m\in(0,1)$, which will be determined later, then for any $\phi\in\mathbf{Y}_+$, system (7) admits a unique nonnegative solution $w(x,t,\phi)$ for $t>0$. Moreover, if $\phi(0)\in$ Int $\mathbb{X}_+$ for any $t\geqslant 0$, then $w(t)\in$ Int $\mathbb{X}_+$ for all $t>0$, and $w_t\in$ Int $\mathbf{Y}_+$ for $t>r$.*

Proof. Let $\sigma_j(x)=\mu_j(x)+D_j>0,\ j=1,2,3$ and

$$\begin{aligned} &\mathcal{Q}_1[w](x,t)\\ =&D_1\int_\Omega J(x-y)w_1(y,t)dy+h(w_1(x,t))-f(w_1(x,t),w_3(x,t))-g(w_1(x,t),w_2(x,t)),\\ \mathcal{Q}_2[w](x,t)=&D_2\int_\Omega J(x-y)w_2(y,t)dy+\Pi(r)\int_\Omega H(x-y)[f(w_1(x,t-r),w_3(x,t-r))\\ &+g(w_1(x,t-r),w_2(x,t-r))]dy,\\ \mathcal{Q}_3[w](x,t)=&D_3\int_\Omega J(x-y)w_3(y,t)dy+q(x)w_2(x,t). \end{aligned}$$

Then the solution of system (7) can be expressed as

$$\mathscr{H}_i[w](x,t)=\begin{cases} w_i(x,t)=e^{-\sigma_i(x)t}w_{i0}(x,t)+\displaystyle\int_0^t e^{-\sigma_i(x)(t-s)}\mathcal{Q}_i[w](x,s)ds, t>0,\\ w_i(x,t)=\phi_i(x,t),\ t\in[-r,0]. \end{cases} \tag{8}$$

Define $\mathbb{D}_+ := C([-r,\infty)\times\overline{\Omega},[0,\infty))$, and $\mathscr{H}=(\mathscr{H}_1,\mathscr{H}_2,\mathscr{H}_3)^T:\mathbb{D}_+\to\mathbb{D}_+$. For any $\eta>0$, define $\Gamma_\eta := \{w(x,t): w\in[C([-r,\infty)\times\overline{\Omega},\mathbb{R})]^3,\ \sup_{(x,t)\in\overline{\Omega}\times[-r,\infty)}|w(x,t)|e^{-\eta t}<\infty\}$ with a norm $||w||_\eta=\sum_{j=1}^3\sup_{(x,t)\in\overline{\Omega}\times[-r,\infty)}|w_i(x,t)|e^{-\eta t}$, then $(\Gamma_\eta,||\cdot||_\eta)$ is a Banach space. Choosing a subset $\mathbb{S}$ in Γ_η as follows

$$\mathbb{S}:=\{w\in\Gamma_\eta : w(x,\theta)=\phi(x,\theta) \text{ for } (x,\theta)\in\overline{\Omega}\times[-r,0]\}.$$

We will show that there exists a fixed point of $\mathscr{H}$ in $\mathbb{S}$. It is obvious that $\mathscr{H}(\mathbb{S})\subset\mathbb{S}$. It is sufficient to show that for any $w,v\in\mathbb{S}$, $||\mathscr{H}[w]-\mathscr{H}[v]||_\eta\leqslant m||w-v||_\eta$, $0<m<1$. In fact, for any $w,v\in\mathbb{S}$, it follows from Eq. (8)that

$$\begin{aligned} \Big|\mathscr{H}[w]-\mathscr{H}[v]\Big|&=\sum_{j=1}^3\Big|\int_0^t e^{-\sigma_j(x)(t-s)}(\mathscr{H}_j(w)(x,s)-\mathscr{H}_j(v)(x,s))ds\Big|\\ &=\Big|\int_0^t e^{-\sigma_j(x)(t-s)}D_j\int_\Omega J(x-y)[w_j(y,s)-v_j(y,s)]dyds \end{aligned}$$

$$+\int_0^t e^{-\sigma_1(x)(t-s)}(h(w_1)-h(v_1)+f(w_1,w_3)-f(v_1,v_3)+g(w_1,w_2)-g(v_1,v_2))(x,s)ds$$
$$+\int_0^t e^{-\sigma_2(x)(t-s)}\int_\Omega H(x-y)[f(w_1,w_3)-f(v_1,v_3)+g(w_1,w_2)-g(v_1,v_2)](y,s-r)dyds$$
$$+\int_0^t e^{-\sigma_3(x)(t-s)}q(x)(w_2(x,s)-v_2(x,s))ds\Big|.$$

Therefore, if $\eta>0$ is large enough, then we can obtain from Assumptions 2.3 (3) – (4) that

$$\begin{aligned}\Big|\mathscr{H}[w]-\mathscr{H}[v]\Big|e^{-\eta t}\leqslant&\sum_{j=1}^{3}\int_0^t e^{-(\sigma_j^-+\eta)(t-s)}dsD_i||w_j(y,s)-v_i(y,s)||_\eta\\
&+\int_0^t e^{-(\sigma_1^-+\eta)(t-s)}ds\left((4\bar\beta+\tilde h)||w_1-v_1||_\eta+2\bar\beta||w_2-v_2||_\eta+2\bar\beta||w_3-v_3||_\eta\right)\\
&+e^{-\eta s}\int_0^t e^{-(\sigma_2^-+\eta)(t-s)}ds\left(4\bar\beta||w_1-v_1||_\eta+2\bar\beta||w_2-v_2||_\eta+2\bar\beta||w_3-v_3||_\eta\right)\\
&+q^+\int_0^t e^{-(\sigma_3^-+\eta)(t-s)}ds||w_2-v_2||_\eta\\
\leqslant&\left[(D_1+4\bar\beta+\tilde h)\int_0^t e^{-(\sigma_1^-+\eta)(t-s)}ds+4\bar\beta e^{-\eta s}\int_0^t e^{-(\sigma_2^-+\eta)(t-s)}ds\right]||w_1-v_1||_\eta\\
&+\left[(D_2+2\bar\beta e^{-\eta s})\int_0^t e^{-(\sigma_2^-+\eta)(t-s)}ds+2\bar\beta\int_0^t e^{-(\sigma_1^-+\eta)(t-s)}ds\right.\\
&\left.+q^+\int_0^t e^{-(\sigma_3^-+\eta)(t-s)}ds\right]||w_2-v_2||_\eta+\left[2\bar\beta e^{-\eta s}\int_0^t e^{-(\sigma_2^-+\eta)(t-s)}ds\right.\\
&\left.+D_3\int_0^t e^{-(\sigma_3^-+\eta)(t-s)}ds+2\bar\beta\int_0^t e^{-(\sigma_1^-+\eta)(t-s)}ds\right]||w_3-v_3||_\eta\\
\leqslant&\left(\frac{D_1+4\bar\beta+\tilde h}{\sigma_1^-+\eta}+\frac{4\bar\beta e^{-\eta s}}{\sigma_2^-+\eta}\right)||w_1-v_1||_\eta+\left(\frac{2\bar\beta e^{-\eta s}}{\sigma_2^-+\eta}+\frac{D_3}{\sigma_3^-+\eta}+\frac{2\bar\beta}{\sigma_1^-+\eta}\right)||w_3-v_3||_\eta\\
&+\left(\frac{D_2+2\bar\beta e^{-\eta s}}{\sigma_2^-+\eta}+\frac{2\bar\beta}{\sigma_1^-+\eta}+\frac{q^+}{\sigma_3^-+\eta}\right)||w_2-v_2||_\eta\leqslant m||w-v||_\eta,\end{aligned}$$

where

$$m=\max\left\{\tfrac{D_1+4\bar\beta+\tilde h}{\sigma_1^-+\eta}+\tfrac{4\bar\beta e^{-\eta s}}{\sigma_2^-+\eta},\tfrac{2\bar\beta e^{-\eta s}}{\sigma_2^-+\eta}+\tfrac{D_3}{\sigma_3^-+\eta}+\tfrac{2\bar\beta}{\sigma_1^-+\eta},\tfrac{D_2+2\bar\beta e^{-\eta s}}{\sigma_2^-+\eta}+\tfrac{2\bar\beta}{\sigma_1^-+\eta}+\tfrac{q^+}{\sigma_3^-+\eta}\right\}\in(0,1).$$

Applying Banach contracting theorem, one admits a unique fixed point of $\mathscr{H}$ in $\mathbb{S}$. If $\phi(0)\in\mathrm{Int}\mathbb{X}_+$, then we have from (8) and the property of $\mathcal{T}$ that $w(t)\in\mathrm{Int}\mathbb{X}_+$ for all $t\geqslant 0$; if $\phi(0)>0$, then $w(t)\in\mathrm{Int}\mathbb{X}_+$ for $t>0$, and $w_t\in\mathrm{Int}\mathbf{Y}_+$ for $t>r$. This completes the proof. □

Theorem 3.2. *Suppose that Assumption 2.3 holds and let $w(x,t;\psi)$ be the solution of system (7) with a nonnegative initial condition. Then $w(x,t;\phi)$ is positive and bounded.*

Proof. For any $x\in\overline{\Omega}, t>0$, it follows from Eq. (8) that

$$\begin{aligned}w_1(x,t;\phi_1)=&\mathcal{T}_1(t)\phi_1(0)(x)+\int_0^t\mathcal{T}_1(t-s)\mathcal{Q}_1(w_s)(x)ds\leqslant\mathcal{T}_1(t)\phi_1(0)(x)+\int_0^t\mathcal{T}_1(t-s)\Lambda(x)ds\\
\leqslant&\varphi_1e^{-D_1t}+\frac{\lambda^+}{D_1}(1-e^{-D_1t})\leqslant\max\left\{\varphi_1,\frac{\Lambda^+}{D_1}\right\},\end{aligned}$$

where $\varphi_j=\max_{(x,\theta)\in\overline{\Omega}\times[-r,0]}\sum\phi_j(x,\theta), j=1,2,3$. Define

$W(t) = \int_\Omega \int_\Omega H(x-y)\Pi(r)w_1(x,t)dydx + \int_\Omega w_2(x,t+r)dx,$

then we can obtain from system (5) that

$$\begin{aligned}\frac{dW(t)}{dt} &= D_1 \int_\Omega \Pi(r)H(x-y)dy \int_\Omega \int_\Omega J(x-z)w_1(z,t)dzdx \\ &- D_1 \int_\Omega \Pi(r)H(x-y)dy \int_\Omega w_1(x,t)dx - \Pi(r) \int_\Omega H(x-y)dy \int_\Omega [f(w_1, w_3) \\ &+ g(w_1, w_2)](x,t)dx + \Pi(r) \int_\Omega H(x-y)dy \int_\Omega h(w_1(x,t))dx \\ &- \Pi(r) \int_\Omega H(x-y)dy \int_\Omega \mu_1(x)w_1(x,t)dx \\ &+ D_2 \int_\Omega \int_\Omega J(x-y)w_2(y,t+r)dydx - D_2 \int_\Omega w_2(x,t+r)dx \\ &+ \Pi(r) \int_\Omega H(x-y)dy \int_\Omega [f(w_1, w_3) + g(w_1, w_2)](x,t)dx - \int_\Omega \mu_2(x)w_2(x,t+r)dx \\ &\leqslant \Pi(r)\Lambda^+|\Omega| - \mu_{\min}W(t),\end{aligned}$$

where $\mu_{\min} = \min\{\mu_1^-, \mu_2^-\}$, it implies that $W(t) \leqslant \frac{\Pi(r)\Lambda^+|\Omega|}{\mu_{\min}} := \mathcal{B}$, is bounded. Consequently, the bounded of $w_2(x,t;\phi)$ is obtained. Finally, we can obtain from Eq. (8) that

$$\begin{aligned}&w_3(x,t;\phi_3) \\ =&\mathcal{T}_3(t)\phi_3(0)(x) + \int_0^t \mathcal{T}_3(t-s)\mathcal{Q}_3(w_s)(x)ds \leqslant \mathcal{T}_3(t)\phi_3(0)(x) + \int_0^t \mathcal{T}_3(t-s)q(x)\mathcal{B}ds \\ \leqslant&\varphi_3 e^{-D_3 t} + \frac{q^+\mathcal{B}}{D_3}(1-e^{-D_3 t}) \leqslant \max\left\{\varphi_3, \frac{q^+\mathcal{B}}{D_3}\right\}.\end{aligned}$$

Thus, the boundedness of $w_3(x,t;\phi_3)$ follows. This completes the proof. □

§4 Conclusion

In this paper, we present a novel HIV infection model with nonlocal time delay and dispersal. To the best of our knowledge, few HIV infection models are formulated to study the basic properties of the solution for the model. As a novel HIV infection model, its threshold dynamics is worth to be investigated in the follow-up work. In the current model, we have not performed the dynamical analysis, this would be done in the future work.

Acknowledgements

We are deeply indebted to the editor and the anonymous referees whose insightful comments and suggestions helped us a lot to improve the quality of this paper.

Declarations

Conflict of interest The authors declare no conflict of interest.

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