Appl. Math. J. Chinese Univ.
2024, 39(3): 398-415

The joint Laplace transforms for killed diffusion occupation times

LI Ying-qiu[1,*]   CHEN Ye[2]

Abstract. The approach of Li and Zhou (2014) is adopted to find the Laplace transform of occupation time over interval $(0, a)$ and joint occupation times over semi-infinite intervals $(-\infty, a)$ and (b, ∞) for a time-homogeneous diffusion process up to an independent exponential time e_q for $0 < a < b$. The results are expressed in terms of solutions to the differential equations associated with the diffusion generator. Applying these results, we obtain explicit expressions on the Laplace transform of occupation time and joint occupation time for Brownian motion with drift.

§1 Introduction

Occupation time is one of the hot issues in the theoretical research on the stochastic process, and is widely used in risk theory and financial models, see for example [1]-[12]. Some classical results related to the Laplace transform on occupation time for examples of diffusion process can be found in Borodin and Salminen [1]. In Cai et al [2] the Laplace transform of occupation time is found for a jump-diffusion with two-sided exponential jumps by a standard approach of specifying and solving the associated partial-integro-differential equation. Pitman and Yor [12, 13], applied the excursion theory to obtain a formula for the joint Laplace transform of the occupation times spent by the process either above or below a level up to a suitable random time. In Landriault et al [6], an approximation scheme was proposed for the first time, which combined with the fluctuation theory, gave th Laplace transform of occupation time for a spectrally negative Lévy process(in short, SNLP). Such a method is also applied to study the joint Laplace transforms of occupation time for time-homogeneous diffusion processes in Li and Zhou [7]. Using the same approach, Li et al [8] found expressions of double Laplace transforms for diffusion processes.

Received: 2019-03-26. Revised: 2022-01-07.
MR Subject Classification: 60J60, 60G17, 60G40.
Keywords: time-homogeneous diffusion process, occupation time, joint occupation time, Laplace transform, Brownian motion with drift.
Digital Object Identifier(DOI): https://doi.org/10.1007/s11766-024-3792-6.
Supported by the National Natural Science Foundation of China(12271062,11731012), by the Hunan Provincial National Natural Science Foundation of China (2019JJ50405).
*Corresponding author.

In order to overcome the difficulty caused by the possible infinite activity of SNLP, Li and Zhou [10] used a new approach of identifying the Laplace transforms with a fluctuation result on the SNLP observed at independent Poisson arrival times to study the Laplace transforms of pre-exit joint occupation time for SNLP. Li et al [11] used the Poisson approach to obtain expressions of two-sided discounted potential measures for SNLP. For diffusion processes, the approach also works as well. It was adopted to find expressions of potential measures that are discounted by their joint occupation times over semi-infinite intervals $(-\infty, a)$ and (a, ∞) in Chen et al [3], and to obtain their joint Laplace transforms of occupation times over intervals (a, r) and (r, b) before it first exits from either a or b for $a < r < b$ in Chen et al [4].

In this paper, we adopt the Poisson approach of [10] to consider the Laplace transform of occupation time over interval $(0, a)$ and joint occupation times over semi-infinite intervals $(-\infty, a)$ and (b, ∞) for one-dimensional time-homogeneous diffusion processes up to an independent exponential time e_q for $0 < a < b$. The results are expressed in terms of solutions to the differential equations associated with the diffusion generator. Our results generalize those in [7] and our approach is different.

The rest of this paper is arranged as follows. In Section 2, we briefly introduce some relevant knowledge about the diffusion process and some classical conclusions of exit problems. In Section 3, we obtain our main results of occupation time related Laplace transforms and some corollaries. In Section 4, using the conclusion of Section 3, we get the result of Brownian motion with drift.

§2 Preliminaries

In this paper, we consider a one-dimensional diffusion process $X = (X_t)_{t\geq 0}$ defined on a filtered probability space $(\Omega, \mathbb{P}, (\mathscr{F}_t)_{t\geq 0})$, and take values in the interval I with endpoints $-\infty \leq l_1 \leq l_2 \leq \infty$, which is specified by the following stochastic differential equation

$$\mathrm{d}X_t = \mu(X_t)\mathrm{d}t + \sigma(X_t)\mathrm{d}W_t, \tag{2.1}$$

where $W = \{W_t, t \geq 0\}$ is a one-dimensional standard *Brownian* motion and $X_0 = x_0$ is the initial value. Throughout the paper, we assume that equation (2.1) allows a unique weak solution, i.e. there exists a constant $K > 0$ such that

$$|\mu(x) - \mu(y)| + |\sigma(x) - \sigma(y)| \leq K|x - y|, \qquad \mu^2(x) + \sigma^2(x) \leq K^2(1 + x^2).$$

Two basic characteristics of diffusion processes X, the speed measure m and the scale function s, are given by

$$m(\mathrm{d}x) = m(x)\mathrm{d}x := \frac{2e^{B(x)}}{\sigma^2(x)}\mathrm{d}x, \quad s(x) := \int^x e^{-B(y)}\mathrm{d}y$$

for $l_1 < x < l_2$, where

$$B(x) := \int^x 2\mu(y)/\sigma^2(y)\mathrm{d}y.$$

Let $p(\cdot; \cdot, \cdot)$ be the transition density of X with respect to the speed measure for diffusion processes, i.e.

$$\mathbb{P}_x(X_t \in \mathrm{d}y) = p(t; x, y)m(\mathrm{d}y).$$

For $\lambda > 0$, let $g_{-,\lambda}(\cdot)$ and $g_{+,\lambda}(\cdot)$ be two independent positive solutions to the (generalized) differential equation associated to the generator of X

$$\frac{1}{2}\sigma^2(x)g''(x) + \mu(x)g'(x) = \lambda g(x), \tag{2.2}$$

with $g_{-,\lambda}(\cdot)$ decreasing and $g_{+,\lambda}(\cdot)$ increasing. Here a solution $g(x)$ to equation (2.2) satisfies

$$\lambda \int_{[a,b)} g(x)m(x)\mathrm{d}x = g^-(b) - g^-(a),$$

where

$$g^-(x) := \lim_{h\to 0_+} \frac{g(x) - g(x-h)}{s(x) - s(x-h)}.$$

The Green *function* for X is

$$G_\lambda(x,y) := \int_0^\infty e^{-\lambda t} p(t;x,y)\mathrm{d}t.$$

Then

$$G_\lambda(x,y) = \begin{cases} \omega_\lambda^{-1} g_{+,\lambda}(x)g_{-,\lambda}(y), & x \le y, \\ \omega_\lambda^{-1} g_{+,\lambda}(y)g_{-,\lambda}(x), & x \ge y, \end{cases}$$

where

$$\omega_\lambda := g_{+,\lambda}^+(x)g_{-,\lambda}(x) - g_{+,\lambda}(x)g_{-,\lambda}^+(x) = g_{+,\lambda}^-(x)g_{-,\lambda}(x) - g_{+,\lambda}(x)g_{-,\lambda}^-(x)$$

is the so-called *Wronskian* with

$$g^+(x) := \lim_{h\to 0+} \frac{g(x+h) - g(x)}{s(x+h) - s(x)}.$$

It is known that ω_λ is independent of x.

We refer to Chapter II of Borodin and Salminen [1] for the facts and more details about diffusion processes.

Further, for $\lambda > 0$, define

$$f_\lambda(y,z) := g_{-,\lambda}(y)g_{+,\lambda}(z) - g_{-,\lambda}(z)g_{+,\lambda}(y).$$

Let $\tau_x := \inf\{t \ge 0 : X_t = x\}$ be the first passage time of X at level x with the convention $\inf \phi = \infty$. Write

$$\mathbb{E}_{x_0}[\,\cdot\,] = \mathbb{E}[\,\cdot\,|\, X_0 = x_0], \quad \mathbb{E}_{x_0}[\,\cdot\,; C] = \mathbb{E}_{x_0}[\,\cdot\, 1_C]$$

with 1_C denoting the indicator function of an event, and we drop the subscript for initial value if $x_0 = 0$.

For $a < x < b$ and $\lambda > 0$, we have the following well-known solutions to the exit problem:

$$\mathbb{E}_x e^{-\lambda\tau_a} = \frac{g_{-,\lambda}(x)}{g_{-,\lambda}(a)}, \quad \mathbb{E}_x e^{-\lambda\tau_b} = \frac{g_{+,\lambda}(x)}{g_{+,\lambda}(b)}, \tag{2.3}$$

$$\mathbb{E}_x[e^{-\lambda\tau_a}; \tau_a < \tau_b] = \frac{f_\lambda(x,b)}{f_\lambda(a,b)}, \quad \mathbb{E}_x[e^{-\lambda\tau_b}; \tau_b < \tau_a] = \frac{f_\lambda(a,x)}{f_\lambda(a,b)}, \tag{2.4}$$

see e.g. Borodin and Salminen [1], Feller [5] and Li and Zhou [7].

The potential measure for diffusion processes is needed for our main results in Section 3. By definition, its expression is easily given as follows

$$\int_0^\infty \mathbb{P}_x\{X_t \in \mathrm{d}y\} e^{-\lambda t}\mathrm{d}t = G_\lambda(x,y)m(\mathrm{d}y), \quad \lambda \ge 0. \tag{2.5}$$

§3 Main results

Throughout the paper, take e_q to be an independent exponential random variable with rate q. We first obtain the Laplace transform of occupation time denoted by

$$f_-(y) := \mathbb{E}_y e^{-\lambda\int_0^{e_q}\mathbf{1}_{(0,a)}(X_s)\mathrm{d}s}.$$

Theorem 3.1 For any $\lambda > 0$, we have for any $y < 0$,

$$f_-(y) = 1 - \frac{g_{+,q}(y)}{g_{+,q}(0)} + \frac{g_{+,q}(y)}{g_{+,q}(0)} f_-(0), \tag{3.1}$$

for any $y > a$,

$$f_-(y) = 1 - \frac{g_{-,q}(y)}{g_{-,q}(a)} + \frac{g_{-,q}(y)}{g_{-,q}(a)} f_-(a), \tag{3.2}$$

for any $0 < y < a$,

$$f_-(y) = \frac{q}{q+\lambda}\big(1 - \frac{f_{q+\lambda}(y,a) + f_{q+\lambda}(0,y)}{f_{q+\lambda}(0,a)}\big) + \frac{f_{q+\lambda}(y,a)}{f_{q+\lambda}(0,a)} f_-(0) + \frac{f_{q+\lambda}(0,y)}{f_{q+\lambda}(0,a)} f_-(a), \tag{3.3}$$

where

$$f_-(0) = \frac{A}{C}, \qquad f_-(a) = \frac{B}{C},$$

$$A := \big(1 - \lambda\int_a^\infty G_{q+\lambda}(a,y)\frac{g_{-,q}(y)}{g_{-,q}(a)}m(\mathrm{d}y)\big)\cdot U(0) + \lambda\int_a^\infty G_{q+\lambda}(0,y)\frac{g_{-,q}(y)}{g_{-,q}(a)}m(\mathrm{d}y)\cdot U(a),$$

$$B := \big(1 - \lambda\int_{-\infty}^0 G_{q+\lambda}(0,y)\frac{g_{+,q}(y)}{g_{+,q}(0)}m(\mathrm{d}y)\big)\cdot U(a) + \lambda\int_{-\infty}^0 G_{q+\lambda}(a,y)\frac{g_{+,q}(y)}{g_{+,q}(0)}m(\mathrm{d}y)\cdot U(0),$$

$$\begin{aligned} C := &\big(1 - \lambda\int_a^\infty G_{q+\lambda}(a,y)\frac{g_{-,q}(y)}{g_{-,q}(a)}m(\mathrm{d}y)\big)\cdot\big(1 - \lambda\int_{-\infty}^0 G_{q+\lambda}(0,y)\frac{g_{+,q}(y)}{g_{+,q}(0)}m(\mathrm{d}y)\big) \\ &-\lambda\int_a^\infty G_{q+\lambda}(0,y)\frac{g_{-,q}(y)}{g_{-,q}(a)}m(\mathrm{d}y)\cdot\lambda\int_{-\infty}^0 G_{q+\lambda}(a,y)\frac{g_{+,q}(y)}{g_{+,q}(0)}m(\mathrm{d}y), \end{aligned}$$

$$\begin{aligned} U(\cdot) := &\frac{q}{q+\lambda} + \lambda\int_{-\infty}^0 G_{q+\lambda}(\cdot,y)m(\mathrm{d}y) - \lambda\int_{-\infty}^0 G_{q+\lambda}(\cdot,y)\frac{g_{+,q}(y)}{g_{+,q}(0)}m(\mathrm{d}y) \\ &+\lambda\int_a^\infty G_{q+\lambda}(\cdot,y)m(\mathrm{d}y) - \lambda\int_a^\infty G_{q+\lambda}(\cdot,y)\frac{g_{-,q}(y)}{g_{-,q}(a)}m(\mathrm{d}y). \end{aligned}$$

Proof: Write $0 < T_1 < T_2 < \cdots$ for the arrival time of independent Poisson processes with rate λ, We also assume that these Poisson processes are independent of process X. By a property of Poisson process we observe that $f_-(y) = \mathbb{P}_y\{D_-\}$, for event

$$D_- := \{\{T_i\} \cap \{s < e_q : 0 < X_s < a\} = \varnothing\}.$$

For independent exponential random variables T with rate λ, using (2.3) and (2.5), we get

$$\begin{aligned} f_-(0) &= \mathbb{P}\{e_q < T\} + \int_{-\infty}^0 \mathbb{P}\{T < e_q, X_T \in \mathrm{d}y\}f_-(y) + \int_a^\infty \mathbb{P}\{T < e_q, X_T \in \mathrm{d}y\}f_-(y) \\ &= \mathbb{E}e^{-\lambda e_q} + \lambda\int_{-\infty}^0\int_0^\infty e^{-(q+\lambda)t}\mathbb{P}\{X_t \in \mathrm{d}y\}f_-(y)\mathrm{d}t \\ &\quad + \lambda\int_a^\infty\int_0^\infty e^{-(q+\lambda)t}\mathbb{P}\{X_t \in \mathrm{d}y\}f_-(y)\mathrm{d}t \\ &= \frac{q}{q+\lambda} + \lambda\int_{-\infty}^0 G_{q+\lambda}(0,y)f_-(y)m(\mathrm{d}y) + \lambda\int_a^\infty G_{q+\lambda}(0,y)f_-(y)m(\mathrm{d}y), \end{aligned} \tag{3.4}$$

$$
\begin{aligned}
f_-(a) &= \mathbb{P}_a\{e_q < T\} + \int_{-\infty}^{0} \mathbb{P}_a\{T < e_q, X_T \in \mathrm{d}y\} f_-(y) + \int_a^{\infty} \mathbb{P}_a\{T < e_q, X_T \in \mathrm{d}y\} f_-(y) \\
&= \mathbb{E}e^{-\lambda e_q} + \lambda \int_{-\infty}^{0}\int_0^{\infty} e^{-(q+\lambda)t}\mathbb{P}_a\{X_t \in \mathrm{d}y\} f_-(y)\mathrm{d}t \\
&\quad + \lambda \int_a^{\infty}\int_0^{\infty} e^{-(q+\lambda)t}\mathbb{P}_a\{X_t \in \mathrm{d}y\} f_-(y)\mathrm{d}t \\
&= \frac{q}{q+\lambda} + \lambda \int_{-\infty}^{0} G_{q+\lambda}(a,y) f_-(y) m(\mathrm{d}y) + \lambda \int_a^{\infty} G_{q+\lambda}(a,y) f_-(y) m(\mathrm{d}y). \qquad (3.5)
\end{aligned}
$$

For $y < 0$, using (2.3), we get

$$
\begin{aligned}
f_-(y) &= \mathbb{P}_y\{e_q < \tau_0\} + \mathbb{P}_y\{\tau_0 < e_q\} f_-(0) \\
&= 1 - \mathbb{E}_y e^{-q\tau_0} + \mathbb{E}_y e^{-q\tau_0} f_-(0) = 1 - \frac{g_{+,q}(y)}{g_{+,q}(0)} + \frac{g_{+,q}(y)}{g_{+,q}(0)} f_-(0); \qquad (3.6)
\end{aligned}
$$

for $y > a$, using (2.3), we get

$$
\begin{aligned}
f_-(y) &= \mathbb{P}_y\{e_q < \tau_a\} + \mathbb{P}_y\{\tau_a < e_q\} f_-(a) \\
&= 1 - \mathbb{E}_y e^{-q\tau_a} + \mathbb{E}_y e^{-q\tau_a} f_-(a) = 1 - \frac{g_{-,q}(y)}{g_{-,q}(a)} + \frac{g_{-,q}(y)}{g_{-,q}(a)} f_-(a); \qquad (3.7)
\end{aligned}
$$

for $0 < y < a$, using (2.4), we get

$$
\begin{aligned}
f_-(y) &= \mathbb{E}_y[e^{-\lambda e_q}; e_q < \tau_0 \wedge \tau_a] + \mathbb{E}_y[e^{-\lambda \tau_0}; \tau_0 < e_q \wedge \tau_a] f_-(0) \\
&\quad + \mathbb{E}_y[e^{-\lambda \tau_a}; \tau_a < e_q \wedge \tau_0] f_-(a) \\
&= \frac{q}{q+\lambda}(1 - \mathbb{E}_y e^{-(q+\lambda)(\tau_0 \wedge \tau_a)}) + \mathbb{E}_y[e^{-(q+\lambda)\tau_0}; \tau_0 < \tau_a] f_-(0) \\
&\quad + \mathbb{E}_y[e^{-(q+\lambda)\tau_a}; \tau_a < \tau_0] f_-(a) \\
&= \frac{q}{q+\lambda}(1 - \frac{f_{q+\lambda}(y,a) + f_{q+\lambda}(0,y)}{f_{q+\lambda}(0,a)}) + \frac{f_{q+\lambda}(y,a)}{f_{q+\lambda}(0,a)} f_-(0) + \frac{f_{q+\lambda}(0,y)}{f_{q+\lambda}(0,a)} f_-(a). \qquad (3.8)
\end{aligned}
$$

Putting (3.6), (3.7) into (3.4), (3.5), respectively, we can get the first-order equations of $f_-(0)$ and $f_-(a)$ for two variables:

$$
\begin{aligned}
f_-(0) = \frac{q}{q+\lambda} &+ \lambda \int_{-\infty}^{0} G_{q+\lambda}(0,y)(1 - \frac{g_{+,q}(y)}{g_{+,q}(0)}) m(\mathrm{d}y) + \lambda \int_a^{\infty} G_{q+\lambda}(0,y)(1 - \frac{g_{-,q}(y)}{g_{-,q}(a)}) m(\mathrm{d}y) \\
&+ \lambda \int_{-\infty}^{0} G_{q+\lambda}(0,y) \frac{g_{+,q}(y)}{g_{+,q}(0)} m(\mathrm{d}y) f_-(0) + \lambda \int_a^{\infty} G_{q+\lambda}(0,y) \frac{g_{-,q}(y)}{g_{-,q}(a)} m(\mathrm{d}y) f_-(a),
\end{aligned}
$$

$$
\begin{aligned}
f_-(a) = \frac{q}{q+\lambda} &+ \lambda \int_{-\infty}^{0} G_{q+\lambda}(a,y)(1 - \frac{g_{+,q}(y)}{g_{+,q}(0)}) m(\mathrm{d}y) + \lambda \int_a^{\infty} G_{q+\lambda}(a,y)(1 - \frac{g_{-,q}(y)}{g_{-,q}(a)}) m(\mathrm{d}y) \\
&+ \lambda \int_{-\infty}^{0} G_{q+\lambda}(a,y) \frac{g_{+,q}(y)}{g_{+,q}(0)} m(\mathrm{d}y) f_-(0) + \lambda \int_a^{\infty} G_{q+\lambda}(a,y) \frac{g_{-,q}(y)}{g_{-,q}(a)} m(\mathrm{d}y) f_-(a).
\end{aligned}
$$

We can obtain,

$$
Cf_-(0) = A, \qquad Cf_-(a) = B,
$$

with A, B and C given by

$$\begin{aligned}A=&\big(1-\lambda\int_a^\infty G_{q+\lambda}(a,y)\frac{g_{-,q}(y)}{g_{-,q}(a)}m(\mathrm{d}y)\big)\\&\times\left[\frac{q}{q+\lambda}+\lambda\int_{-\infty}^0 G_{q+\lambda}(0,y)(1-\frac{g_{+,q}(y)}{g_{+,q}(0)})m(\mathrm{d}y)+\lambda\int_a^\infty G_{q+\lambda}(0,y)(1-\frac{g_{-,q}(y)}{g_{-,q}(a)})m(\mathrm{d}y)\right]\\&+\lambda\int_a^\infty G_{q+\lambda}(0,y)\frac{g_{-,q}(y)}{g_{-,q}(a)}m(\mathrm{d}y)\\&\times\left[\frac{q}{q+\lambda}+\lambda\int_{-\infty}^0 G_{q+\lambda}(a,y)(1-\frac{g_{+,q}(y)}{g_{+,q}(0)})m(\mathrm{d}y)+\lambda\int_a^\infty G_{q+\lambda}(a,y)(1-\frac{g_{-,q}(y)}{g_{-,q}(a)})m(\mathrm{d}y)\right]\\&=\big(1-\lambda\int_a^\infty G_{q+\lambda}(a,y)\frac{g_{-,q}(y)}{g_{-,q}(a)}m(\mathrm{d}y)\big)\cdot U(0)+\lambda\int_a^\infty G_{q+\lambda}(0,y)\frac{g_{-,q}(y)}{g_{-,q}(a)}m(\mathrm{d}y)\cdot U(a),\end{aligned}$$

$$\begin{aligned}B=&\big(1-\lambda\int_{-\infty}^0 G_{q+\lambda}(0,y)\frac{g_{+,q}(y)}{g_{+,q}(0)}m(\mathrm{d}y)\big)\\&\times\left[\frac{q}{q+\lambda}+\lambda\int_{-\infty}^0 G_{q+\lambda}(a,y)(1-\frac{g_{+,q}(y)}{g_{+,q}(0)})m(\mathrm{d}y)+\lambda\int_a^\infty G_{q+\lambda}(a,y)(1-\frac{g_{-,q}(y)}{g_{-,q}(a)})m(\mathrm{d}y)\right]\\&+\lambda\int_{-\infty}^0 G_{q+\lambda}(a,y)\frac{g_{+,q}(y)}{g_{+,q}(0)}m(\mathrm{d}y)\\&\times\left[\frac{q}{q+\lambda}+\lambda\int_{-\infty}^0 G_{q+\lambda}(0,y)(1-\frac{g_{+,q}(y)}{g_{+,q}(0)})m(\mathrm{d}y)+\lambda\int_a^\infty G_{q+\lambda}(0,y)(1-\frac{g_{-,q}(y)}{g_{-,q}(a)})m(\mathrm{d}y)\right]\\&=\big(1-\lambda\int_{-\infty}^0 G_{q+\lambda}(0,y)\frac{g_{+,q}(y)}{g_{+,q}(0)}m(\mathrm{d}y)\big)\cdot U(a)+\lambda\int_{-\infty}^0 G_{q+\lambda}(a,y)\frac{g_{+,q}(y)}{g_{+,q}(0)}m(\mathrm{d}y)\cdot U(0),\end{aligned}$$

and

$$\begin{aligned}C=&\big(1-\lambda\int_a^\infty G_{q+\lambda}(a,y)\frac{g_{-,q}(y)}{g_{-,q}(a)}m(\mathrm{d}y)\big)\cdot\big(1-\lambda\int_{-\infty}^0 G_{q+\lambda}(0,y)\frac{g_{+,q}(y)}{g_{+,q}(0)}m(\mathrm{d}y)\big)\\&-\lambda\int_a^\infty G_{q+\lambda}(0,y)\frac{g_{-,q}(y)}{g_{-,q}(a)}m(\mathrm{d}y)\cdot\lambda\int_{-\infty}^0 G_{q+\lambda}(a,y)\frac{g_{+,q}(y)}{g_{+,q}(0)}m(\mathrm{d}y).\end{aligned}$$

Putting $f_-(0)$ and $f_-(a)$ into (3.6),(3.7),(3.8), respectively, we can get the results of theorem 3.1.

For $0<a<b$, we consider the Laplace transform of joint occupation times over semi-infinite intervals $(-\infty,a)$ and (b,∞) for one-dimensional time-homogeneous diffusion processes killed at an exponential rate. Define $f_+(y):=\mathbb{E}_y e^{-\lambda_-\int_0^{e_q}\mathbf{1}_{(-\infty,a)}(X_s)\mathrm{d}s-\lambda_+\int_0^{e_q}\mathbf{1}_{(b,\infty)}(X_s)\mathrm{d}s}$.

Theorem 3.2 For any $\lambda_-,\lambda_+>0$, with $\lambda_-\neq\lambda_+$, we have
for any $y<a$,

$$f_+(y)=\frac{q}{q+\lambda_-}\big(1-\frac{g_{+,q+\lambda_-}(y)}{g_{+,q+\lambda_-}(a)}\big)+\frac{g_{+,q+\lambda_-}(y)}{g_{+,q+\lambda_-}(a)}f_+(a);\tag{3.9}$$

for any $y>b$,

$$f_+(y)=\frac{q}{q+\lambda_+}\big(1-\frac{g_{-,q+\lambda_+}(y)}{g_{-,q+\lambda_+}(b)}\big)+\frac{g_{-,q+\lambda_+}(y)}{g_{-,q+\lambda_+}(b)}f_+(b);\tag{3.10}$$

for any $a < y < b$,

$$f_+(y) = 1 - \frac{f_q(y,b) + f_q(a,y)}{f_q(a,b)} + \frac{f_q(y,b)}{f_q(a,b)} f_+(a) + \frac{f_q(a,y)}{f_q(a,b)} f_+(b), \tag{3.11}$$

where $f_+(a) = D/F$,$f_+(b) = E/F$,

$$\begin{aligned}
D &:= \Big[1 - (\lambda_- + \lambda_+) \int_a^b H(b;a,y) m(\mathrm{d}y) - \lambda_- \int_b^\infty I_-(b) m(\mathrm{d}y)\Big] \cdot V(a) \\
&\quad + \Big[\lambda_- \int_b^\infty I_-(a) m(\mathrm{d}y) + (\lambda_- + \lambda_+) \int_a^b H(a;a,y) m(\mathrm{d}y)\Big] \cdot V(b), \\
E &:= \Big[(\lambda_- + \lambda_+) \int_a^b H(b;y,b) m(\mathrm{d}y) + \lambda_+ \int_{-\infty}^a I_+(b) m(\mathrm{d}y)\Big] \cdot V(a) \\
&\quad + \Big[1 - (\lambda_- + \lambda_+) \int_a^b H(a;y,b) m(\mathrm{d}y) - \lambda_+ \int_{-\infty}^a I_+(a) m(\mathrm{d}y)\Big] \cdot V(b), \\
F &:= \Big[1 - (\lambda_- + \lambda_+) \int_a^b H(a;y,b) m(\mathrm{d}y) - \lambda_+ \int_{-\infty}^a I_+(a) m(\mathrm{d}y)\Big] \\
&\quad \times \Big[1 - (\lambda_- + \lambda_+) \int_a^b H(b;a,y) m(\mathrm{d}y) - \lambda_- \int_b^\infty I_-(b) m(\mathrm{d}y)\Big] \\
&\quad - \Big[\lambda_- \int_b^\infty I_-(a) m(\mathrm{d}y) + (\lambda_- + \lambda_+) \int_a^b H(a;a,y) m(\mathrm{d}y)\Big] \\
&\quad \times \Big[(\lambda_- + \lambda_+) \int_a^b H(b;y,b) m(\mathrm{d}y) + \lambda_+ \int_{-\infty}^a I_+(b) m(\mathrm{d}y)\Big], \\
H(x;u,v) &:= G_{q+\lambda_-+\lambda_+}(x,y) \frac{f_q(u,v)}{f_q(a,b)}, \quad I_-(x) := G_{q+\lambda_-+\lambda_+}(x,y) \frac{g_{-,q+\lambda_+}(y)}{g_{-,q+\lambda_+}(b)}, \\
I_+(x) &:= G_{q+\lambda_-+\lambda_+}(x,y) \frac{g_{+,q+\lambda_-}(y)}{g_{+,q+\lambda_-}(a)}, \\
V(\cdot) &:= \frac{q}{q+\lambda_-+\lambda_+} + \frac{q\lambda_-}{q+\lambda_+} \int_b^\infty G_{q+\lambda_-+\lambda_+}(\cdot,y) \big(1 - \frac{g_{-,q+\lambda_+}(y)}{g_{-,q+\lambda_+}(b)}\big) m(\mathrm{d}y) \\
&\quad + \frac{q\lambda_+}{q+\lambda_-} \int_{-\infty}^a G_{q+\lambda_-+\lambda_+}(\cdot,y) \big(1 - \frac{g_{+,q+\lambda_-}(y)}{g_{+,q+\lambda_-}(a)}\big) m(\mathrm{d}y) \\
&\quad + (\lambda_- + \lambda_+) \int_a^b G_{q+\lambda_-+\lambda_+}(\cdot,y) \big(1 - \frac{f_q(y,b) + f_q(a,y)}{f_q(a,b)}\big) m(\mathrm{d}y).
\end{aligned}$$

Proof. Write $0 < T_1^- < T_2^- < \cdots$ and $0 < T_1^+ < T_2^+ < \cdots$ for the arrived times of two independent Poisson processes with rates λ_- and λ_+, respectively. We also assume that Poisson processes are independent of process X. Define

$$D_+ := \{\{T_i^-\} \cap \{s < e_q : -\infty < X_s < a\} = \varnothing = \{T_i^+\} \cap \{s < e_q : b < X_s < \infty\}\},$$

using a property of Poisson processes that $f_+(y) = \mathbb{P}_y\{D_+\}$.

For independent exponential randow variables T_- and T_+ with rates λ_- and λ_+, respectively, using (2.3) and (2.5), we have

$$\begin{aligned}
f_+(a) &= \mathbb{P}_a\{e_q < T_- \wedge T_+\} + \int_{-\infty}^b \mathbb{P}_a\{T_+ < e_q \wedge T_-, X_{T_+} \in \mathrm{d}y\} f_+(y) \\
&\quad + \int_a^\infty \mathbb{P}_a\{T_- < e_q \wedge T_+, X_{T_-} \in \mathrm{d}y\} f_+(y)
\end{aligned}$$

$$
\begin{aligned}
&= \int_0^\infty e^{-(\lambda_-+\lambda_+)t} q e^{-qt} \mathrm{d}t + \lambda_- \int_a^b \int_0^\infty e^{-(q+\lambda_-+\lambda_+)t} \mathbb{P}_a\{X_t \in \mathrm{d}y\} f_+(y) \mathrm{d}t \\
&\quad + \lambda_- \int_b^\infty \int_0^\infty e^{-(q+\lambda_-+\lambda_+)t} \mathbb{P}_a\{X_t \in \mathrm{d}y\} f_+(y) \mathrm{d}t \\
&\quad + \lambda_+ \int_a^b \int_0^\infty e^{-(q+\lambda_-+\lambda_+)t} \mathbb{P}_a\{X_t \in \mathrm{d}y\} f_+(y) \mathrm{d}t \\
&\quad + \lambda_+ \int_{-\infty}^a \int_0^\infty e^{-(q+\lambda_-+\lambda_+)t} \mathbb{P}_a\{X_t \in \mathrm{d}y\} f_+(y) \mathrm{d}t \\
&= \frac{q}{q+\lambda_-+\lambda_+} + (\lambda_-+\lambda_+) \int_a^b G_{q+\lambda_-+\lambda_+}(a,y) f_+(y) m(\mathrm{d}y) \\
&\quad + \lambda_+ \int_{-\infty}^a G_{q+\lambda_-+\lambda_+}(a,y) f_+(y) m(\mathrm{d}y) + \lambda_- \int_b^\infty G_{q+\lambda_-+\lambda_+}(a,y) f_+(y) m(\mathrm{d}y), \qquad (3.12)
\end{aligned}
$$

$$
\begin{aligned}
f_+(b) &= \mathbb{P}_b\{e_q < T_- \wedge T_+\} + \int_{-\infty}^b \mathbb{P}_b\{T_+ < e_q \wedge T_-, X_{T_+} \in \mathrm{d}y\} f_+(y) \\
&\quad + \int_a^\infty \mathbb{P}_b\{T_- < e_q \wedge T_+, X_{T_-} \in \mathrm{d}y\} f_+(y) \\
&= \int_0^\infty e^{-(\lambda_-+\lambda_+)t} q e^{-qt} \mathrm{d}t + \lambda_- \int_a^b \int_0^\infty e^{-(q+\lambda_-+\lambda_+)t} \mathbb{P}_b\{X_t \in \mathrm{d}y\} f_+(y) \mathrm{d}t \\
&\quad + \lambda_- \int_b^\infty \int_0^\infty e^{-(q+\lambda_-+\lambda_+)t} \mathbb{P}_b\{X_t \in \mathrm{d}y\} f_+(y) \mathrm{d}t \\
&\quad + \lambda_+ \int_a^b \int_0^\infty e^{-(q+\lambda_-+\lambda_+)t} \mathbb{P}_b\{X_t \in \mathrm{d}y\} f_+(y) \mathrm{d}t \\
&\quad + \lambda_+ \int_{-\infty}^a \int_0^\infty e^{-(q+\lambda_-+\lambda_+)t} \mathbb{P}_b\{X_t \in \mathrm{d}y\} f_+(y) \mathrm{d}t \\
&= \frac{q}{q+\lambda_-+\lambda_+} + (\lambda_-+\lambda_+) \int_a^b G_{q+\lambda_-+\lambda_+}(b,y) f_+(y) m(\mathrm{d}y) \\
&\quad + \lambda_+ \int_{-\infty}^a G_{q+\lambda_-+\lambda_+}(b,y) f_+(y) m(\mathrm{d}y) + \lambda_- \int_b^\infty G_{q+\lambda_-+\lambda_+}(b,y) f_+(y) m(\mathrm{d}y). \qquad (3.13)
\end{aligned}
$$

For any $y < a$, using (2.3), we have

$$
\begin{aligned}
f_+(y) &= \mathbb{E}_y[e^{-\lambda_- e_q}; e_q < \tau_a] + \mathbb{E}_y[e^{-\lambda_- \tau_a}; \tau_a < e_q] f_+(a) \\
&= \mathbb{E}_y e^{-\lambda_- e_q} - \mathbb{E}_y[e^{-\lambda_- e_q}; \tau_a < e_q] + \mathbb{E}_y e^{-(q+\lambda_-)\tau_a} f_+(a) \\
&= \frac{q}{q+\lambda_-}\big(1 - \mathbb{E}_y e^{-(q+\lambda_-)\tau_a}\big) + \mathbb{E}_y e^{-(q+\lambda_-)\tau_a} f_+(a) \\
&= \frac{q}{q+\lambda_-}\big(1 - \frac{g_{+,q+\lambda_-}(y)}{g_{+,q+\lambda_-}(a)}\big) + \frac{g_{+,q+\lambda_-}(y)}{g_{+,q+\lambda_-}(a)} f_+(a); \qquad (3.14)
\end{aligned}
$$

for any $y > b$, using (2.3), we have

$$
\begin{aligned}
f_+(y) &= \mathbb{E}_y[e^{-\lambda_+ e_q}; e_q < \tau_b] + \mathbb{E}_y[e^{-\lambda_+ \tau_b}; \tau_b < e_q] f_+(b) \\
&= \frac{q}{q+\lambda_+}\big(1 - \mathbb{E}_y e^{-(q+\lambda_+)\tau_b}\big) + \mathbb{E}_y e^{-(q+\lambda_+)\tau_b} f_+(b) \\
&= \frac{q}{q+\lambda_+}\big(1 - \frac{g_{-,q+\lambda_+}(y)}{g_{-,q+\lambda_+}(b)}\big) + \frac{g_{-,q+\lambda_+}(y)}{g_{-,q+\lambda_+}(b)} f_+(b); \qquad (3.15)
\end{aligned}
$$

for any $a < y < b$, using (2.4), we have

$$\begin{aligned} f_+(y) &= \mathbb{P}_y(e_q < \tau_a \wedge \tau_b) + \mathbb{P}_y(\tau_a < e_q \wedge \tau_b)f_+(a) + \mathbb{P}_y(\tau_b < e_q \wedge \tau_a)f_+(b) \\ &= 1 - \mathbb{E}_y(e^{-q\tau_a}; \tau_a < \tau_b) - \mathbb{E}_y(e^{-q\tau_b}; \tau_b < \tau_a) \\ &\quad + \mathbb{E}_y(e^{-q\tau_a}; \tau_a < \tau_b)f_+(a) + \mathbb{E}_y(e^{-q\tau_b}; \tau_b < \tau_a)f_+(b) \\ &= 1 - \frac{f_q(y,b) + f_q(a,y)}{f_q(a,b)} + \frac{f_q(y,b)}{f_q(a,b)}f_+(a) + \frac{f_q(a,y)}{f_q(a,b)}f_+(b). \end{aligned} \tag{3.16}$$

Combining (3.12), (3.13), (3.14),(3.15)and (3.16), after some algebras, we can get :

$$\begin{aligned} & f_+(a)\Big[1 - (\lambda_- + \lambda_+)\int_a^b H(a;y,b)m(\mathrm{d}y) - \lambda_+\int_{-\infty}^a I_+(a)m(\mathrm{d}y)\Big] \\ &= \frac{q}{q+\lambda_-+\lambda_+} + (\lambda_- + \lambda_+)\int_a^b G_{q+\lambda_-+\lambda_+}(a,y)\big(1 - \frac{f_q(y,b)+f_q(a,y)}{f_q(a,b)}\big)m(\mathrm{d}y) \\ &\quad + \frac{q\lambda_-}{q+\lambda_+}\int_b^\infty G_{q+\lambda_-+\lambda_+}(a,y)\big(1 - \frac{g_{-,q+\lambda_+}(y)}{g_{-,q+\lambda_+}(b)}\big)m(\mathrm{d}y) \\ &\quad + \frac{q\lambda_+}{q+\lambda_-}\int_{-\infty}^a G_{q+\lambda_-+\lambda_+}(a,y)\big(1 - \frac{g_{+,q+\lambda_-}(y)}{g_{+,q+\lambda_-}(a)}\big)m(\mathrm{d}y) \\ &+ \Big[\lambda_-\int_b^\infty I_-(a)m(\mathrm{d}y) + (\lambda_- + \lambda_+)\int_a^b H(a;a,y)m(\mathrm{d}y)\Big]f_+(b) \\ &= V(a) + \Big[\lambda_-\int_b^\infty I_-(a)m(\mathrm{d}y) + (\lambda_- + \lambda_+)\int_a^b H(a;a,y)m(\mathrm{d}y)\Big]f_+(b), \end{aligned}$$

and

$$\begin{aligned} & f_+(b)\Big[1 - (\lambda_- + \lambda_+)\int_a^b H(b;a,y)m(\mathrm{d}y) - \lambda_-\int_b^\infty I_-(b)m(\mathrm{d}y)\Big] \\ &= \frac{q}{q+\lambda_-+\lambda_+} + (\lambda_- + \lambda_+)\int_a^b G_{q+\lambda_-+\lambda_+}(b,y)\big(1 - \frac{f_q(y,b)+f_q(a,y)}{f_q(a,b)}\big)m(\mathrm{d}y) \\ &\quad + \frac{q\lambda_-}{q+\lambda_+}\int_b^\infty G_{q+\lambda_-+\lambda_+}(b,y)\big(1 - \frac{g_{-,q+\lambda_+}(y)}{g_{-,q+\lambda_+}(b)}\big)m(\mathrm{d}y) \\ &\quad + \frac{q\lambda_+}{q+\lambda_-}\int_{-\infty}^a G_{q+\lambda_-+\lambda_+}(b,y)\big(1 - \frac{g_{+,q+\lambda_-}(y)}{g_{+,q+\lambda_-}(a)}\big)m(\mathrm{d}y) \\ &\quad + \Big[(\lambda_- + \lambda_+)\int_a^b H(b;y,b)m(\mathrm{d}y) + \lambda_+\int_{-\infty}^a I_+(b)m(\mathrm{d}y)\Big]f_+(a) \\ &= V(b) + \Big[(\lambda_- + \lambda_+)\int_a^b H(b;y,b)m(\mathrm{d}y) + \lambda_+\int_{-\infty}^a I_+(b)m(\mathrm{d}y)\Big]f_+(a). \end{aligned}$$

By the above equations, we have $f_+(a) = \frac{D}{F}$ and $f_+(b) = \frac{E}{F}$, where

$$\begin{aligned} D &= \Big[1 - (\lambda_- + \lambda_+)\int_a^b H(b;a,y)m(\mathrm{d}y) - \lambda_-\int_b^\infty I_-(b)m(\mathrm{d}y)\Big]\cdot V(a) \\ &\quad + \Big[\lambda_-\int_b^\infty I_-(a)m(\mathrm{d}y) + (\lambda_- + \lambda_+)\int_a^b H(a;a,y)m(\mathrm{d}y)\Big]\cdot V(b), \\ E &= \Big[(\lambda_- + \lambda_+)\int_a^b H(b;y,b)m(\mathrm{d}y) + \lambda_+\int_{-\infty}^a I_+(b)m(\mathrm{d}y)\Big]\cdot V(a) \end{aligned}$$

$$+\left[1-(\lambda_-+\lambda_+)\int_a^b H(a;y,b)m(\mathrm{d}y)-\lambda_+\int_{-\infty}^a I_+(a)m(\mathrm{d}y)\right]\cdot V(b),$$

$$F=\left[1-(\lambda_-+\lambda_+)\int_a^b H(a;y,b)m(\mathrm{d}y)-\lambda_+\int_{-\infty}^a I_+(a)m(\mathrm{d}y)\right]$$
$$\times\left[1-(\lambda_-+\lambda_+)\int_a^b H(b;a,y)m(\mathrm{d}y)-\lambda_-\int_b^\infty I_-(b)m(\mathrm{d}y)\right]$$
$$-\left[\lambda_-\int_b^\infty I_-(a)m(\mathrm{d}y)+(\lambda_-+\lambda_+)\int_a^b H(a;a,y)m(\mathrm{d}y)\right]$$
$$\times\left[(\lambda_-+\lambda_+)\int_a^b H(b;y,b)m(\mathrm{d}y)+\lambda_+\int_{-\infty}^a I_+(b)m(\mathrm{d}y)\right]$$

and

$$V(\cdot)=\frac{q}{q+\lambda_-+\lambda_+}+\frac{q\lambda_-}{q+\lambda_+}\int_b^\infty G_{q+\lambda_-+\lambda_+}(\cdot,y)\big(1-\frac{g_{-,q+\lambda_+}(y)}{g_{-,q+\lambda_+}(b)}\big)m(\mathrm{d}y)$$
$$+\frac{q\lambda_+}{q+\lambda_-}\int_{-\infty}^a G_{q+\lambda_-+\lambda_+}(\cdot,y)\big(1-\frac{g_{+,q+\lambda_-}(y)}{g_{+,q+\lambda_-}(a)}\big)m(\mathrm{d}y)$$
$$+(\lambda_-+\lambda_+)\int_a^b G_{q+\lambda_-+\lambda_+}(\cdot,y)\big(1-\frac{f_q(y,b)+f_q(a,y)}{f_q(a,b)}\big)m(\mathrm{d}y).$$

Putting $f_+(a)$ and $f_+(b)$ into (3.9), (3.10) and (3.11), respectively, we can obtain the results of Theorem 3.2.

Remark. Let $y=0$, $a\to 0+$ and $y=b$, $a\to 0+$, respectively, in Theorem 3.2, we can get the Theorem 3.2 in Li and Zhou [7]. We refer to Li and Zhou [7] for more details.

Define

$$g_+(y):=\mathbb{E}_y e^{-\lambda_-\int_0^{e_q}\mathbf{1}_{(-\infty,a)}(X_s)\mathrm{d}s-\lambda_+\int_0^{e_q}\mathbf{1}_{(a,\infty)}(X_s)\mathrm{d}s}.$$

Let $b\to a$ in Theorem 3.2, we obtain the following result.

Corollary 3.1 For any $y<a$,

$$g_+(y)=\frac{q}{q+\lambda_-}\big(1-\frac{g_{+,q+\lambda_-}(y)}{g_{+,q+\lambda_-}(a)}\big)+\frac{g_{+,q+\lambda_-}(y)}{g_{+,q+\lambda_-}(a)}g_+(a); \tag{3.17}$$

for any $y>a$,

$$g_+(y)=\frac{q}{q+\lambda_+}\big(1-\frac{g_{-,q+\lambda_+}(y)}{g_{-,q+\lambda_+}(a)}\big)+\frac{g_{-,q+\lambda_+}(y)}{g_{-,q+\lambda_+}(a)}g_+(a), \tag{3.18}$$

where

$$g_+(a)=\frac{M}{N},$$

$$M:=\frac{q}{q+\lambda_-+\lambda_+}+\frac{q\lambda_+}{q+\lambda_-}\int_{-\infty}^a G_{q+\lambda_-+\lambda_+}(a,y)\big(1-\frac{g_{+,q+\lambda_-}(y)}{g_{+,q+\lambda_-}(a)}\big)m(\mathrm{d}y)$$
$$+\frac{q\lambda_-}{q+\lambda_+}\int_a^\infty G_{q+\lambda_-+\lambda_+}(a,y)\big(1-\frac{g_{-,q+\lambda_+}(y)}{g_{-,q+\lambda_+}(a)}\big)m(\mathrm{d}y),$$

$$N:=1-\lambda_+\int_{-\infty}^a G_{q+\lambda_-+\lambda_+}(a,y)\frac{g_{+,q+\lambda_-}(y)}{g_{+,q+\lambda_-}(a)}m(\mathrm{d}y)-$$

$$\lambda_- \int_a^\infty G_{q+\lambda_-+\lambda_+}(a,y)\frac{g_{-,q+\lambda_+}(y)}{g_{-,q+\lambda_+}(a)}m(\mathrm{d}y).$$

Letting $\lambda_+ = 0$, in Theorem 3.2, we obtain the following simplified expressions.

Corollary 3.2 For $y = a$,

$$f_+(a) := \mathbb{E}_a e^{-\lambda_- \int_0^{e_q} \mathbf{1}_{(-\infty,a)}(X_s)\mathrm{d}s} = 1 + \frac{\lambda_- \int_a^\infty G_{q+\lambda_-}(a,y)m(\mathrm{d}y) - \frac{\lambda_-}{q+\lambda_-}}{1-\lambda_- \int_a^\infty G_{q+\lambda_-}(a,y)\frac{g_{-,q}(y)}{g_{-,q}(a)}m(\mathrm{d}y)};$$

for any $y > a$,

$$\mathbb{E}_y e^{-\lambda_- \int_0^{e_q} \mathbf{1}_{(-\infty,a)}(X_s)\mathrm{d}s} = 1 - \frac{g_{-,q}(y)}{g_{-,q}(a)} + \frac{g_{-,q}(y)}{g_{-,q}(a)}f_+(a);$$

for any $y < a$,

$$\mathbb{E}_y e^{-\lambda_- \int_0^{e_q} \mathbf{1}_{(-\infty,a)}(X_s)\mathrm{d}s} = \frac{q}{q+\lambda_-}\big(1 - \frac{g_{+,q+\lambda_-}(y)}{g_{+,q+\lambda_-}(a)}\big) + \frac{g_{+,q+\lambda_-}(y)}{g_{+,q+\lambda_-}(a)}f_+(a).$$

Letting $\lambda_- = 0$, in Theorem 3.2, we obtain the following simplified expressions.

Corollary 3.3 For $y = b$,

$$f_+(b) := \mathbb{E}_b e^{-\lambda_+ \int_0^{e_q} \mathbf{1}_{(b,\infty)}(X_s)\mathrm{d}s} = 1 + \frac{\lambda_+ \int_{-\infty}^b G_{q+\lambda_+}(b,y)m(\mathrm{d}y) - \frac{\lambda_+}{q+\lambda_+}}{1-\lambda_+ \int_{-\infty}^b G_{q+\lambda_+}(b,y)\frac{g_{+,q}(y)}{g_{+,q}(b)}m(\mathrm{d}y)};$$

for any $y > b$,

$$\mathbb{E}_y e^{-\lambda_+ \int_0^{e_q} \mathbf{1}_{(b,\infty)}(X_s)\mathrm{d}s} = \frac{q}{q+\lambda_+}\big(1 - \frac{g_{-,q+\lambda_+}(y)}{g_{-,q+\lambda_+}(b)}\big) + \frac{g_{-,q+\lambda_+}(y)}{g_{-,q+\lambda_+}(b)}f_+(b);$$

for any $y < b$,

$$\mathbb{E}_y e^{-\lambda_+ \int_0^{e_q} \mathbf{1}_{(b,\infty)}(X_s)\mathrm{d}s} = 1 - \frac{g_{+,q}(y)}{g_{+,q}(b)} + \frac{g_{+,q}(y)}{g_{+,q}(b)}f_+(b).$$

§4 Examples

In the following section, we apply the results in Section 3 to Brownian motion with drift to compare with the known result. Let $X_t = \mu t + W_t$ be a Brownian motion with drift. The corresponding differential equation is reduced to

$$\frac{1}{2}g''(x) + \mu g'(x) = \lambda g(x), \qquad \lambda > 0,$$

with two independent solutions $g_{+,\lambda}(x) = e^{(-\mu+\sqrt{\mu^2+2\lambda})x}, g_{-,\lambda}(x) = e^{(-\mu-\sqrt{\mu^2+2\lambda})x}$, see pages 127-128 of [1]. We also have $m(\mathrm{d}x) = 2e^{2\mu x}\mathrm{d}x, \omega_\lambda = 2\sqrt{\mu^2+2\lambda}$, and

$$G_\lambda(x,y) = \begin{cases} \omega_\lambda^{-1} e^{-(\mu+\sqrt{\mu^2+2\lambda})x} e^{(-\mu+\sqrt{\mu^2+2\lambda})y}, & y \le x, \\ \omega_\lambda^{-1} e^{-(\mu+\sqrt{\mu^2+2\lambda})y} e^{(-\mu+\sqrt{\mu^2+2\lambda})x}, & y \ge x. \end{cases}$$

Moreover,

$$f_\lambda(x,y) = e^{-\mu(x+y)}\big(e^{\sqrt{\mu^2+2\lambda}(y-x)} - e^{-\sqrt{\mu^2+2\lambda}(y-x)}\big) = e^{-\mu(x+y)} sh\big(\sqrt{\mu^2+2\lambda}(y-x)\big).$$

Denote $\Upsilon_q := \sqrt{\mu^2+2q}, \Upsilon_{q+\lambda} := \sqrt{\mu^2+2(q+\lambda)}$.

By Theorem 3.1, with some computations, we have

$$\frac{g_{+,q}(y)}{g_{+,q}(0)} = e^{(-\mu+\sqrt{\mu^2+2q})y}, \qquad \frac{g_{-,q}(y)}{g_{-,q}(a)} = e^{(\mu+\sqrt{\mu^2+2q})(a-y)},$$

$$f_{q+\lambda}(0,y) = e^{-\mu y} sh\big(\sqrt{\mu^2+2(q+\lambda)}y\big) = e^{-\mu y} sh(\Upsilon_{q+\lambda} y),$$

$$f_{q+\lambda}(y,a) = e^{-\mu(a+y)} sh\big(\sqrt{\mu^2+2(q+\lambda)}(a-y)\big) = e^{-\mu(a+y)} sh\big(\Upsilon_{q+\lambda}(a-y)\big),$$

$$f_{q+\lambda}(0,a) = e^{-\mu a} sh\big(\sqrt{\mu^2+2(q+\lambda)}a\big) = e^{-\mu a} sh(\Upsilon_{q+\lambda} a).$$

For any $y<0$, we have

$$G_{q+\lambda}(0,y) = \frac{1}{2\Upsilon_{q+\lambda}} e^{(-\mu+\Upsilon_{q+\lambda})y},$$

$$\lambda \int_{-\infty}^{0} G_{q+\lambda}(0,y) m(\mathrm{d}y) = \frac{\lambda}{2(q+\lambda)} \frac{\Upsilon_{q+\lambda}-\mu}{\Upsilon_{q+\lambda}},$$

$$\lambda \int_{-\infty}^{0} G_{q+\lambda}(0,y) \frac{g_{+,q}(y)}{g_{+,q}(0)} m(\mathrm{d}y) = \frac{\Upsilon_{q+\lambda}-\Upsilon_q}{2\Upsilon_{q+\lambda}}.$$

$$G_{q+\lambda}(a,y) = \frac{1}{2\Upsilon_{q+\lambda}} e^{-(\mu+\Upsilon_{q+\lambda})a} e^{(-\mu+\Upsilon_{q+\lambda})y},$$

$$\lambda \int_{-\infty}^{0} G_{q+\lambda}(a,y) m(\mathrm{d}y) = \frac{\lambda}{2(q+\lambda)} \frac{\Upsilon_{q+\lambda}-\mu}{\Upsilon_{q+\lambda}} e^{-(\mu+\Upsilon_{q+\lambda})a},$$

$$\lambda \int_{-\infty}^{0} G_{q+\lambda}(a,y) \frac{g_{+,q}(y)}{g_{+,q}(0)} m(\mathrm{d}y) = \frac{\Upsilon_{q+\lambda}-\Upsilon_q}{2\Upsilon_{q+\lambda}} e^{-(\mu+\Upsilon_{q+\lambda})a}.$$

For any $y>a$, we have

$$G_{q+\lambda}(0,y) = \frac{1}{2\Upsilon_{q+\lambda}} e^{-(\mu+\Upsilon_{q+\lambda})y},$$

$$\lambda \int_{a}^{\infty} G_{q+\lambda}(0,y) m(\mathrm{d}y) = \frac{\lambda}{2(q+\lambda)} \frac{\Upsilon_{q+\lambda}+\mu}{\Upsilon_{q+\lambda}} e^{(\mu-\Upsilon_{q+\lambda})a},$$

$$\lambda \int_{a}^{\infty} G_{q+\lambda}(0,y) \frac{g_{-,q}(y)}{g_{-,q}(a)} m(\mathrm{d}y) = \frac{\Upsilon_{q+\lambda}-\Upsilon_q}{2\Upsilon_{q+\lambda}} e^{(\mu-\Upsilon_{q+\lambda})a}.$$

$$G_{q+\lambda}(a,y) = \frac{1}{2\Upsilon_{q+\lambda}} e^{-(\mu+\Upsilon_{q+\lambda})y} e^{(-\mu+\Upsilon_{q+\lambda})a},$$

$$\lambda \int_{a}^{\infty} G_{q+\lambda}(a,y) m(\mathrm{d}y) = \frac{\lambda}{2(q+\lambda)} \frac{\Upsilon_{q+\lambda}+\mu}{\Upsilon_{q+\lambda}},$$

$$\lambda \int_{a}^{\infty} G_{q+\lambda}(a,y) \frac{g_{-,q}(y)}{g_{-,q}(a)} m(\mathrm{d}y) = \frac{\Upsilon_{q+\lambda}-\Upsilon_q}{2\Upsilon_{q+\lambda}}.$$

Further, we can compute

$$\begin{aligned} U(0) &= \frac{q}{q+\lambda} + \lambda \int_{-\infty}^{0} G_{q+\lambda}(0,y) m(\mathrm{d}y) - \lambda \int_{-\infty}^{0} G_{q+\lambda}(0,y) \frac{g_{+,q}(y)}{g_{+,q}(0)} m(\mathrm{d}y) \\ &\quad + \lambda \int_{a}^{\infty} G_{q+\lambda}(0,y) m(\mathrm{d}y) - \lambda \int_{a}^{\infty} G_{q+\lambda}(0,y) \frac{g_{-,q}(y)}{g_{-,q}(a)} m(\mathrm{d}y) \\ &= \frac{q}{q+\lambda} + \frac{\lambda}{2(q+\lambda)} \frac{\Upsilon_{q+\lambda}-\mu}{\Upsilon_{q+\lambda}} - \frac{\Upsilon_{q+\lambda}-\Upsilon_q}{2\Upsilon_{q+\lambda}} \end{aligned}$$

$$
\begin{aligned}
&+\frac{\lambda}{2(q+\lambda)}\frac{\Upsilon_{q+\lambda}+\mu}{\Upsilon_{q+\lambda}}e^{(\mu-\Upsilon_{q+\lambda})a}-\frac{\Upsilon_{q+\lambda}-\Upsilon_q}{2\Upsilon_{q+\lambda}}e^{(\mu-\Upsilon_{q+\lambda})a}\\
&=\frac{1}{2(q+\lambda)\Upsilon_{q+\lambda}}\Big[\big((q+\lambda)\Upsilon_q+q\Upsilon_{q+\lambda}-\lambda\mu\big)+\big((q+\lambda)\Upsilon_q-q\Upsilon_{q+\lambda}+\lambda\mu\big)e^{(\mu-\Upsilon_{q+\lambda})a}\Big],
\end{aligned}
$$

$$
\begin{aligned}
U(a)&=\frac{q}{q+\lambda}+\lambda\int_{-\infty}^{0}G_{q+\lambda}(a,y)m(\mathrm{d}y)-\lambda\int_{-\infty}^{0}G_{q+\lambda}(a,y)\frac{g_{+,q}(y)}{g_{+,q}(0)}m(\mathrm{d}y)\\
&\qquad+\lambda\int_{a}^{\infty}G_{q+\lambda}(a,y)m(\mathrm{d}y)-\lambda\int_{a}^{\infty}G_{q+\lambda}(a,y)\frac{g_{-,q}(y)}{g_{-,q}(a)}m(\mathrm{d}y)\\
&=\frac{q}{q+\lambda}+\frac{\lambda}{2(q+\lambda)}\frac{\Upsilon_{q+\lambda}-\mu}{\Upsilon_{q+\lambda}}e^{-(\mu+\Upsilon_{q+\lambda})a}-\frac{\Upsilon_{q+\lambda}-\Upsilon_q}{2\Upsilon_{q+\lambda}}e^{-(\mu+\Upsilon_{q+\lambda})a}\\
&\quad+\frac{\lambda}{2(q+\lambda)}\frac{\Upsilon_{q+\lambda}+\mu}{\Upsilon_{q+\lambda}}-\frac{\Upsilon_{q+\lambda}-\Upsilon_q}{2\Upsilon_{q+\lambda}}\\
&=\frac{1}{2(q+\lambda)\Upsilon_{q+\lambda}}\Big[\big((q+\lambda)\Upsilon_q+q\Upsilon_{q+\lambda}+\lambda\mu\big)+\big((q+\lambda)\Upsilon_q-q\Upsilon_{q+\lambda}-\lambda\mu\big)e^{-(\mu+\Upsilon_{q+\lambda})a}\Big].
\end{aligned}
$$

$$
\begin{aligned}
A&=\big(1-\lambda\int_{a}^{\infty}G_{q+\lambda}(a,y)\frac{g_{-,q}(y)}{g_{-,q}(a)}m(\mathrm{d}y)\big)\cdot U(0)+\lambda\int_{a}^{\infty}G_{q+\lambda}(0,y)\frac{g_{-,q}(y)}{g_{-,q}(a)}m(\mathrm{d}y)\cdot U(a)\\
&=\frac{\Upsilon_{q+\lambda}+\Upsilon_q}{2\Upsilon_{q+\lambda}}\frac{1}{2(q+\lambda)\Upsilon_{q+\lambda}}\big[(q+\lambda)\Upsilon_q+q\Upsilon_{q+\lambda}-\lambda\mu\big]\\
&+\frac{\Upsilon_{q+\lambda}+\Upsilon_q}{2\Upsilon_{q+\lambda}}\frac{1}{2(q+\lambda)\Upsilon_{q+\lambda}}\big[(q+\lambda)\Upsilon_q-q\Upsilon_{q+\lambda}+\lambda\mu\big]e^{(\mu-\Upsilon_{q+\lambda})a}\\
&+\frac{\Upsilon_{q+\lambda}-\Upsilon_q}{2\Upsilon_{q+\lambda}}e^{(\mu-\Upsilon_{q+\lambda})a}\frac{1}{2(q+\lambda)\Upsilon_{q+\lambda}}\big[(q+\lambda)\Upsilon_q+q\Upsilon_{q+\lambda}+\lambda\mu\big]\\
&+\frac{\Upsilon_{q+\lambda}-\Upsilon_q}{2\Upsilon_{q+\lambda}}e^{(\mu-\Upsilon_{q+\lambda})a}\frac{1}{2(q+\lambda)\Upsilon_{q+\lambda}}\big[(q+\lambda)\Upsilon_q-q\Upsilon_{q+\lambda}-\lambda\mu\big]e^{-(\mu+\Upsilon_{q+\lambda})a},
\end{aligned}
$$

$$
\begin{aligned}
\big(4\Upsilon_{q+\lambda}^2(q+\lambda)e^{\Upsilon_{q+\lambda}a}\big)A&=(\Upsilon_{q+\lambda}+\Upsilon_q)\big[(q+\lambda)\Upsilon_q+q\Upsilon_{q+\lambda}-\lambda\mu\big]e^{\Upsilon_{q+\lambda}a}\\
&+(\Upsilon_{q+\lambda}+\Upsilon_q)\big[(q+\lambda)\Upsilon_q-q\Upsilon_{q+\lambda}+\lambda\mu\big]e^{\mu a}\\
&+(\Upsilon_{q+\lambda}-\Upsilon_q)\big[(q+\lambda)\Upsilon_q+q\Upsilon_{q+\lambda}+\lambda\mu\big]e^{\mu a}\\
&+(\Upsilon_{q+\lambda}-\Upsilon_q)\big[(q+\lambda)\Upsilon_q-q\Upsilon_{q+\lambda}-\lambda\mu\big]e^{-\Upsilon_{q+\lambda}a}\\
&=2(q+\lambda)(\mu^2+2q+\lambda)sh(\Upsilon_{q+\lambda}a)+2(q+\lambda)\Upsilon_{q+\lambda}\Upsilon_q ch(\Upsilon_{q+\lambda}a)\\
&-\lambda(\Upsilon_{q+\lambda}-\mu)(\Upsilon_{q+\lambda}-\Upsilon_q)sh(\Upsilon_{q+\lambda}a)\\
&-2\lambda\Upsilon_{q+\lambda}(\Upsilon_q+\mu)(e^{\Upsilon_{q+\lambda}a}-e^{\mu a}).
\end{aligned}
$$

$$
\begin{aligned}
B&=\big(1-\lambda\int_{-\infty}^{0}G_{q+\lambda}(0,y)\frac{g_{+,q}(y)}{g_{+,q}(0)}m(\mathrm{d}y)\big)\cdot U(a)+U(0)\lambda\int_{-\infty}^{0}G_{q+\lambda}(a,y)\frac{g_{+,q}(y)}{g_{+,q}(0)}m(\mathrm{d}y)\\
&=\frac{\Upsilon_{q+\lambda}+\Upsilon_q}{2\Upsilon_{q+\lambda}}\frac{1}{2(q+\lambda)\Upsilon_{q+\lambda}}\big[(q+\lambda)\Upsilon_q+q\Upsilon_{q+\lambda}+\lambda\mu\big]\\
&+\frac{\Upsilon_{q+\lambda}+\Upsilon_q}{2\Upsilon_{q+\lambda}}\frac{1}{2(q+\lambda)\Upsilon_{q+\lambda}}\big[(q+\lambda)\Upsilon_q-q\Upsilon_{q+\lambda}-\lambda\mu\big]e^{-(\mu+\Upsilon_{q+\lambda})a}\\
&+\frac{\Upsilon_{q+\lambda}-\Upsilon_q}{2\Upsilon_{q+\lambda}}e^{-(\mu+\Upsilon_{q+\lambda})a}\frac{1}{2(q+\lambda)\Upsilon_{q+\lambda}}\big[(q+\lambda)\Upsilon_q+q\Upsilon_{q+\lambda}-\lambda\mu\big]
\end{aligned}
$$

$$+\frac{\Upsilon_{q+\lambda}-\Upsilon_q}{2\Upsilon_{q+\lambda}}e^{-(\mu+\Upsilon_{q+\lambda})a}\frac{1}{2(q+\lambda)\Upsilon_{q+\lambda}}\big[(q+\lambda)\Upsilon_q-q\Upsilon_{q+\lambda}+\lambda\mu\big]e^{(\mu-\Upsilon_{q+\lambda})a},$$

$$\begin{aligned}\big(4\Upsilon_{q+\lambda}^2(q+\lambda)e^{\Upsilon_{q+\lambda}a}\big)B&=(\Upsilon_{q+\lambda}+\Upsilon_q)\big[(q+\lambda)\Upsilon_q+q\Upsilon_{q+\lambda}+\lambda\mu\big]e^{\Upsilon_{q+\lambda}a}\\&+(\Upsilon_{q+\lambda}+\Upsilon_q)\big[(q+\lambda)\Upsilon_q-q\Upsilon_{q+\lambda}-\lambda\mu\big]e^{-\mu a}\\&+(\Upsilon_{q+\lambda}-\Upsilon_q)\big[(q+\lambda)\Upsilon_q+q\Upsilon_{q+\lambda}-\lambda\mu\big]e^{-\mu a}\\&+(\Upsilon_{q+\lambda}-\Upsilon_q)\big[(q+\lambda)\Upsilon_q-q\Upsilon_{q+\lambda}+\lambda\mu\big]e^{-\Upsilon_{q+\lambda}a}\\&=2(q+\lambda)(\mu^2+2q+\lambda)sh(\Upsilon_{q+\lambda}a)+2(q+\lambda)\Upsilon_{q+\lambda}\Upsilon_q ch(\Upsilon_{q+\lambda}a)\\&-\lambda(\Upsilon_{q+\lambda}+\mu)(\Upsilon_{q+\lambda}-\Upsilon_q)sh(\Upsilon_{q+\lambda}a)\\&-2\lambda\Upsilon_{q+\lambda}(\Upsilon_q-\mu)(e^{\Upsilon_{q+\lambda}a}-e^{-\mu a}).\end{aligned}$$

$$\begin{aligned}C&=\big(1-\lambda\int_a^{\infty}G_{q+\lambda}(a,y)\frac{g_{-,q}(y)}{g_{-,q}(a)}m(\mathrm{d}y)\big)\cdot\big(1-\lambda\int_{-\infty}^{0}G_{q+\lambda}(0,y)\frac{g_{+,q}(y)}{g_{+,q}(0)}m(\mathrm{d}y)\big)\\&-\lambda\int_a^{\infty}G_{q+\lambda}(0,y)\frac{g_{-,q}(y)}{g_{-,q}(a)}m(\mathrm{d}y)\cdot\lambda\int_{-\infty}^{0}G_{q+\lambda}(a,y)\frac{g_{+,q}(y)}{g_{+,q}(0)}m(\mathrm{d}y)\\&=\frac{\Upsilon_{q+\lambda}+\Upsilon_q}{2\Upsilon_{q+\lambda}}\cdot\frac{\Upsilon_{q+\lambda}+\Upsilon_q}{2\Upsilon_{q+\lambda}}-\frac{\Upsilon_{q+\lambda}-\Upsilon_q}{2\Upsilon_{q+\lambda}}e^{(\mu-\Upsilon_{q+\lambda})a}\cdot\frac{\Upsilon_{q+\lambda}-\Upsilon_q}{2\Upsilon_{q+\lambda}}e^{-(\mu+\Upsilon_{q+\lambda})a}\\&=\frac{(\Upsilon_{q+\lambda}+\Upsilon_q)^2}{4\Upsilon_{q+\lambda}^2}-\frac{(\Upsilon_{q+\lambda}-\Upsilon_q)^2}{4\Upsilon_{q+\lambda}^2}e^{-2\Upsilon_{q+\lambda}a}\\&=\frac{2\mu^2+4q+2\lambda+2\Upsilon_{q+\lambda}\Upsilon_q}{4\Upsilon_{q+\lambda}^2}-\frac{2\mu^2+4q+2\lambda-2\Upsilon_{q+\lambda}\Upsilon_q}{4\Upsilon_{q+\lambda}^2}e^{-2\Upsilon_{q+\lambda}a},\end{aligned}$$

$$\begin{aligned}\big(4\Upsilon_{q+\lambda}^2(q+\lambda)e^{\Upsilon_{q+\lambda}a}\big)C&=(q+\lambda)(2\mu^2+4q+2\lambda+2\Upsilon_{q+\lambda}\Upsilon_q)e^{\Upsilon_{q+\lambda}a}\\&-(q+\lambda)(2\mu^2+4q+2\lambda-2\Upsilon_{q+\lambda}\Upsilon_q)e^{-\Upsilon_{q+\lambda}a}\\&=2(q+\lambda)(\mu^2+2q+\lambda)sh(\Upsilon_{q+\lambda}a)+2(q+\lambda)\Upsilon_{q+\lambda}\Upsilon_q ch(\Upsilon_{q+\lambda}a).\end{aligned}$$

So

$$\begin{aligned}f_-(0)&=\frac{A}{C}=\frac{\big(4\Upsilon_{q+\lambda}^2(q+\lambda)e^{\Upsilon_{q+\lambda}a}\big)A}{\big(4\Upsilon_{q+\lambda}^2(q+\lambda)e^{\Upsilon_{q+\lambda}a}\big)C}\\&=\Big[2(q+\lambda)(\mu^2+2q+\lambda)sh(\Upsilon_{q+\lambda}a)+2(q+\lambda)\Upsilon_{q+\lambda}\Upsilon_q ch(\Upsilon_{q+\lambda}a)\\&-\lambda(\Upsilon_{q+\lambda}-\mu)(\Upsilon_{q+\lambda}-\Upsilon_q)sh(\Upsilon_{q+\lambda}a)-2\lambda\Upsilon_{q+\lambda}(\Upsilon_q+\mu)(e^{\Upsilon_{q+\lambda}a}-e^{\mu a})\Big]\\&\div\Big[2(q+\lambda)(\mu^2+2q+\lambda)sh(\Upsilon_{q+\lambda}a)+2(q+\lambda)\Upsilon_{q+\lambda}\Upsilon_q ch(\Upsilon_{q+\lambda}a)\Big]\\&=1-\frac{\lambda(\Upsilon_{q+\lambda}-\mu)(\Upsilon_{q+\lambda}-\Upsilon_q)sh(\Upsilon_{q+\lambda}a)+2\lambda\Upsilon_{q+\lambda}(\Upsilon_q+\mu)(e^{\Upsilon_{q+\lambda}a}-e^{\mu a})}{2(q+\lambda)(\mu^2+2q+\lambda)sh(\Upsilon_{q+\lambda}a)+2(q+\lambda)\Upsilon_{q+\lambda}\Upsilon_q ch(\Upsilon_{q+\lambda}a)}\\&=1-\frac{\lambda}{q+\lambda}S_0,\end{aligned}\tag{4.1}$$

$$\begin{aligned}f_-(a)&=\frac{B}{C}=\frac{\big(4\Upsilon_{q+\lambda}^2(q+\lambda)e^{\Upsilon_{q+\lambda}a}\big)B}{\big(4\Upsilon_{q+\lambda}^2(q+\lambda)e^{\Upsilon_{q+\lambda}a}\big)C}\\&=\Big[2(q+\lambda)(\mu^2+2q+\lambda)sh(\Upsilon_{q+\lambda}a)+2(q+\lambda)\Upsilon_{q+\lambda}\Upsilon_q ch(\Upsilon_{q+\lambda}a)\end{aligned}$$

$$- \lambda(\Upsilon_{q+\lambda}+\mu)(\Upsilon_{q+\lambda}-\Upsilon_q)sh(\Upsilon_{q+\lambda}a) - 2\lambda\Upsilon_{q+\lambda}(\Upsilon_q-\mu)(e^{\Upsilon_{q+\lambda}a}-e^{-\mu a})\Big]$$
$$\div \Big[2(q+\lambda)(\mu^2+2q+\lambda)sh(\Upsilon_{q+\lambda}a) + 2(q+\lambda)\Upsilon_{q+\lambda}\Upsilon_q ch(\Upsilon_{q+\lambda}a)\Big]$$
$$= 1 - \frac{\lambda(\Upsilon_{q+\lambda}+\mu)(\Upsilon_{q+\lambda}-\Upsilon_q)sh(\Upsilon_{q+\lambda}a) + 2\lambda\Upsilon_{q+\lambda}(\Upsilon_q-\mu)(e^{\Upsilon_{q+\lambda}a}-e^{-\mu a})}{2(q+\lambda)(\mu^2+2q+\lambda)sh(\Upsilon_{q+\lambda}a) + 2(q+\lambda)\Upsilon_{q+\lambda}\Upsilon_q ch(\Upsilon_{q+\lambda}a)}$$
$$= 1 - \frac{\lambda}{q+\lambda}S_a, \tag{4.2}$$

where

$$S_0 := \frac{(\Upsilon_{q+\lambda}-\mu)(\Upsilon_{q+\lambda}-\Upsilon_q)sh(\Upsilon_{q+\lambda}a) + 2\Upsilon_{q+\lambda}(\Upsilon_q+\mu)(e^{\Upsilon_{q+\lambda}a}-e^{\mu a})}{2(\mu^2+2q+\lambda)sh(\Upsilon_{q+\lambda}a) + 2\Upsilon_{q+\lambda}\Upsilon_q ch(\Upsilon_{q+\lambda}a)},$$
$$S_a := \frac{(\Upsilon_{q+\lambda}+\mu)(\Upsilon_{q+\lambda}-\Upsilon_q)sh(\Upsilon_{q+\lambda}a) + 2\Upsilon_{q+\lambda}(\Upsilon_q-\mu)(e^{\Upsilon_{q+\lambda}a}-e^{-\mu a})}{2(\mu^2+2q+\lambda)sh(\Upsilon_{q+\lambda}a) + 2\Upsilon_{q+\lambda}\Upsilon_q ch(\Upsilon_{q+\lambda}a)}.$$

Putting (4.1) and (4.2) in (3.6), (3.7) and (3.8), respectively, for any $y<0$, we have

$$\begin{aligned}\mathbb{E}_y e^{-\lambda\int_0^{e_q}\mathbf{1}_{(0,a)}(X_s)\mathrm{d}s} &= 1 - \frac{g_{+,q}(y)}{g_{+,q}(0)} + \frac{g_{+,q}(y)}{g_{+,q}(0)}f_-(0)\\ &= 1 - e^{(-\mu+\sqrt{\mu^2+2q})y} + e^{(-\mu+\sqrt{\mu^2+2q})y}(1-\frac{\lambda}{q+\lambda}S_0)\\ &= 1 - \frac{\lambda S_0}{q+\lambda}e^{(-\mu+\sqrt{\mu^2+2q})y};\end{aligned} \tag{4.3}$$

for any $y>a$, we have

$$\begin{aligned}\mathbb{E}_y e^{-\lambda\int_0^{e_q}\mathbf{1}_{(0,a)}(X_s)\mathrm{d}s} &= 1 - \frac{g_{-,q}(y)}{g_{-,q}(a)} + \frac{g_{-,q}(y)}{g_{-,q}(a)}f_-(a)\\ &= 1 - e^{(\mu+\sqrt{\mu^2+2q})(a-y)} + e^{(\mu+\sqrt{\mu^2+2q})(a-y)}(1-\frac{\lambda}{q+\lambda}S_a)\\ &= 1 - \frac{\lambda S_a}{q+\lambda}e^{(\mu+\sqrt{\mu^2+2q})(a-y)};\end{aligned} \tag{4.4}$$

for any $0<y<a$, we have

$$\begin{aligned}&\mathbb{E}_y e^{-\lambda\int_0^{e_q}\mathbf{1}_{(0,a)}(X_s)\mathrm{d}s}\\ &= \frac{q}{q+\lambda}(1-\frac{f_{q+\lambda}(y,a)+f_{q+\lambda}(0,y)}{f_{q+\lambda}(0,a)}) + \frac{f_{q+\lambda}(y,a)}{f_{q+\lambda}(0,a)}f_-(0) + \frac{f_{q+\lambda}(0,y)}{f_{q+\lambda}(0,a)}f_-(a)\\ &= \frac{q}{q+\lambda}(1-\frac{f_{q+\lambda}(y,a)+f_{q+\lambda}(0,y)}{f_{q+\lambda}(0,a)}) + \frac{f_{q+\lambda}(y,a)}{f_{q+\lambda}(0,a)}(1-\frac{\lambda}{q+\lambda}S_0) + \frac{f_{q+\lambda}(0,y)}{f_{q+\lambda}(0,a)}(1-\frac{\lambda}{q+\lambda}S_a)\\ &= \frac{q}{q+\lambda} + \frac{\lambda(1-S_0)}{q+\lambda}\frac{f_{q+\lambda}(y,a)}{f_{q+\lambda}(0,a)} + \frac{\lambda(1-S_a)}{q+\lambda}\frac{f_{q+\lambda}(0,y)}{f_{q+\lambda}(0,a)}\\ &= \frac{q}{q+\lambda} + \frac{\lambda(1-S_0)}{q+\lambda}\frac{sh\big(\Upsilon_{q+\lambda}(a-y)\big)e^{-\mu y}}{sh\big(\Upsilon_{q+\lambda}a\big)} + \frac{\lambda(1-S_a)}{q+\lambda}\frac{sh\big(\Upsilon_{q+\lambda}y\big)e^{\mu(a-y)}}{sh\big(\Upsilon_{q+\lambda}a\big)}.\end{aligned} \tag{4.5}$$

By Corollary 3.1, with some computations, we have

$$\frac{g_{+,q+\lambda_-}(y)}{g_{+,q+\lambda_-}(a)} = e^{(-\mu+\sqrt{\mu^2+2(q+\lambda_-)})(y-a)} = e^{(-\mu+\Upsilon_{q+\lambda_-})(y-a)},$$

$$\frac{g_{-,q+\lambda_+}(y)}{g_{-,q+\lambda_+}(a)} = e^{(\mu+\sqrt{\mu^2+2(q+\lambda_+)})(a-y)} = e^{(\mu+\Upsilon_{q+\lambda_+})(a-y)},$$

for any $y < a$, we have

$$G_{q+\lambda_-+\lambda_+}(a,y) = \frac{1}{2\Upsilon_{q+\lambda_-+\lambda_+}} e^{-(\mu+\Upsilon_{q+\lambda_-+\lambda_+})a} e^{(-\mu+\Upsilon_{q+\lambda_-+\lambda_+})y},$$

$$\int_{-\infty}^{a} G_{q+\lambda_-+\lambda_+}(a,y)m(\mathrm{d}y) = \frac{\Upsilon_{q+\lambda_-+\lambda_+}-\mu}{2(q+\lambda_-+\lambda_+)\Upsilon_{q+\lambda_-+\lambda_+}},$$

$$\int_{-\infty}^{a} G_{q+\lambda_-+\lambda_+}(a,y)\frac{g_{+,q+\lambda_-}(y)}{g_{+,q+\lambda_-}(a)}m(\mathrm{d}y) = \frac{\Upsilon_{q+\lambda_-+\lambda_+}-\Upsilon_{q+\lambda_-}}{2\lambda_+\Upsilon_{q+\lambda_-+\lambda_+}};$$

for any $y > a$, we have

$$G_{q+\lambda_-+\lambda_+}(a,y) = \frac{1}{2\Upsilon_{q+\lambda_-+\lambda_+}} e^{(-\mu+\Upsilon_{q+\lambda_-+\lambda_+})a} e^{-(\mu+\Upsilon_{q+\lambda_-+\lambda_+})y},$$

$$\int_{a}^{\infty} G_{q+\lambda_-+\lambda_+}(a,y)m(\mathrm{d}y) = \frac{\Upsilon_{q+\lambda_-+\lambda_+}+\mu}{2(q+\lambda_-+\lambda_+)\Upsilon_{q+\lambda_-+\lambda_+}},$$

$$\int_{a}^{\infty} G_{q+\lambda_-+\lambda_+}(a,y)\frac{g_{-,q+\lambda_+}(y)}{g_{-,q+\lambda_+}(a)}m(\mathrm{d}y) = \frac{\Upsilon_{q+\lambda_-+\lambda_+}-\Upsilon_{q+\lambda_+}}{2\lambda_-\Upsilon_{q+\lambda_-+\lambda_+}}.$$

Further, we can compute

$$\begin{aligned}
M &= \frac{q}{q+\lambda_-+\lambda_+} + \frac{q\lambda_+}{q+\lambda_-}\int_{-\infty}^{a} G_{q+\lambda_-+\lambda_+}(a,y)\big(1-\frac{g_{+,q+\lambda_-}(y)}{g_{+,q+\lambda_-}(a)}\big)m(\mathrm{d}y)\\
&\quad + \frac{q\lambda_-}{q+\lambda_+}\int_{a}^{\infty} G_{q+\lambda_-+\lambda_+}(a,y)\big(1-\frac{g_{-,q+\lambda_+}(y)}{g_{-,q+\lambda_+}(a)}\big)m(\mathrm{d}y)\\
&= \frac{q}{q+\lambda_-+\lambda_+} + \frac{q\lambda_+}{q+\lambda_-}\left(\frac{\Upsilon_{q+\lambda_-+\lambda_+}-\mu}{2(q+\lambda_-+\lambda_+)\Upsilon_{q+\lambda_-+\lambda_+}} - \frac{\Upsilon_{q+\lambda_-+\lambda_+}-\Upsilon_{q+\lambda_-}}{2\lambda_+\Upsilon_{q+\lambda_-+\lambda_+}}\right)\\
&\quad + \frac{q\lambda_-}{q+\lambda_+}\left(\frac{\Upsilon_{q+\lambda_-+\lambda_+}+\mu}{2(q+\lambda_-+\lambda_+)\Upsilon_{q+\lambda_-+\lambda_+}} - \frac{\Upsilon_{q+\lambda_-+\lambda_+}-\Upsilon_{q+\lambda_+}}{2\lambda_-\Upsilon_{q+\lambda_-+\lambda_+}}\right)\\
&= \frac{q}{(\Upsilon_{q+\lambda_-}+\mu)\Upsilon_{q+\lambda_-+\lambda_+}} + \frac{q}{(\Upsilon_{q+\lambda_+}-\mu)\Upsilon_{q+\lambda_-+\lambda_+}}\\
&= \frac{q\big(\Upsilon_{q+\lambda_-}+\Upsilon_{q+\lambda_+}\big)}{(\Upsilon_{q+\lambda_-}+\mu)(\Upsilon_{q+\lambda_+}-\mu)\Upsilon_{q+\lambda_-+\lambda_+}},
\end{aligned}$$

$$N = 1-\lambda_+\int_{-\infty}^{a} G_{q+\lambda_-+\lambda_+}(a,y)\frac{g_{+,q+\lambda_-}(y)}{g_{+,q+\lambda_-}(a)}m(\mathrm{d}y)$$

$$- \lambda_- \int_a^{\infty} G_{q+\lambda_-+\lambda_+}(a,y)\frac{g_{-,q+\lambda_+}(y)}{g_{-,q+\lambda_+}(a)}m(\mathrm{d}y)$$
$$= 1 - \lambda_+ \frac{\Upsilon_{q+\lambda_-+\lambda_+} - \Upsilon_{q+\lambda_-}}{2\lambda_+\Upsilon_{q+\lambda_-+\lambda_+}} - \lambda_- \frac{\Upsilon_{q+\lambda_-+\lambda_+} - \Upsilon_{q+\lambda_+}}{2\lambda_-\Upsilon_{q+\lambda_-+\lambda_+}} = \frac{\Upsilon_{q+\lambda_-} + \Upsilon_{q+\lambda_+}}{2\Upsilon_{q+\lambda_-+\lambda_+}}.$$

So

$$g_+(a) = \mathbb{E}_a e^{-\lambda_- \int_0^{e_q} \mathbf{1}_{(-\infty,a)}(X_s)\mathrm{d}s - \lambda_+ \int_0^{e_q} \mathbf{1}_{(a,\infty)}(X_s)\mathrm{d}s} = \frac{M}{N} = \frac{2q}{(\Upsilon_{q+\lambda_-} + \mu)(\Upsilon_{q+\lambda_+} - \mu)}. \quad (4.6)$$

Putting (4.6) in (3.17) and (3.18), respectively, for any $y < a$, we have

$$\mathbb{E}_y e^{-\lambda_- \int_0^{e_q} \mathbf{1}_{(-\infty,a)}(X_s)\mathrm{d}s - \lambda_+ \int_0^{e_q} \mathbf{1}_{(a,\infty)}(X_s)\mathrm{d}s} = \frac{q}{q+\lambda_-}\big(1 - \frac{g_{+,q+\lambda_-}(y)}{g_{+,q+\lambda_-}(a)}\big) + \frac{g_{+,q+\lambda_-}(y)}{g_{+,q+\lambda_-}(a)} f_+(a)$$
$$= \frac{q}{q+\lambda_-}\big(1 - e^{(-\mu+\Upsilon_{q+\lambda_-})(y-a)}\big) + \frac{2q}{(\Upsilon_{q+\lambda_-} + \mu)(\Upsilon_{q+\lambda_+} - \mu)} e^{(-\mu+\Upsilon_{q+\lambda_-})(y-a)}$$
$$= \frac{q}{q+\lambda_-} - \big(\frac{q}{q+\lambda_-} - \frac{2q}{(\Upsilon_{q+\lambda_-} + \mu)(\Upsilon_{q+\lambda_+} - \mu)}\big) e^{(-\mu+\Upsilon_{q+\lambda_-})(y-a)}; \quad (4.7)$$

for any $y > a$, we have

$$\mathbb{E}_y e^{-\lambda_- \int_0^{e_q} \mathbf{1}_{(-\infty,a)}(X_s)\mathrm{d}s - \lambda_+ \int_0^{e_q} \mathbf{1}_{(a,\infty)}(X_s)\mathrm{d}s}$$
$$= \frac{q}{q+\lambda_+}\big(1 - \frac{g_{-,q+\lambda_+}(y)}{g_{-,q+\lambda_+}(a)}\big) + \frac{g_{-,q+\lambda_+}(y)}{g_{-,q+\lambda_+}(a)} f_+(a)$$
$$= \frac{q}{q+\lambda_+}\big(1 - e^{(\mu+\Upsilon_{q+\lambda_+})(a-y)}\big) + \frac{2q}{(\Upsilon_{q+\lambda_-} + \mu)(\Upsilon_{q+\lambda_+} - \mu)} e^{(\mu+\Upsilon_{q+\lambda_+})(a-y)}$$
$$= \frac{q}{q+\lambda_+} - \big(\frac{q}{q+\lambda_+} - \frac{2q}{(\Upsilon_{q+\lambda_-} + \mu)(\Upsilon_{q+\lambda_+} - \mu)}\big) e^{(\mu+\Upsilon_{q+\lambda_+})(a-y)}. \quad (4.8)$$

By Corollary 3.2, we have

$$\mathbb{E}_y e^{-\lambda_- \int_0^{e_q} \mathbf{1}_{(-\infty,a)}(X_s)\mathrm{d}s} = \begin{cases} \frac{\Upsilon_q+\mu}{\Upsilon_{q+\lambda_-}+\mu}, & y = a, \\ \frac{q}{q+\lambda_-} + \frac{\lambda_-(\Upsilon_q+\mu)}{(q+\lambda_-)(\Upsilon_{q+\lambda_-}+\Upsilon_q)} e^{(-\mu+\Upsilon_{q+\lambda_-})(y-a)}, & y < a, \\ 1 - \frac{\lambda_-(\Upsilon_{q+\lambda_-}-\mu)}{(q+\lambda_-)(\Upsilon_{q+\lambda_-}+\Upsilon_q)} e^{(\mu+\Upsilon_q)(a-y)}, & y > a. \end{cases}$$

By Corollary 3.3, we have

$$\mathbb{E}_y e^{-\lambda_+ \int_0^{e_q} \mathbf{1}_{(b,\infty)}(X_s)\mathrm{d}s} = \begin{cases} \frac{\Upsilon_q-\mu}{\Upsilon_{q+\lambda_+}-\mu}, & y = b, \\ 1 - \frac{\lambda_+(\Upsilon_{q+\lambda_+}+\mu)}{(q+\lambda_+)(\Upsilon_{q+\lambda_+}+\Upsilon_q)} e^{(-\mu+\Upsilon_q)(y-b)}, & y < b, \\ \frac{q}{q+\lambda_+} + \frac{\lambda_+(\Upsilon_q-\mu)}{(q+\lambda_+)(\Upsilon_{q+\lambda_+}+\Upsilon_q)} e^{(\mu+\Upsilon_{q+\lambda_+})(b-y)}, & y > b. \end{cases}$$

The above results are consistent with pages 155-162 in [1].

Declarations

Conflict of interest The authors declare no conflict of interest.

References

[1] A N Borodin, P Salminen. *Handbook of Brownian Motion-Facts and Formulae*, Springer Science & Business Media, 2002.

[2] N Cai, N Chen, X Wan. *Occupation times of jump-diffusion processes with double exponential jumps and the pricing of options*, Mathematics of Operations Research, 2010, 35(2): 412-437.

[3] Y Chen, Y Li, X Zhou. *An occupation time related potential measure for Diffusion processes*, Front Math China, 2017, 12(3): 559-582.

[4] Y Chen, X Yang, Y Li, et al. *A Joint Laplace Transform for Pre-exit Diffusion of Occupation Times*, Acta Math Sinica, English Series, 2017, 33(4): 509-525.

[5] W Feller. *Diffusion processes in one dimension*, Transactions of the American Mathematical Society, 1954, 77(1): 1-31.

[6] D Landriault, J F Renaud, X Zhou. *Occupation times of spectrally negative Lévy processes with applications*, Stochastic processes and their applications, 2011, 121(11): 2629-2641.

[7] B Li, X Zhou. *The joint Laplace transforms for diffusion occupation times*, Advances in Applied Probability, 2013, 45(4): 1049-1067.

[8] Y Li, S Wang, X Zhou, et al. *Diffusion occupation time before exiting*, Frontiers of Mathematics in China, 2014, 9(4): 843-861.

[9] Y Li, C Yin, X Zhou. *On the last exit times for spectrally negative Levy processes*, Journal of Applied Probability, 2017, 54(2): 474-489.

[10] Y Li, X Zhou. *On pre-exit joint occupation times for spectrally negative Lévy processes*, Statistics & Probability Letters, 2014, 94(1): 48-55.

[11] Y Li, X Zhou, N Zhu. *Two-sided discounted potential measures for spectrally negative Lévy processes*, Statistics & Probability Letters, 2015, 100(1): 67-76.

[12] J Pitman, M Yor. *Hitting, occupation and inverse local times of one-dimensional diffusions: martingale and excursion approaches*, Bernoulli, 2003, 9(1): 1-24.

[13] J Pitman, M Yor. *Laplace transforms related to excursions of a one-dimensional diffusion*, Bernoulli, 1999, 5(2): 249-255.

[1]School of Mathematics and Statistics, Changsha University of Science and Technology, Changsha 410114, China.

[2]College of Mathematics and Computational Science, Hunan University of Arts and Science, Changde 415000, China.

Email: liyq-2001@163.com