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Some convergence theorems of fuzzy concave integral on fuzzy σ-algebra

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Abstract. In this paper, we consider the extension of the concave integral from classical crisp σ-algebra to fuzzy σ-algebra of fuzzy sets. Firstly, the concept of fuzzy concave integral on a fuzzy set is introduced. Secondly, some important properties of such integral are discussed. Finally, various kinds of convergence theorems of a sequence of fuzzy concave integrals are proved.

§1 Introduction

The concave integral defined on classical crisp σ-algebra was first introduced by Lehrer [1], which differs from the Choquet integral when the capacity is not convex (super modular). The integral stemmed from the concavification of cooperative game that was first proposed by Weber and later appeared in Azrieli and Lehrer, the most prominent feature of the integral is concavity [2,3]. In the context of a decision under uncertainty, this property might be interpreted as uncertainty aversion. Lehrer and Teper investigated the concave integral for capacities defined over large spaces. A non-additive version of the Levi theorem and the Fatou lemma and other convergence theorems for capacities with large cores were proven [4]. Teper studied the continuity of the concave integral, and got the Dominated Convergence Theorem of the concave integral for capacities[5]. Amarante obtained the necessary and sufficient conditions for the existence of an additive set function sandwiched between two arbitrary set functions by concave and convex integral [6]. From the perspective of risk assessment, integral function is an important tool for the comprehensive weighted average of risk value. However, their discussion was limited to a classical crisp σ-algebra, and by their theory, we can't deal with the decision for the uncertainty caused by fuzzy events in many economic activities. Therefore, we attempt to establish a new fuzzy integral on fuzzy sets to tackle such a problem.

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Zhong proposed many definitions related to fuzzy sets and fuzzy measures, and studied the abstract integral on fuzzy sets and L-fuzzy sets[7-11]. Based on the classical Lebesgue integral idea, Butnariu proposed a fuzzy integral on σ-additive class of fuzzy sets. He proved that some fundamental results of Lebesgue integral theory can be carried over to the general fuzzy case [12]. Huang and Wu studied Choquet integrals with respect to fuzzy measure on fuzzy σ-algebra[13]. Dvorák and Holcapek explored fuzzy measures and integrals defined on algebras of fuzzy subsets over complete residuated lattices[14]. Khalid etc proposed the definition of fuzzy neutrosophic soft σ-algebra and fuzzy neutrosophic soft measure. Khalid and Samet put forth the definition of picture fuzzy soft σ-algebra and picture fuzzy soft measure[15,16]. These definitions lay a theoretical foundation for the expansion of fuzzy integrals on fuzzy sets. Different from the above research, based on the idea of the concavification of cooperative game, we propose a fuzzy concave integral of fuzzy capacity on fuzzy σ-algebra of fuzzy sets, and get some properties and convergence theorems in this paper. The integral and its convergence theorems may provide one tool for making decisions under uncertainty caused by fuzzy events in economic activities.

The remainder of this paper is organized as follows. In Section 2, we provide some basic notions and definitions about fuzzy measure and fuzzy σ-algebra that are needed later. In Section 3, we recall some concepts of convergence for a sequence of measurable functions, then we introduce the concept of the fuzzy concave integral on a fuzzy set and discuss some of its properties. In Section 4, some convergence theorems of a sequence of fuzzy concave integrals on fuzzy σ-algebra are proved. Finally, conclusions are drawn in section 5.

§2 Preliminaries

Let X be a nonempty set and I the real number interval [0,1]. $\mathcal{F}(X)$ is the family of all fuzzy sets on X. If $\tilde{A}$ denotes a fuzzy set and x is a point in X, then $A_d(x)$ denotes the membership function of $\tilde{A}$, $\tilde{A}^c$ denotes complement set of $\tilde{A}$, $A_d^c(x) = 1 - A_d(x)$ for every x. In order to distinguish from fuzzy sets, a crisp set is denoted as A, its characteristic function is denoted as $I_A(\cdot)$. Let $R^+=[0,\infty)$ and $\bar{R}^+=[0,\infty]$ denote the set of nonnegative real numbers and extended nonnegative real numbers respectively.

Definition 2.1. [12] Let $\tilde{A}$ and $\tilde{B}$ be two fuzzy sets.

(a) The *sum* $\tilde{A} \oplus \tilde{B}$ is the fuzzy set whose membership function is given by

$(A \oplus B)_d(x) = \min(1, A_d(x) + B_d(x)) \qquad (x \in X)$

(b) The *product* of $\tilde{A}$ and $\tilde{B}$ is defined by

$(A \bullet B)_d(x) = A_d(x) \bullet B_d(x) \qquad (x \in X)$

Definition 2.2. [10] Let $\mathcal{K}$ be a sub-family of $\mathcal{F}(X)$. If a set function $v : \mathcal{K} \to \bar{R}^+$ satisfies the conditions:

(1) $v(\varnothing) = 0$;

(2) $\tilde{A}, \tilde{B} \in \mathcal{K}, \tilde{A} \subset \tilde{B} \Rightarrow v(\tilde{A}) \leq v(\tilde{B})$.

v is called a $fuzzy\ measure$.

If a fuzzy measure v satisfies the conditions:

(3) $v(X) = 1$.

v is called a *fuzzy capacity*.

Definition 2.3. [10] Let v be a fuzzy capacity.

(1) v is called *lower semicontinuous* if $\tilde{A}_n, \tilde{A} \in \mathcal{F}(X)$, $\tilde{A}_n \uparrow \tilde{A}$, then $\lim\limits_{n\to\infty} v(A_n) = v(A)$.

(2) v is called *upper semicontinuous* if $\tilde{A}_n, \tilde{A} \in \mathcal{F}(X)$, $\tilde{A}_n \downarrow \tilde{A}$, then $\lim\limits_{n\to\infty} v(A_n) = v(A)$.

(3) v is called *continuous* if v is lower semicontinuous and upper semicontinuous.

(4) v is called *autocontinuous from below* if $v(\tilde{B}_n) \to 0$ implies $v(\tilde{A}\cap\tilde{B}_n^c) \to v(\tilde{A})$ whenever $\tilde{A}, \tilde{B} \in \mathcal{F}(X)$.

Definition 2.4. Let v be a fuzzy capacity, v is called *subadditive* if $v(\tilde{A}\oplus\tilde{B}) \leq v(\tilde{A})+v(\tilde{B})$ whenever $\tilde{A}, \tilde{B} \in \mathcal{F}(X)$.

Remark 1.

The definition of subadditivity is different from the general definition of that. In generally, $v(\tilde{A} \oplus \tilde{B}) \leq v(\tilde{A}) + v(\tilde{B}) \Rightarrow v(\tilde{A} \cup \tilde{B}) \leq v(\tilde{A}) + v(\tilde{B})$, but the reverse is not necessarily true. If there is at least one crisp set between $\tilde{A}$ and $\tilde{B}$, then two definitions are equivalent. Since if there is at least one crisp set between $\tilde{A}$ and $\tilde{B}$, we have $v(\tilde{A}\oplus\tilde{B}) \leq v(\tilde{A})+v(\tilde{B}) \Leftrightarrow v(\tilde{A}\cup\tilde{B}) \leq v(\tilde{A}) + v(\tilde{B})$. The main reason for adopting such a definition here is to obtain ideal properties for the fuzzy concave integral defined later.

Definition 2.5. [10] A sub-family $\mathcal{K}$ of $\mathcal{F}(X)$ is called a *fuzzy σ-algebra*, if it satisfies the following conditions:

(1) $\varnothing, X \in \mathcal{K}$,

(2) $\tilde{A} \in \mathcal{K} \Rightarrow \tilde{A}^c \in \mathcal{K}$,

(3) $\left\{\tilde{A}_n\right\} \in \mathcal{K} \Rightarrow \bigcup\limits_{n=1}^{\infty} \tilde{A}_n \in \mathcal{K}$.

§3 Fuzzy concave integral on fuzzy σ-algebra

Let $\mathcal{K}$ be a fuzzy σ-algebra over X. A mapping $f : X \to (-\infty, \infty)$ is called a *measurable function on fuzzy σ-algebra* $\mathcal{K}$ if $\{x; f(x) \geq \alpha\} \in \mathcal{K}$ for every $\alpha \in [-\infty, \infty]$. Denote by $\mathscr{B} := \{f;\, f$ is a measurable function on $\mathcal{K}\}$, $\mathscr{B}_+ := \{f; f \in \mathscr{B}, f \geq 0\}$. An extended functional $H : \mathscr{B}_+ \to [0, \infty]$ is concave if $H(\alpha f + (1 - \alpha)g) \geq \alpha H(f) + (1 - \alpha)H(g)$ for every $\alpha \in (0, 1)$, $f, g \in \mathscr{B}$, and it is positive homogeneous iff $H(\alpha f) = \alpha H(f)$ for every $\alpha \in (0, 1)$ and $f, g \in \mathscr{B}_+$.

Definition 3.1. A property D over X is a crisp subset of X. A property D *holds everywhere on* $\tilde{A}$ if $\tilde{A} \subset D$, denote it by D on $\tilde{A}$; A property D *holds almost everywhere on* $\tilde{A}$, if there exists $\tilde{E} \in \mathcal{K}$ with $v(\tilde{E}) = 0$ and $\tilde{A} \cap \tilde{E}^c \subset D$, denote it by D on $\tilde{A}$ *v-a.e.*, if $\tilde{A} = X$, property D *holds almost everywhere*, denote it by D *v-a.e.*; A property D *holds pseudo − almost everywhere*, if there exists $\tilde{E} \in \mathcal{K}$ with $v(\tilde{E}) = 0$ such that $\tilde{A}\cap\tilde{E}^c \subset D$ and $v(\tilde{A}\cap\tilde{E}^c) = v(\tilde{A})$ for every $\tilde{A} \in \mathcal{K}$, denote it by D *v-p.a.e.*(In order to simplify the symbols, such as *v-a.e.* and *v-p.a.e.*, when the fuzzy measure v is obvious, we often omit it).

Remark 2.

From **Definition 3.1**, We can draw the following conclusions.

(1) If D v-$p.a.e.$, then D v-$a.e.$ holds;

(2) if D v-$a.e.$, then $v(D^c) = 0$;

(3) If D v-$p.a.e.$, then $D^c \subset \widetilde{E}$ and $v(\widetilde{A} \bigcap D) = v(\widetilde{A})$ for every $\tilde{A} \in \mathcal{F}$;

By **Definition 3.1**, the following definitions are easy to be understand.

Definition 3.2. Let $\{f, f_n\} \subset \mathscr{B}$, $\tilde{A} \in \mathcal{K}$, and $L = \{x : f_n \to f\}$.

(1) If $\tilde{A} \subset L$, then we say $\{f_n\}$ *converges to* f *everywhere on* $\tilde{A}$, and denote it by $f_n \to f$ on $\tilde{A}$.

(2) If there exists $\tilde{E} \in \mathcal{K}$ with $v(\tilde{E}) = 0$, such that $f_n \to f$ on $\tilde{A} \cap \tilde{E}$. then we say $\{f_n\}$ *converges to* f *almost everywhere on* $\tilde{A}$, and denote it by $f_n \to f$ on $\tilde{A}$ $a.e.$ If $\tilde{A} = X$, we say $\{f_n\}$ *converges to* f *almost everywhere*, and denote it by $f_n \to f$ $a.e.$.

(3) If there exists $\tilde{E} \in \mathcal{K}$ with $v(\tilde{A} \cap \tilde{E}^c) = v(\tilde{A})$, such that $f_n \to f$ on $\tilde{A} \cap \tilde{E}$. then we say $\{f_n\}$ $converges\ to\ f\ pseudo-almost\ everywhere\ on\ \tilde{A}$, and denote it by $f_n \to f$ on $\tilde{A}$ $p.a.e.$; If $\tilde{A} = X$, we say $\{f_n\}$ $converges\ to\ f\ pseudo-almost\ everywhere$, and denote it by $f_n \to f$ $p.a.e.$.

Definition 3.3.[14]. Let $\{f, f_n\} \subset \mathscr{B}$. If for any given $\varepsilon > 0$, when $n \to \infty$, we have $v(|f_n - f| \geq \varepsilon) \to 0$, then we say $\{f_n\}$ $converges\ in\ fuzzy\ measure\ v\ to\ f$, and denote it by $f_n \xrightarrow{v} f$.

Theorem 3.4.(Riesz's Theorem)[14]. *Let* $\{f, f_n\} \subset B_+$, $if\ v\ is\ autocontinuous\ from$ *below and* $f_n \xrightarrow{v} f$ *on* X, *then there exists a subsequence* $\{f_{n_k}\}$ *of* $\{f_n\}$, *such that* $f_n \to f$ on $\tilde{A}$ $p.a.e.$, *whenever* $\tilde{A} \in \mathcal{K}$.

Fix a fuzzy capacity v defined on $\mathcal{K}$ and $f \in \mathscr{B}_+$.

Definition 3.5. The fuzzy convex integral of f over $\tilde{A}$ w.r.t. v is defined by

$\int_{\tilde{A}}^{cav} f dv := \inf \{H(f \cdot A_d)\}$

where the infimum is taken over all convex and positive homogeneous extended functionals $H : \mathscr{B}_+ \to [0, \infty]$ that satisfy $H(E_d) \geq v(\tilde{E})$ for all $(\tilde{E}) \in \mathcal{K}$.

Remark 3.

Similar to the proof **Proposition 1**of [4], we have

$$\int_{\tilde{A}}^{Cav} f dv = \sup \left\{ \sum_{i=1}^{N} \lambda_i v(\tilde{A}_i); \lambda_i A_{id}(x) \leq f(x) \cdot A_d(x), \tilde{A}_1, \cdots, \tilde{A}_N \in \mathcal{K}, \lambda_i > 0, i \in N, \forall x \in X \right\}.$$

We say that f is *integrable over* $\tilde{A} \in \mathcal{K}$ if $\int_{\tilde{A}}^{Cav} f dv < \infty$. If f is integrable over X we say that it is *integrable*. In order to express the Vitali theorem of the version of fuzzy concave integral conveniently, let $1 \leq p < \infty$, denote by $L^p(v) := \left\{f : f \in B, \int |f|^p dv < \infty\right\}$. Unless stated otherwise, for expressional convenience, next we will denote $\int dv$ as the fuzzy concave integral.

Proposition 3.6. $The\ fuzzy\ integrals\ on\ fuzzy\ sets\ satisfy\ the\ following\ properties:$

(1) $\int_{\tilde{A}} 0 dv = 0$ $for\ every$ $\tilde{A} \in \mathcal{K}$;

(2) $\int_{\tilde{A}} dv$ $is\ a\ positive\ functional\ on$ $\mathscr{B}_+$ $for\ every$ $\tilde{A} \in \mathcal{K}$;

(3) If $\mu, v : \mathcal{K} \to [0, \infty)$ and $\mu \leq v$, $then$ $\int_{\tilde{A}} f d\mu \leq \int_{\tilde{A}} f dv$ for $f \in \mathscr{B}_+$ and $\tilde{A} \in \mathcal{K}$;

(4) $\int_{\tilde{A}} f dv + \int_{\tilde{A}} g dv \leq \int_{\tilde{A}} (f + g) dv$ $for\ every$ $f, g \in \mathscr{B}_+$ and $\tilde{A} \in \mathcal{K}$;

(5) If $f \leq g$, $then$ $\int_{\tilde{A}} f dv \leq \int_{\tilde{A}} g dv$ $for\ every$ $f, g \in \mathscr{B}_+$ and $\tilde{A} \in \mathcal{K}$;

(6) $If\ a\ property$ D $p.a.e.$, $then$ $\int_{\tilde{A}} f dv = \int_{\tilde{A} \cap D} f dv$ $for\ every$ $f \in \mathscr{B}_+$ and $\tilde{A} \in \mathcal{K}$;

(7) *If* $f \leq g$ *p.a.e., then* $\int_{\tilde{A}} fdv \leq \int_{\tilde{A}} gdv$ *for every* $f, g \in \mathscr{B}_+$ *and* $\tilde{A} \in \mathcal{K}$;

(8) $\int_{\tilde{A} \cap B} fdv = 0$ *for every* $f \in \mathscr{B}_+$ *and* $\tilde{A}, B \in \mathcal{K}$ *with* $v(B) = 0$;

(9) *If* v *is subadditive, then* $\int_{\tilde{A}} fdv = \int_{\tilde{A}\cdot\tilde{B}} fdv + \int_{\tilde{A}\cdot\tilde{B}^c} fdv$ *for every* $f \in \mathscr{B}_+$ *and* $\tilde{A}, \tilde{B} \in \mathcal{K}$;

(10) *If* v *is subadditive, then* $\int_{\tilde{A}\cdot\tilde{B}} fdv = 0$ *and* $\int_{\tilde{A}} fdv = \int_{\tilde{A}\cdot\tilde{B}^c} fdv$ *for every* $f \in \mathscr{B}_+$ *and* $\tilde{A}, \tilde{B} \in \mathcal{K}$, $v(\tilde{B}) = 0$;

(11) *If* v *is subadditive,* $f \leq g$ *a.e., then* $\int_{\tilde{A}} fdv \leq \int_{\tilde{A}} gdv$ *for every* $f, g \in \mathscr{B}_+$ *and* $\tilde{A}, \tilde{B} \in \mathcal{K}$.

Proof. By **Remark 3, (1)** to **(7)** are obvious, we only prove (8) to (11).

(8) For every $f \in \mathscr{B}_+$ and $\sum_{i=1}^{k} \lambda_i A_{id} \leq f \cdot \tilde{A} \cdot I_B$, it follows that $A_{id}(x) = 0$ for every $i \in N, x \in X \cap B$. Thus $A_{id} \cdot I_B(x) = A_{id}(x)$ for every $i \in N, x \in X$. Since $(\tilde{A}_i \cap B) \subset B$ and $v(B) = 0$, we have $v(\tilde{A}_i) = 0$. Thus $\int_{\tilde{A} \cap B} fdv = 0$ by **Remark 3**. It ensures the conclusion holds.

(9) First, it is obvious that $\int_{\tilde{A}} fdv \geq \int_{\tilde{A}\cdot\tilde{B}} fdv + \int_{\tilde{A}\cdot\tilde{B}^c} fdv$ by **Remark 3**. Next, we will show that $\int_{\tilde{A}} fdv \leq \int_{\tilde{A}\cdot\tilde{B}} fdv + \int_{\tilde{A}\cdot\tilde{B}^c} fdv$. For any $\sum_{i=1}^{k} \lambda_i A_{id} \leq f \cdot A_d$, we have $\sum_{i=1}^{k} \lambda_i A_{id} \cdot B_d \leq f \cdot A_d \cdot B_d$ and $\sum_{i=1}^{k} \lambda_i A_{id} \cdot B_d^c \leq f \cdot A_d \cdot B_d^c$. Since $A_{id} = A_{id} \cdot B_d + A_{id} \cdot B_d^c$ and v is subadditive, then $v(\tilde{A}_i) \leq v(\tilde{A}_i \cdot \tilde{B}) + v(\tilde{A}_i \cdot \tilde{B}^c)$. Thus, $\int_{\tilde{A}} fdv \leq \int_{\tilde{A}\cdot\tilde{B}} fdv + \int_{\tilde{A}\cdot\tilde{B}^c} fdv$, which implies $\int_{\tilde{A}} fdv = \int_{\tilde{A}\cdot\tilde{B}} fdv + \int_{\tilde{A}\cdot\tilde{B}^c} fdv$.

(10) For any $\sum_{i=1}^{k} \lambda_i A_{id} \leq f \cdot A_d$, we have $\sum_{i=1}^{k} \lambda_i A_{id} \cdot B_d^c \leq f \cdot A_d \cdot B_d^c$. Since v is subadditive, then

$v(\tilde{A}_i) \leq v(\tilde{A}_i \cdot \tilde{B}) + (\tilde{A}_i \cdot \tilde{B}^c)$.

Since $v(\tilde{B}) = 0$, $\tilde{A}_i \cdot \tilde{B} \subset \tilde{B}$, then $v(\tilde{A}_i) \leq v(\tilde{A}_i \cdot \tilde{B}^c)$. This means that

$\sum_{i=1}^{k} \lambda_i v(\tilde{A}_i) \leq \sum_{i=1}^{k} \lambda_i v(\tilde{A}_i \cdot \tilde{B}^c) \leq \int_{\tilde{A}\cdot\tilde{B}^c} fdv$.

Therefore $\int_{\tilde{A}} fdv \leq \int_{\tilde{A}\cdot\tilde{B}^c} fdv$.

However $\int_{\tilde{A}} fdv \geq \int_{\tilde{A}\cdot\tilde{B}^c} fdv$, which means that $\int fdv = \int_{\tilde{A}^c} fdv$.

From (9) it also follows that $\int_{\tilde{A}\cdot\tilde{B}} fdv = 0$.

(11) Denote $J := \{x : f(x) \leq g(x)\}$. If D *a.e.*, it follows that $\int fdv = \int_J fdv$ and $\int gdv = \int_J g\, dv$ from **Remark 2** and (10). Since $fI_J \leq gI_J$ for every $x \in X$, by(4) we have

$\int_{\tilde{A}} fI_J dv \leq \int_{\tilde{A}} gI_J dv$.

This implies $\int_{\tilde{A}} fdv \leq \int_{\tilde{A}} gdv$.

Remark 4.

If give a fuzzy measure v over $\mathcal{K}$, define that $\hat{v}(\tilde{A}) := \int_{\tilde{A}} 1dv$. Similar to the proof of **Lemma 1** of [7] we have that

(1) $\hat{v} \geq v$;

(2) $\int_{\tilde{A}} fdv = \int_{\tilde{A}} fd\hat{v}$ for every $f \in \mathscr{B}_+$ *and* $\tilde{A}, \tilde{B} \in \mathcal{K}$.

§4 Convergence theorems of fuzzy concave integral on fuzzy σ-algebra

In this section, various integral convergence theorems will be discussed.

Lemma 4.1. *Let* $\{f, f_n\} \subset \mathscr{B}_+$, *if* $f_n \uparrow$ *and* $\lim_{n\to\infty} f_n \geq f$ *a.e.* (*respect p.a.e.*), v *is lower semicontinuous and subadditive*(*respect lower semicontinuous*), *then for every* $\tilde{A} \in \mathcal{K}$

$\lim\limits_{n\to\infty} \int_{\tilde{A}} f_n dv \geq \int_{\tilde{A}} f dv.$

Proof. Denote by $D := \left\{x : f_n(x) \uparrow, \lim\limits_{n\to\infty} f_n(x) \geq f(x)\right\}$. If D *a.e.*, there exists $\tilde{E} \in \mathcal{K}$ with $v(\tilde{E}^c) = 0$ and $D^c \subset \tilde{E}^c$. Denote by $g_n(x) := f_n I_D(x)$ and $g(x) := f I_D(x)$. By **Definition 3.1.**, we have $g_n(x) = f_n(x)$, $g(x) = f(x)$ *a.e.*, and also we have $g_n \uparrow$ and $\lim\limits_{n\to\infty} g_n \geq g$ for each x. Since v is subadditive, without loss generality, we can assume that $f_n \uparrow$ and $\lim\limits_{n\to\infty} f_n \geq f$ for every x by (2) of **Remark 2** and (10) of **Proposition 3.6**. For every $\alpha \in (0, 1)$, denote by $B_n(\alpha) := \{f_n \geq \alpha f\}$. Since $f_n \cdot I_{B_n(\alpha)} \geq \alpha f \cdot I_{B_n(\alpha)}$, by **Remark 3**, for every $\sum\limits_{i=1}^{k} \lambda_i A_{id} \leq f \cdot A_d$, we have $\alpha \sum\limits_{i=1}^{k} \lambda_i A_{id} \leq \alpha f \cdot A_d$ and $\alpha \sum\limits_{i=1}^{k} \lambda_i A_{id} \cdot I_{B_n(\alpha)} \leq f_n \cdot A_d \cdot I_{B_n(\alpha)}$. Thus by **Remark 3**, $\int_{\tilde{A}} f_n dv \geq \int_{\tilde{A}} f_n I_{B_n(\alpha)} dv \geq \alpha \sum\limits_{i=1}^{k} \lambda_i v(\tilde{A}_i \cap B_n(\alpha))$. Furthermore, from $f_n \uparrow$ and $\lim_{n\to\infty} f_n \geq f$ it follows that $B_n(\alpha) \uparrow X$. Since v is lower semicontinuous, $v(\tilde{A}_i \cap B_n(\alpha)) \uparrow v(\tilde{A}_i)$, we have

$\lim\limits_{n\to\infty} \int_{\tilde{A}} f_n dv \geq \alpha \sum\limits_{i=1}^{k} \lim\limits_{n\to\infty} \lambda_i v(\tilde{A}_i \cap B_n(\alpha)) = \alpha \sum\limits_{i=1}^{k} \lambda_i v(\tilde{A}_i).$

Thus

$\lim\limits_{n\to\infty} \int_{\tilde{A}} f_n dv \geq \alpha \int f dv.$

Let $\alpha \to 1$, the result follows.

Since $f_n \cdot A_d \cdot I_{D\cap B_n(\alpha)} \geq \alpha f \cdot A_d \cdot I_{D\cap B_n(\alpha)}$ and $\int_{\tilde{A}} f_n dv \geq \int_{\tilde{A}} f_n I_{D\cap B_n(\alpha)} dv$, If D *p.a.e.*, we have $D \subset \bigcup\limits_{n=1}^{\infty} B_n(\alpha)$ and $v(\tilde{A} \cap D \cap \bigcup\limits_{n=1}^{\infty} B_n(\alpha)) = v(\tilde{A})$ for every $\tilde{A} \in \mathcal{F}$ by (3) of **Remark 2.** Therefore, the proof of the rest is similar to the one above.

Theorem 4.2. (Pseudo-Almost Everywhere Monotone Convergence Theorem) *Let* $\{f, f_n\} \subset \mathscr{B}_+$, $\lim\limits_{n\to\infty} \int_{\tilde{A}} f_n dv = \int_{\tilde{A}} f dv$ *for every increasing sequence of nonnegative measurable functions* $\{f_n\} \subset \mathscr{B}_+$ *converging to a function* f *p.a.e. and* $\tilde{A} \in \mathcal{K}$ *iff* $\hat{v}$ *is lower semicontinuous.*

Proof. Sufficiency: Denote by $D := \left\{x : f_n(x) \uparrow \lim\limits_{n\to\infty} f_n(x) = f(x)\right\}$. If D *p.a.e.* by **Lemma 4.1**, we have $\lim\limits_{n\to\infty} \int_{\tilde{A}} f_n d\hat{v} \geq \int_{\tilde{A}} f d\hat{v}$. On the other hand, let $g_n = f_n I_D$ and $g = f I_D$. we have $g_n \leq g (\forall n \in N)$. It ensures that $\lim\limits_{n\to\infty} \int_{\tilde{A}} g_n d\hat{v} \leq \int_{\tilde{A}} g d\hat{v}$. Then it follows that $\lim\limits_{n\to\infty} \int_{\tilde{A}} f_n d\hat{v} = \int_{\tilde{A}} f d\hat{v}$ from (6) of **Proposition 3.6**. This means that $\lim\limits_{n\to\infty} \int_{\tilde{A}} f_n dv = \int_{\tilde{A}} f dv$.

Necessity: It follows by analogy to the proof of the necessity of **Proposition 1**of [4].

Theorem 4.3. (Almost Everywhere Monotone Convergence Theorem) *Let* $\{f, f_n\} \subset \mathscr{B}_+$, *if* v *is lower semicontinuous and subadditive, then* $\lim\limits_{n\to\infty} \int_{\tilde{A}} f_n dv = \int_{\tilde{A}} f dv$ *for every increasing sequence of nonnegative measurable functions* $\{f_n\} \subset \mathscr{B}_+$ *converging to a* *function* f *a.e. and* $\tilde{A} \in \mathcal{K}$.

Proof. It is easily proved according to **Lemma 4.1** and (10) of **Proposition 3.6**.

Definition 4.4. Let $\mathscr{F}$ be a nonempty subset of $\mathscr{B}$.

(1) $\mathscr{F}$ is called *uniformly v-integrable* if $\lim\limits_{c\to\infty}\sup_{f\in\mathscr{F}}\int |f|I_{\{|f|>c\}}dv = 0$.

(2) $\mathscr{F}$ is called *uniformly v-integral bounded* if $\sup_{f\in\mathscr{F}}\int |f|dv < \infty$.

(3) $\mathscr{F}$ is called *uniformly v-absolutely continuous* if for any $\varepsilon > 0$, there is a $\delta > 0$ such that $\sup_{f\in\mathscr{F}}\int_{\tilde{A}} |f|dv < \varepsilon$ for every $\tilde{A} \in \mathcal{K}$ with $\hat{v}(\tilde{A}) < \delta$.

Theorem 4.5. *Let* $\mathscr{F}$ *be a nonempty subset of* $\mathscr{B}_+$. *If* v *is subadditive and* $\hat{v}(X) < \infty$, *then* $\mathscr{F}$ *is uniformly v-integral bounded and uniformly v-absolutely continuous if* $\mathscr{F}$ *is uniformly v-integrable.*

Proof. Sufficiency: Since v is subadditive, by (9) of **Proposition 3.6**, for every $\tilde{A} \in \mathcal{K}$ and any $c > 0$ we have

$\sup_{f\in\mathscr{F}}\int_{\tilde{A}} fdv \le \sup_{f\in\mathscr{F}}\int_{\tilde{A}} fI_{\{f\le c\}}dv + \sup_{f\in\mathscr{F}}\int_{\tilde{A}} fI_{\{f>c\}}dv \le c\hat{v}(\tilde{A}) + \sup_{f\in\mathscr{F}}\int fI_{\{f>c\}}dv$.

Fix $\varepsilon > 0$. Since $\mathscr{F}$ is uniformly v-integrable, there is a $c_0 > 0$ such that $\sup_{f\in\mathscr{F}}\int fI_{\{f>c_0\}}dv < \frac{\varepsilon}{2}$. Moreover, there is a $\delta > 0$ such that $c_0\hat{v}(\tilde{A}) < \frac{\varepsilon}{2}$ for every $\hat{v}(\tilde{A}) < \delta$. Thus take any $\tilde{A} \in \mathcal{K}$ with $v(\tilde{A}) < \delta$, $\sup_{f\in F}\int_{\tilde{A}} fdv < \varepsilon$.

Then, let $\tilde{A} = X$, $c = c_0$. Since $\hat{v}(X) < \infty$, from the proof above it follows that

$\sup_{f\in F}\int fdv \le c\hat{v}(X) + \frac{\varepsilon}{2} < \infty$.

Necessity: Since $\mathscr{F}$ is uniformly v-integral bounded, for any $c > 0$ we have

$\sup_{f\in\mathscr{F}}\int cI_{\{f>c\}}dv \le \sup_{f\in\mathscr{F}}\int fI_{\{f>c\}}dv \le \sup_{f\in\mathscr{F}}\int fdv < \infty$.

Thus from $\sup_{f\in\mathscr{F}}c\hat{v}(f > c) < \infty$, it follows that $\lim\limits_{c\to\infty}\sup_{f\in\mathscr{F}}\hat{v}(f > c) = 0$, this is, for any $\varepsilon > 0$, there is $c_0 > 0$ such that $\sup_{f\in\mathscr{F}}\hat{v}(f > c) < \delta$ for any $c \ge c_0$. Moreover, $\mathscr{F}$ is uniformly v-absolutely continuous, this implies that $\sup_{f\in\mathscr{F}}\int fI_{\{f>c\}}dv < \varepsilon$. Thus $\mathscr{F}$ is uniformly uniformly v-integrable.

Unless stated otherwise, for a fuzzy capacity v, we always assume $\hat{v}(X) < \infty$ in the following.

Lemma 4.6. *Let* $\{f, f_n\} \subset \mathscr{B}_+$, *if* $f_n \downarrow$ *and* $\lim\limits_{n\to\infty} f_n \le f$ *a.e.* (or *p.a.e.*), $\lim\limits_{c\to\infty}\int f_1 I_{\{f_1>c\}}dv = 0$, *v is lower semicontinuous and subadditive*, $\hat{v}$ *is upper semicontinuous*, *denote by* $Q := \{x : f_n(x) > 0 (\forall n \in N),\ f(x) = 0\}$ *with* $\hat{v}(Q) = 0$, *then for every* $\tilde{A} \in \mathcal{K}$

$\lim\limits_{n\to\infty}\int_{\tilde{A}} f_n dv \le \int_{\tilde{A}} fdv$.

Proof. Denote by $D_1 := \left\{x : f_n(x)\downarrow, \lim\limits_{n\to\infty} f_n(x) \le f(x)\right\}$.If D_1 *a.e.*, Since v is subadditive, without loss generality, we can assume that $f_n \downarrow$ and $\lim\limits_{n\to\infty} f_n \le f$ for every x. For every $\alpha \in (0,1)$, denote by $C_n(\alpha) := \{\alpha f_n \le f\}$, then $\alpha\int_{\tilde{A}} f_n I_{C_n(\alpha)}dv \le \int_{\tilde{A}} fI_{C_n(\alpha)}dv$. From (9) of **Proposition 3.6**, it follows that

$\alpha\int_{\tilde{A}} f_n dv = \alpha\int_{\tilde{A}} f_n I_{C_n(\alpha)}dv + \alpha\int_{\tilde{A}} f_n I_{C_n^c(\alpha)}dv$.

Since $f_n \downarrow$, $\lim\limits_{n\to\infty} f_n \le f$, $C_n(\alpha) \uparrow Q^c$ and $C_n^c(\alpha) \downarrow Q$, we have $fI_{C_n(\alpha)} \uparrow fI_{F^c}$. Thus $\lim\limits_{n\to\infty}\int_{\tilde{A}} fI_{C_n(\alpha)}dv = \int_{\tilde{A}} fI_{Q^c}dv = \int_{\tilde{A}} fdv$, where the first equation holds due to **Theorem4.3**, the second equation holds due to (10) of **Proposition 3.6.** Moreover, $\lim\limits_{c\to\infty}\int f_1 I_{\{f_1>c\}}dv = 0$, we have that $\{f_n\}$ is uniformly v-integrable. It follows that $\alpha\lim\limits_{n\to\infty}\int_{\tilde{A}} f_n I_{C_n^c(\alpha)}dv = 0$ by **Theorem 4.5**, as $\hat{v}$ *is upper semicontinuous* and $\hat{v}(Q) = 0$. Thus $\lim\limits_{n\to\infty}\alpha\int_{\tilde{A}} f_n dv \le \int_{\tilde{A}} fdv$.

Let $\alpha \to 1$, the result follows.

If D_1 *p.a.e.*, we have that $\int_{\tilde{A}} f_n dv = \int_{\tilde{A}} f_n I_{D_1} dv = \int_{\tilde{A}} f_n I_{D_1 \cap C_n(\alpha)} dv + \int_{\tilde{A}} f_n I_{D_1 \cap C_n^c(\alpha)} dv$ and $\int_{\tilde{A}} f dv = \int_{\tilde{A}} f I_{D_1} dv = \int_{\tilde{A}} f I_{D_1 \cap F^c} dv$ by (7) and (10) of **Proposition 3.7**, also we have $D_1 \cap C_n(\alpha) \uparrow D_1 \cap F^c$ and $D_1 \cap C_n^c(\alpha) \downarrow D_1 \cap F$. Therefore, the proof of the rest is similar to the one above.

Theorem 4.7. (Absolute continuity theorem of integral) *If $f \in \mathscr{B}_+$ is integrable, then for any $\varepsilon > 0$, there exists $\delta > 0$ for every $\tilde{A} \in \mathcal{K}$* with $\hat{v}(\tilde{A}) < \delta$ *such that $\int_{\tilde{A}} f dv < \varepsilon$.*

Proof. Since $f \in B_+$, there exists nonnegative increasing simple measurable function series $\{f_n\}$ which converge to f. By **Theorem 4.2**, it follows that $\int_{\tilde{A}} f_n dv \uparrow \int_{\tilde{A}} f dv$ for every $\tilde{A} \in F$. Thus for any $\varepsilon > 0$, there exists positive integer n_ε, such that $\int f dv - \int f_{n_\varepsilon} dv < \frac{\varepsilon}{2}$. Denote by $M := \max_{x \in X} f_{n_\varepsilon}(x)$, then $\int_{\tilde{A}} f dv < \frac{\varepsilon}{2} + \int_{\tilde{A}} f_{n_\varepsilon} dv \le \frac{\varepsilon}{2} + M\hat{v}(\tilde{A})$. Therefore, when take $\delta = \frac{\varepsilon}{(2M)}$, the conclusion holds.

Lemma 4.8. *Let $\{f, f_n\} \subset \mathscr{B}_+$, if $f_n \downarrow$ and $\lim\limits_{n\to\infty} f_n \le f$ a.e.* (or *p.a.e.*), *there exists a integrable function $g \in \mathscr{B}_+$, such that $f_n \le g$ $(\forall n \in N)$ a.e.* (or *p.a.e.*), *v is lower semicontinuous and subadditive, $\hat{v}$ is upper semicontinuous, denote by*

$F := \{x : f_n(x) > 0 (\forall n \in N), f(x) = 0\}$ *with* $\hat{v}(F) = 0$,

then for every $\tilde{A} \in \mathcal{K}$, $\lim\limits_{n\to\infty} \int_{\tilde{A}} f_n dv \le \int_{\tilde{A}} f dv$.

Proof. By (7), (10) of **Proposition 3.6**, **Remark 4** and **Theorem 4.7**, it follows from similar proof of **Lemma 4.6**.

Lemma 4.9. (Fatou Lemma) *If $\{f_n\} \subset \mathscr{B}_+$ is a nonnegative measurable function series, v is lower semicontinuous, then for every $\tilde{A} \in \mathcal{K}$*

$\int_{\tilde{A}} (\liminf\limits_{n\to\infty} f_n) dv \le \liminf\limits_{n\to\infty} \int_{\tilde{A}} f dv$.

Proof. It is easily proved by **Theorem 4.2.**

Theorem 4.10. (Almost Everywhere Convergence Theorem) *Let $\{f, f_n\} \subset \mathscr{B}_+$, if $\lim\limits_{n\to\infty} f_n = f$ a.e. , there exists a integrable function $g \in \mathscr{B}_+$, such that $f_n \le g$ $(\forall n \in N)$ a.e.* (or *p.a.e.*), *v is lower semicontinuous and subadditive, $\hat{v}$ is upper semicontinuous, denoted by $F_b := \{x : \sup_{k\ge n} f_k(x) > 0 (\forall n \in N), f(x) = 0\}$ with $\hat{v}(F_b) = 0$, then for every $\tilde{A} \in \mathcal{K}$*

$\lim\limits_{n\to\infty} \int_{\tilde{A}} f_n dv = \int_{\tilde{A}} f dv$.

Proof. Denote by $D_2 := \left\{ x : \lim\limits_{n\to\infty} f_n = f \right\}$, $e_n := \inf_{k\ge n} f_k$ and $g_n := \sup_{k\ge n} f_k$ $(\forall k, n \in N)$. Since $e_n \le f_n \le g_n (\forall n \in N)$.we have $\int_{\tilde{A}} e_n d\hat{v} \le \int_{\tilde{A}} f_n d\hat{v} \le \int_{\tilde{A}} g_n d\hat{v}$ $(\forall n \in N)$. If D_2 *a.e.*, we have $g_n \uparrow f$, $e_n \downarrow f$ *a.e.* and $g_n \le g$ $(\forall n \in N)$ *a.e.*. Thus $\lim\limits_{n\to\infty} \int_{\tilde{A}} e_n dv = \lim\limits_{n\to\infty} \int_{\tilde{A}} g_n dv = \int_{\tilde{A}} f dv$ by **Remark 4**, **Theorem 4.3** and **Lemma 4.8**, which implies that $\lim\limits_{n\to\infty} \int_{\tilde{A}} f_n dv = \int_{\tilde{A}} f dv$.

Remark 5. From the proof of **Theorem 4.10**, we know that if replace $\lim\limits_{n\to\infty} f_n = f$ a.e. with $\lim\limits_{n\to\infty} f_n = f$ p.a.e. in **Theorem 4.10**, the conclusion is still valid.

Theorem 4.11. (Convergence in fuzzy measure theorem)

Let $\{f, f_n\} \subset \mathscr{B}_+$, $f_n \xrightarrow{v} f$, if $\lim_{c\to\infty} \int \sup_{n\ge1} f_n I_{\{\sup_{n\ge1} f_n > c\}} dv = 0$, v is autocontinuous from below, lower semicontinuous and subadditive, $\hat{v}$ is upper semicontinuous, denote by $F_b := \{x : \sup_{k\ge n} f_k(x) > 0 (\forall n \in N), f(x) = 0\}$ with $\hat{v}(F_b) = 0$, then for every $\tilde{A} \in \mathcal{K}$

$\lim\limits_{n\to\infty} \int_{\tilde{A}} f_n dv = \int_{\tilde{A}} f dv$.

Proof. If $f_n \xrightarrow{v} f$, since v is autocontinuous from below, by **Theorem 3.4**, for any subsequence $\{f_{n_k}\}$ of $\{f_n\}$ there exists subsequence $\{f_{n_{k_j}}\}$ of $\{f_{n_k}\}$ such that $f_{n_{k_j}} \to f$ *p.a.e.*. By **Theorem 4.2** and **Lemma 4.6**, it follows that $\lim_{j\to\infty} \int_{\tilde{A}} f_{n_{k_j}} dv = \int_{\tilde{A}} f dv$. By **Remark 4**, this means that $\lim\limits_{n\to\infty} \int_{\tilde{A}} f_n dv = \int_{\tilde{A}} f dv$.

To the abstract Lebesgue integral, the Vitali convergence in measure theorem is very important, which ensures the convergence of a sequence of Lebesgue integrals without a dominant function. We will give the Vitali theorem of the version of fuzzy concave integral in the following.

Theorem 4.12. (The Vitali convergence theorem) *Let* $\{f, f_n\} \subset \mathscr{B}_+$, $f_n \xrightarrow{\hat{v}} f$, $\{f_n\} \in L^p(v)(1 \le p < \infty)$, $\{f_n^p\}$ *is uniformly* v*-integrable,* v *is autocontinuous from below, lower semicontinuous and subadditive, then*

$$f \in L^p(v)$$

and

$$\int |f_n - f|^p dv \to 0$$

Proof. Since $f_n \xrightarrow{\hat{v}} f$, it follows that $f_n \xrightarrow{v} f$ by **Lemma 3.8**. Moreover, v is autocontinuous from below, by **Theorem 3.4**, for any subsequence $\{f_{n_k}\}$ of $\{f_n\}$ there exists subsequence $\left\{f_{n_{k_j}}\right\}$ of $\{f_{n_k}\}$ such that $f_{n_{k_j}} \to f$ *p.a.e.* by **Theorem 4.2** and **Lemma 4.6**. Since $\{f_n\}$ is uniformly v-integrable and v is lower semicontinuous, we have

$$\int f^p dv = \int \lim_{j\to\infty} f_{n_{k_j}}^p dv \le \liminf_{j\to\infty} \int f_{n_{k_j}}^p dv \le \sup_{n\ge 1} \int f_n^p dv < \infty$$

by **Lemma 4.9**. This implies that $f \in L^p(v)$. Furthermore, v is subadditive

$$\begin{aligned}
\int |f_n - f|^p dv &= \int_{\{|f_n - f|^p \le \varepsilon\}} |f_n - f|^p + \int_{\{|f_n - f|^p > \varepsilon\}} |f_n - f|^p \\
&= \int_{\{|f_n - f| \le \varepsilon\}} |f_n - f|^p dv + \int_{\{f_n > f+\varepsilon\}} |f_n - f|^p + \int_{\{f_n < f-\varepsilon\}} |f_n - f|^p \\
&\le \varepsilon \hat{v} \{|f_n - f| \le \varepsilon\} + \int_{\{f_n > f+\varepsilon\}} f_n^p dv + \int_{\{f_n < f-\varepsilon\}} f^p dv.
\end{aligned}$$

Since ε is arbitrary,if $f_n \xrightarrow{\hat{v}} f$ and $\{f_n^p\}$ is uniformly v-integrable, it follows that $\int |f_n - f|^p dv \to 0$ from **Theorem 4.5** and **Theorem 4.7**.

§5 Conclusion

Inspired by the idea of the concavification of cooperative game, we have proposed the new concept of fuzzy concave integral on fuzzy set, and have obtained some desirable properties. Based on these, we proved some convergence theorems of a sequence of fuzzy concave integrals on fuzzy σ-algebra. Relevant results can provide new theoretical basis for risk evaluation under uncertainty caused by fuzzy events in economic activities.

Declarations

Conflict of interest The authors declare no conflict of interest.

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