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Strong invariance principle for a counterbalanced random walk

TAN Hui-qun[1] HU Zhi-shui[1] DONG Liang[2,*]

Abstract. We study a counterbalanced random walk $\check{S}_n = \check{X}_1 + \cdots + \check{X}_n$, which is a discrete time non-Markovian process and $\check{X}_n$ are given recursively as follows. For $n \geq 2$, $\check{X}_n$ is a new independent sample from some fixed law $\mu \neq 0$ with a fixed probability p, and $\check{X}_n = -\check{X}_{v(n)}$ with probability $1-p$, where $v(n)$ is a uniform random variable on $\{1, \cdots, n-1\}$. We apply martingale method to obtain a strong invariance principle for $\check{S}_n$.

§1 Introduction

Random walks with long memory have been widely used in applied mathematics, theoretical physics, computer sciences and econometrics. One of them is the so-called elephant random walk introduced by Schütz and Trimper [15]. It is a discrete time random walk on $\mathbb{Z}$ whose increments at each step depend on the whole history of the process. Let $p, q \in [0, 1]$ be fixed constants. At time zero, an elephant starts at the origin. At time $n = 1$, the elephant moves to the right with probability q and to the left with probability $1-q$. At any time $n \geq 2$, the elephant remembers one step chosen uniformly at random from the past, with probability p the elephant repeats it, and with probability $1-p$ it makes a step in the opposite direction.

A wide range of literature is available on the asymptotic behaviour of the elephant random walk and its variants, see, for instance, [1], [2], [4], [5], [6], [7], [8], [13]. The memory parameter p influences the limiting distribution of the elephant random walk ([9], [14], [15]). By using the connection of the elephant random walk to Pólya-type urns, Baur and Bertoin [1] derived the limiting process of the elephant random walk. In the diffusive regime ($0 \leq p < 3/4$) and the critical regime ($p = 3/4$), the limiting process turns out to be Gaussian; in the superdiffusive regime ($3/4 < p \leq 1$), the limit is not Gaussian. Coletti, Gava and Schütz ([7], [8]) studied the central limit theorem and strong invariance principle of the elephant random walk.

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*Corresponding author.

In this paper, we turn our attention to a counterbalanced random walk which is a generalization of the elephant random walk and was introduced in [4]. Let $(X_n)_{n\geq 1}, (\varepsilon_n)_{n\geq 2}$ and $(v(n))_{n\geq 2}$ be random variables which are independent of each other, where $(X_n)_{n\geq 1}$ are i.i.d. random variables with some given non-degenerate law μ, $\varepsilon_1 = 1$ and $\varepsilon_2, \varepsilon_3, \ldots$ are i.i.d. Bernoulli random variables with parameter $p \in [0,1]$, and for each $n \geq 2$, $v(n)$ has the uniform distribution $\{1, \cdots, n-1\}$. We construct a counterbalanced sequence $(\check{X}_n)$ by interpreting each $\{\varepsilon_n = 0\}$ as a counterbalancing event and each $\{\varepsilon_n = 1\}$ as an innovation event. Denote the number of innovations after n steps $\mathrm{i}(n)$ by

$$\mathrm{i}(n) = \sum_{j=1}^{n} \varepsilon_j \qquad \text{for} \quad n \geq 1.$$

Let $\check{X}_1 = X_1$ and for $n \geq 2$, $(\check{X}_n)$ be defined recursively

$$\check{X}_n = \begin{cases} -\check{X}_{v(n)}, & \text{if } \varepsilon_n = 0, \\ X_{\mathrm{i}(n)}, & \text{if } \varepsilon_n = 1. \end{cases} \tag{1}$$

Then we get the counterbalanced random walk:

$$\check{S}_n = \check{X}_1 + \cdots + \check{X}_n, \qquad n \in \mathbb{N}.$$

Bertoin [4] obtained the law of large numbers and the central limit theorem for $\check{S}_n$ by applying a coupling with a reinforcement algorithm due to H. A. Simon and properties of random recursive trees. Bertenghi and Rosales-Ortiz [3] studied the joint weak invariant principles via a method of martingale functional central limit theorem. The main purpose of our work is to establish a strong invariance principle for $\check{S}_n$. As a by-product of our main result Theorem 1.1, we get the law of the iterated logarithm and weak convergence on $D[0,1]$ for the counterbalanced random walk.

For any $k \in \mathbb{N}$, we write

$$m_k = \int_{\mathbb{R}} x^k \mu(dx)$$

for the moment of order k of μ whenever $\int_{\mathbb{R}} |x|^k \mu(dx) < \infty$. Let

$$b_1 = \frac{p}{2-p} m_1, \tag{2}$$

then we observe that $m_2 - b_1^2 > 0$.

Our main results are stated as follows.

Theorem 1.1. *Suppose that $p \in (0,1]$ and $||X_1||_\infty < \infty$. Then we can redefine $\{\check{S}_n, n \geq 1\}$ on a new probability space such that there exists a Brownian motion $\{W(t), t \geq 0\}$ satisfying that*

$$\frac{1}{\sqrt{n \log\log n}} \sup_{0\leq t\leq 1} \left| \sqrt{\frac{3-2p}{m_2 - b_1^2}} \big(\check{S}_{[nt]} - [nt] b_1\big) - (nt)^{p-1} W((nt)^{3-2p}) \right| \longrightarrow 0 \quad a.s.$$

and

$$\frac{1}{\sqrt{n}} \sup_{0\leq t\leq 1} \left| \sqrt{\frac{3-2p}{m_2 - b_1^2}} \big(\check{S}_{[nt]} - [nt] b_1\big) - (nt)^{p-1} W((nt)^{3-2p}) \right| \xrightarrow{p} 0.$$

Corollary 1.1. *Suppose that $p \in (0,1]$ and $||X_1||_\infty < \infty$.*

(a). We have the law of the iterated logarithm:

$$\limsup_{n\to\infty} \frac{|\check{S}_n - nb_1|}{\sqrt{2n\log\log n}} = \sqrt{\frac{m_2 - b_1^2}{3-2p}} \qquad a.s. \tag{3}$$

(b). Let $\{W(t), t \geq 0\}$ be a Brownian motion, then we have weak convergence on $D[0,1]$ with respect to the uniform topology:

$$\sqrt{\frac{3-2p}{m_2 - b_1^2}} \frac{\check{S}_{[nt]} - [nt]b_1}{\sqrt{n}} \Longrightarrow t^{p-1}W(t^{3-2p}). \tag{4}$$

Remark 1.1. *In Theorem 1.1 and Corollary 1.1, we require the restrictive condition $||X_1||_\infty < \infty$ instead of $\mathbb{E}(X_1^2) < \infty$ for technical reasons in the proof. In order to weaken the condition $||X_1||_\infty < \infty$, one should use the truncation method. For weak limit theorems including central limit theorem and functional central limit theorem, Bertenghi and Rosales-Ortiz [3] used a special truncation technique to obtain the results in the centered case(see section 4.2 in [3]). But the truncation technique in [3] is not applicable to strong limit theorems. To our knowledge, there was no strong limit theorem for $\check{S}_n$ except that Bertenghi and Rosales-Ortiz [3] obtained the strong law of large numbers under the condition $||X_1||_\infty < \infty$. It is an open question to get some strong limit theorems for $\check{S}_n$ without the condition $||X_1||_\infty < \infty$.*

The rest of this paper is organised as follows: In section 2 we introduce a martingale associated with the counterbalanced random walk. We obtain the result about asymptotic variance of the partial sum process of the counterbalanced random walk $\check{S}_n$. Section 3 is devoted to several preliminaries. We shall present some key results about strong law of large numbers of $\hat{S}_n$, $\check{S}_n$ and angle bracket process $\langle M\rangle_n$. Finally, in section 4 we apply the martingale method to obtain a strong invariance principle for $\check{S}_n$.

§2 A martingale associated with the counterbalanced random walk

Set $\gamma_n = \frac{n+p-1}{n}$ and $a_1 = 1$. For all $n \geq 2$, we define

$$a_n = \prod_{k=1}^{n-1} \gamma_k = \frac{\Gamma(n+p-1)}{\Gamma(n)\Gamma(p)},$$

where $\Gamma(\cdot)$ stands for the Euler gamma function. Denote by $\mathscr{F}_n = \sigma(\{(X_{\mathrm{i}(j)}, \varepsilon_j, v(j)) : 1 \leq j \leq n\})$ and

$$M_n = \frac{\check{S}_n - nb_1}{a_n}, \tag{5}$$

where b_1 is defined in (2). If $\int_{\mathbb{R}} |x|\mu(dx) < \infty$, then it follows from the definition (1) that for any $n \geq 1$,

$$\mathbb{E}(\check{X}_{n+1}|\mathscr{F}_n) = p\mathbb{E}(X_{\mathrm{i}(n)+1}|\mathscr{F}_n) - (1-p)\frac{\check{X}_1 + \cdots + \check{X}_n}{n} = pm_1 - (1-p)\frac{\check{S}_n}{n}. \tag{6}$$

This implies that $\mathbb{E}(\check{S}_{n+1}|\mathscr{F}_n) = pm_1 + \gamma_n\check{S}_n$ and then

$$\mathbb{E}(\check{S}_{n+1}|\mathscr{F}_n) - (n+1)b_1 = \gamma_n(\check{S}_n - nb_1). \tag{7}$$

Hence $\mathbb{E}(M_{n+1}|\mathscr{F}_n) = M_n$, i.e. M_n is a martingale with respect to the filtration $\mathscr{F}_n$. Note that $a_n \sim n^{p-1}/\Gamma(p)$. Then by applying (7) and recalling $\mathbb{E}(\check{S}_1) = m_1$, we obtain that

$$\frac{\mathbb{E}(\check{S}_n)}{n} \to b_1 \qquad \text{as} \quad n \to \infty.$$

Proposition 2.1. *Assume that* $\int_{\mathbb{R}} |x|^2 \mu(dx) < \infty$, *then*

$$\frac{\mathrm{Var}(\check{S}_n)}{n} \to \frac{m_2 - b_1^2}{3 - 2p}.$$

Proof. Similar arguments as in (6) yields that

$$\mathbb{E}(\check{X}_{n+1}^2|\mathscr{F}_n) = pm_2 + (1-p)\frac{V_n}{n}$$

where $V_n = \sum_{i=1}^n \check{X}_i^2$. Thus,

$$\mathbb{E}(V_{n+1}) - (n+1)m_2 = \frac{n+1-p}{n}\big(\mathbb{E}(V_n) - nm_2\big).$$

With the initial condition $\mathbb{E}(\check{X}_1^2) = m_2$, we deduce that $\mathbb{E}(V_n) = m_2 n$. By applying (6), we have

$$\begin{aligned}
\mathbb{E}(\check{S}_{n+1}^2|\mathscr{F}_n) &= \check{S}_n^2 + 2\check{S}_n\mathbb{E}(\check{X}_{n+1}|\mathscr{F}_n) + \mathbb{E}(\check{X}_{n+1}^2|\mathscr{F}_n)\\
&= \check{S}_n^2 + 2\check{S}_n\big(pm_1 - \frac{1-p}{n}\check{S}_n\big) + \mathbb{E}(\check{X}_{n+1}^2|\mathscr{F}_n)\\
&= \frac{n+2p-2}{n}\check{S}_n^2 + 2pm_1\check{S}_n + \mathbb{E}(\check{X}_{n+1}^2|\mathscr{F}_n).
\end{aligned}$$

Hence

$$\mathbb{E}(\check{S}_{n+1}^2) = \frac{n+2p-2}{n}\mathbb{E}(\check{S}_n^2) + 2pm_1\mathbb{E}(\check{S}_n) + m_2.$$

It follows from (7) that

$$\mathbb{E}^2(\check{S}_{n+1}) = \frac{n+2p-2}{n}\mathbb{E}^2(\check{S}_n) + \left(\frac{p-1}{n}\mathbb{E}(\check{S}_n) + pm_1\right)^2 + 2pm_1\mathbb{E}(\check{S}_n).$$

By letting

$$\gamma_n' = \frac{n+2p-2}{n} \qquad \text{and} \qquad \alpha_n = m_2 - \Big(\frac{p-1}{n}\mathbb{E}(\check{S}_n) + pm_1\Big)^2,$$

we conclude that $\mathrm{Var}(\check{S}_{n+1}) = \gamma_n'\mathrm{Var}(\check{S}_n) + \alpha_n$. Therefore,

$$\mathrm{Var}(\check{S}_{n+1}) = \mathrm{Var}(\check{S}_1)\prod_{k=1}^n \gamma_k' + \sum_{j=1}^{n-1}\prod_{k=j+1}^n \gamma_k'\alpha_j + \alpha_n. \tag{8}$$

Using the fact that

$$\sum_{k=1}^n \frac{1}{k} = \log n + \gamma + O\left(\frac{1}{n}\right)$$

where γ is Euler's constant, we obtain

$$\begin{aligned}
&\textstyle\sum_{j=[n^{1/2}]}^{n-1}\prod_{k=j+1}^n \frac{k+2p-2}{k}\\
&= \textstyle\sum_{j=[n^{1/2}]}^{n-1}\exp\left(\sum_{k=j+1}^n \log\left(1+\frac{2p-2}{k}\right)\right)\\
&= \textstyle\sum_{j=[n^{1/2}]}^{n-1}\exp\left(\sum_{k=j+1}^n\left(\frac{2p-2}{k} + O\left(\frac{1}{k^2}\right)\right)\right)\\
&= \textstyle\sum_{j=[n^{1/2}]}^{n-1}\exp\left((2p-2)\left(\log\frac{n}{j} + O\left(\frac{1}{j}\right)\right) + O\left(\frac{1}{j}\right)\right)
\end{aligned}$$

$$= \sum_{j=[n^{1/2}]}^{n-1} \exp\Big((2p-2)\log\tfrac{n}{j}\Big) + O(n^{1/2}).$$

Noting that

$$\sum_{j=[n^{1/2}]}^{n-1} \exp\left((2p-2)\left(\log\frac{n}{j}\right)\right) = \sum_{j=[n^{1/2}]}^{n-1} \left(\frac{j}{n}\right)^{2-2p} \sim \frac{n}{3-2p}$$

and

$$\sum_{j=1}^{[n^{1/2}]-1} \prod_{k=j+1}^{n} \frac{k+2p-2}{k} = O(n^{1/2}),$$

we have

$$\frac{1}{n}\sum_{j=1}^{n-1}\prod_{k=j+1}^{n}\gamma_k' = \frac{1}{n}\sum_{j=1}^{[n^{1/2}]-1}\prod_{k=j+1}^{n}\frac{k+2p-2}{k} + \frac{1}{n}\sum_{j=[n^{1/2}]}^{n-1}\prod_{k=j+1}^{n}\frac{k+2p-2}{k} \to \frac{1}{3-2p}.$$

This together with the fact that $\alpha_n \to m_2 - b_1^2$ implies that

$$\frac{1}{n+1}\sum_{j=1}^{n-1}\prod_{k=j+1}^{n}\gamma_k'\alpha_k + \frac{\alpha_n}{n+1} \to \frac{m_2-b_1^2}{3-2p}. \tag{9}$$

Note that

$$\prod_{k=1}^{n}\gamma_k' = \frac{\Gamma(n+2p-1)}{\Gamma(n+1)\Gamma(2p-1)} \sim \frac{n^{2p-2}}{\Gamma(2p-1)},$$

which yields

$$\frac{\mathrm{Var}(\check{S}_1)}{n+1}\prod_{k=1}^{n}\gamma_k' \to 0. \tag{10}$$

We can conclude from (8)-(10) that

$$\frac{\mathrm{Var}(\check{S}_{n+1})}{n+1} \to \frac{m_2-b_1^2}{3-2p}.$$

The proof of Proposition 2.1 is completed. □

§3 Preliminaries

Our analysis relies on a natural coupling of the counterbalanced random walk with the step reinforced random walk. Specifically, consider the same sequences $(X_n)_{n\geq 1}$, $(\varepsilon_n)_{n\geq 1}$ and $(v(n))_{n\geq 2}$ in (1). The step reinforced random walk is defined recursively

$$\hat{X}_n = \begin{cases} \hat{X}_{v(n)}, & \text{if } \varepsilon_n = 0, \\ X_{\mathrm{i}(n)}, & \text{if } \varepsilon_n = 1. \end{cases} \tag{11}$$

Then we get the step reinforced random walk:

$$\hat{S}_n = \hat{X}_1 + \cdots + \hat{X}_n, \qquad n \in \mathbb{N}.$$

The following Lemma 3.1 can be found in Bertenghi and Rosales-Ortiz [3] (Proposition 2.5, Lemmas 2.2 and 5.1). It is a law of large numbers of the step reinforced random walk, which will be used in the proof of Lemma 3.3.

Lemma 3.1. *Suppose that* $||X_1||_\infty < \infty$.

(a) *For $p \in (0, 1/2)$, we have*

$$\frac{\hat{S}_n - nm_1}{n^{1-p}} \to W \qquad a.s. \text{ and in } L^2(\mathbb{P}),$$

where $W \in L^2(\mathbb{P})$ is a non-degenerate random variable.

(b) *For $p = 1/2$, we have*

$$\frac{\hat{S}_n - nm_1}{\sqrt{n}\log n} \to 0 \qquad a.s.$$

(c) *For $p \in (1/2, 1)$, we have*

$$\frac{\hat{S}_n - nm_1}{n^p} \to 0 \qquad a.s.$$

Remark 3.1. *In the case $p = 1$, i.e. when no reinforcement events occur, $\hat{S}_n$ is just the partial sum of i.i.d. random variables $(X_n)_{n\geq 1}$ with given law μ. Hence, for $p = 1$, it follows from the classical law of large numbers that $(\hat{S}_n - nm_1)/n \to 0$ a.s.*

Lemma 3.2. *Let $p \in (0, 1]$ and suppose that $||X_1||_\infty < \infty$. We have*

$$\frac{\check{S}_n}{n} \to b_1 \qquad a.s.$$

Remark 3.2. *For the weak law of large numbers, Bertoin [4] obtained $\check{S}_n/n \to b_1$ in probability under the assumption that $\mathbb{E}(|X_1|) < \infty$ (see Proposition 1.1 in [4]).*

Proof. Note that

$$\begin{aligned} |\Delta M_k| &= \left| \frac{\check{S}_k - kb_1}{a_k} - \frac{\check{S}_{k-1} - (k-1)b_1}{a_{k-1}} \right| \\ &\leq \left| \frac{\check{X}_k - b_1}{a_k} + \frac{\check{S}_{k-1} - (k-1)b_1}{k-1} \frac{1-p}{a_k} \right| \leq \frac{C||X_1||_\infty}{a_k}. \end{aligned} \tag{12}$$

Then

$$\sum_{k=1}^{\infty} \frac{a_k^2}{k^2} \mathbb{E}(\Delta M_k^2 | \mathscr{F}_{k-1}) < \infty \qquad \text{a.s.}$$

By Theorem 2.17 of [11], we obtain that $\sum_{k=1}^{\infty} a_k \Delta M_k / k$ converges almost surely. By applying Kronecker's Lemma and $n/a_n \uparrow \infty$, we have

$$\frac{\check{S}_n - nb_1}{n} = \frac{a_n M_n}{n} = \frac{a_n}{n} \sum_{k=1}^{n} \Delta M_k \to 0 \qquad \text{a.s.}$$

The proof of Lemma 3.2 is completed. □

Let M_n be defined in (5). Define

$$\langle M \rangle_n = \sum_{k=1}^{n} \mathbb{E}(\Delta M_k^2 | \mathscr{F}_{k-1}), \quad n \geq 1,$$

where $\{\Delta M_k, k \geq 1\}$ is the martingale difference sequence, which is defined by $\Delta M_1 = M_1$, and for $k \geq 2$, $\Delta M_k = M_k - M_{k-1}$. We set

$$s_n^2 := \sum_{k=1}^{n} \frac{1}{a_k^2}.$$

Then we have

$$a_n \sim \frac{n^{p-1}}{\Gamma(p)}, \quad s_n^2 \sim \frac{\Gamma^2(p)}{3-2p} n^{3-2p} \quad \text{and} \quad s_n a_n \sim \sqrt{\frac{n}{3-2p}}. \tag{13}$$

Lemma 3.3. *Let $p \in (0,1]$ and suppose that $||X_1||_\infty < \infty$. Then*

$$\frac{\langle M \rangle_n}{s_n^2} \to m_2 - b_1^2 \qquad a.s.$$

Proof. Note that

$$\Delta M_n = \frac{\check{X}_n - b_1}{a_n} + \frac{1-p}{n+p-2} \frac{\check{S}_{n-1} - (n-1)b_1}{a_{n-1}},$$

and, by Lemma 3.2 and $a_n/a_{n-1} \to 1$,

$$\frac{1-p}{n+p-2} \frac{\check{S}_{n-1} - (n-1)b_1}{a_{n-1}} = o(a_n^{-1}) \qquad \text{a.s.}$$

We obtain that

$$\mathbb{E}((\Delta M_n)^2 | \mathscr{F}_{n-1}) = \frac{1}{a_n^2} \mathbb{E}((\check{X}_n - b_1)^2 | \mathscr{F}_{n-1}) + o(a_n^{-2}) \mathbb{E}(\check{X}_n - b_1 | \mathscr{F}_{n-1}) + o(a_n^{-2}) \qquad \text{a.s.} \tag{14}$$

By (6) and Lemma 3.2, we have

$$\mathbb{E}(\check{X}_n - b_1 | \mathscr{F}_{n-1}) = pm_1 - (1-p)\frac{\check{S}_n}{n} - b_1 \to 0 \qquad \text{a.s.}, \tag{15}$$

and similarly,

$$\begin{aligned} \mathbb{E}((\check{X}_n - b_1)^2 | \mathscr{F}_{n-1}) &= \mathbb{E}(\check{X}_n^2 | \mathscr{F}_{n-1}) - 2b_1 \mathbb{E}(\check{X}_n - b_1 | \mathscr{F}_{n-1}) - b_1^2 \\ &= pm_2 - b_1^2 + (1-p)\frac{V_{n-1}}{n-1} + o(1) \qquad \text{a.s.}, \end{aligned}$$

where

$$V_{n-1} = \sum_{k=1}^{n-1} \check{X}_k^2 = \sum_{k=1}^{n-1} \hat{X}_n^2.$$

Note that $(\hat{X}_n^2)_{n \geq 1}$ is also a step reinforced random walk which can be defined in (11) by replacing $(X_n)_{n\geq 1}$ with $(X_n^2)_{n\geq 1}$. It follows from Lemma 3.1 and Remark 3.1 that $V_{n-1}/(n-1) \to m_2$ a.s. Hence

$$\mathbb{E}((\check{X}_n - b_1)^2 | \mathscr{F}_{n-1}) \to m_2 - b_1^2 \qquad \text{a.s.}$$

This together with (14) and (15) implies that

$$a_n^2 \mathbb{E}((\Delta M_n)^2 | \mathscr{F}_{n-1}) \to m_2 - b_1^2 \qquad \text{a.s.}$$

By noting that

$$s_n^2 = \sum_{k=1}^{n} \frac{1}{a_k^2} \sim \frac{\Gamma^2(p)}{3-2p} n^{3-2p} \to \infty$$

and applying Toeplitz Lemma, we conclude that

$$\frac{\langle M \rangle_n}{s_n^2} = \frac{1}{s_n^2} \sum_{k=1}^{n} \frac{a_k^2 \mathbb{E}((\Delta M_k)^2 | \mathscr{F}_{k-1})}{a_k^2} \to m_2 - b_1^2 \qquad \text{a.s.}$$

The proof of Lemma 3.3 is completed. □

§4 Proof of main results

Before presenting the proof of Theorem 1.1, we first introduce two lemmas which are minor modifications of Lemmas 1 and 2 in [10]. The detailed proofs are omitted here.

Lemma 4.1. *Assume that $\{W(t), t \geq 0\}$ is a Brownian motion. Let $\{a_n\}, \{a'_n\}, \{s_n\}, \{s'_n\}$ be sequences of non-decreasing real positive numbers. If $a_n \sim a'_n, s_n \sim s'_n$ and $a_n, s_n \to \infty$, then we have*

$$\frac{a_n W(s_n^2) - a'_n W((s'_n)^2)}{a_n s_n \sqrt{\log\log s_n}} \longrightarrow 0 \quad a.s.$$

and

$$\frac{\sup\limits_{1\leq k\leq n} \Big| a_k W(s_k^2) - a'_k W((s'_k)^2) \Big|}{a_n s_n} \stackrel{p}{\longrightarrow} 0.$$

Lemma 4.2. *Assume that $\{W(t), t \geq 0\}$ is a Brownian motion. If $0 < p \leq 1$, then we have*

$$\frac{1}{\sqrt{n}} \sup_{0\leq t\leq n} \Big| [t]^{p-1} W([t]^{3-2p}) - t^{p-1} W(t^{3-2p}) \Big| \longrightarrow 0 \quad a.s.$$

Proof of Theorem 1.1. In the proof, we shall apply the martingale version of the Skorohod embedding theorem which is successfully used in [8]. For $n \geq 1$, we let

$$M'_n = \frac{M_n}{\sqrt{m_2 - b_1^2}}.$$

By (5), $\{M'_n, \mathscr{F}_n, n \geq 1\}$ is a martingale. By applying the martingale version of the Skorohod embedding theorem (see e.g. Theorem 14.16 in [12], we can redefine $\{M'_n, \mathscr{F}_n, n \geq 1\}$ on a new probability space such that there exists a Brownian motion $\{B(t), t \geq 0\}$ with associated stopping times $0 = T_0 \leq T_1 \leq \cdots$ such that $M'_n = B(T_n)$ a.s. for all $n \geq 1$ and

$$\mathbb{E}(\Delta T_n | \mathscr{G}_{n-1}) = \mathbb{E}((\Delta M'_n)^2 | \mathscr{F}_{n-1}), \quad \mathbb{E}((\Delta T_n)^2 | \mathscr{G}_{n-1}) \leq 4\mathbb{E}((\Delta M'_n)^4 | \mathscr{F}_{n-1}), \quad n \geq 1, \tag{16}$$

where $\Delta T_n = T_n - T_{n-1}$, $\Delta M'_n = M'_n - M'_{n-1}$ and $\mathscr{G}_n = \sigma(B(t), t \leq T_n)$ for any $n \geq 1$. By applying Lemma 3.3, we have

$$\sum_{k=1}^{n} \mathbb{E}((\Delta M'_k)^2 | \mathscr{F}_{k-1}) \sim s_n^2 \quad \text{a.s.}$$

Hence it follows from (16) that

$$\sum_{k=1}^{n} \mathbb{E}(\Delta T_k | \mathscr{G}_{k-1}) \sim s_n^2 \quad \text{a.s.} \tag{17}$$

For $1 \leq k \leq n$, we denote $\Delta T'_k = \Delta T_k - \mathbb{E}(\Delta T_k | \mathscr{G}_{k-1})$. Using (12) and (16), then for some $C_1 > 0$, we have

$$\mathbb{E}((\Delta T'_k)^2 | \mathscr{G}_{k-1}) = \mathbb{E}((\Delta T_k)^2 | \mathscr{G}_{k-1}) - \mathbb{E}^2(\Delta T_k | \mathscr{G}_{k-1}) \leq \frac{C_1 ||X_1||_\infty}{a_k^4} \quad \text{a.s.}$$

We conclude from (13) that

$$\sum_{k=1}^{\infty} \mathbb{E}\Big(\frac{(\Delta T'_k)^2}{s_k^4}\Big) \leq \sum_{k=1}^{\infty} \frac{C_1^2 ||X_1||_\infty^2}{(a_k s_k)^4} < \infty.$$

By Theorem 2.18 of [11], we obtain

$$\sum_{k=1}^{\infty} \frac{\Delta T'_k}{s_k^2} < \infty \qquad \text{a.s.}$$

Then, by applying Kronecker's lemma, we have

$$\frac{1}{s_n^2} \sum_{k=1}^{n} \Delta T'_k \to 0 \qquad \text{a.s.}$$

This together with (17) implies that

$$T_n = \sum_{k=1}^{n} \Delta T_k \sim s_n^2 \qquad \text{a.s.}$$

Then from the proof of Theorem 14.6 in [12], we can obtain that

$$B(T_n) - B(s_n^2) = o(\sqrt{s_n^2 \log\log s_n}) \quad \text{a.s.}$$

and

$$s_n^{-1} \sup_{0\le t\le 1} \left| B(T_{[nt]}) - B(s_{[nt]}^2) \right| \xrightarrow{p} 0.$$

Since

$$\frac{\check{S}_n - nb_1}{\sqrt{m_2 - b_1^2}} - a_n B(s_n^2) = a_n M'_n - a_n B(s_n^2) = a_n (B(T_n) - B(s_n^2)) \quad \text{a.s.,}$$

we have

$$\frac{\check{S}_n - nb_1}{\sqrt{m_2 - b_1^2}} - a_n B(s_n^2) = o(a_n s_n \sqrt{\log\log s_n}) \quad \text{a.s.} \tag{18}$$

and

$$(a_n s_n)^{-1} \sup_{0\le t\le 1} \left| \frac{\check{S}_{[nt]} - [nt]b_1}{\sqrt{m_2 - b_1^2}} - a_{[nt]} B(s_{[nt]}^2) \right| \xrightarrow{p} 0.$$

By applying (13), (18) and Lemma 4.1, we obtain

$$\frac{1}{\sqrt{n \log\log n}} \Big(\frac{\check{S}_n - nb_1}{\sqrt{m_2 - b_1^2}} - \frac{n^{p-1}}{\Gamma(p)} B\Big(\frac{(\Gamma(p))^2}{3-2p} n^{3-2p} \Big) \Big) \longrightarrow 0 \quad \text{a.s.} \tag{19}$$

For $0 \le t \le 1$, we define

$$W(t) = \frac{\sqrt{3-2p}}{\Gamma(p)} B\Big(\frac{(\Gamma(p))^2}{3-2p} t \Big).$$

By applying the rescaling property, $\{W(t), t \ge 0\}$ is a standard Brownian motion. By rewriting (19), we obtain

$$\frac{1}{\sqrt{n \log\log n}} \left(\sqrt{\frac{3-2p}{m_2 - b_1^2}} \big(\check{S}_n - nb_1 \big) - n^{p-1} W(n^{3-2p}) \right) \longrightarrow 0 \quad \text{a.s.}$$

This implies that

$$\frac{1}{\sqrt{n \log\log n}} \sup_{0\le t\le 1} \left| \sqrt{\frac{3-2p}{m_2 - b_1^2}} \big(\check{S}_{[nt]} - [nt]b_1 \big) - (nt)^{p-1} W((nt)^{3-2p}) \right| \longrightarrow 0 \quad \text{a.s.} \tag{20}$$

Similarly, we obtain that

$$\frac{1}{\sqrt{n}} \sup_{0\le t\le 1} \left| \sqrt{\frac{3-2p}{m_2 - b_1^2}} \big(\check{S}_{[nt]} - [nt]b_1 \big) - (nt)^{p-1} W((nt)^{3-2p}) \right| \xrightarrow{p} 0. \tag{21}$$

Now Theorem 1.1 follows from (20), (21) and Lemma 4.2. □

Proof of Corollary 1.1. By applying Theorem 1.1, we have

$$\sup_{0\le t\le 1}\left|\sqrt{\frac{3-2p}{m_2-b_1^2}}\frac{\check{S}_{[nt]}-[nt]b_1}{\sqrt{n}}-n^{p-3/2}t^{p-1}W((nt)^{3-2p})\right|\xrightarrow{p}0.$$

Since $\{n^{p-3/2}W((nt)^{3-2p}), t\ge 0\}$ is also a Brownian motion, we have

$$\{n^{p-3/2}t^{p-1}W((nt)^{3-2p}), t\ge 0\}\overset{d}{=}\{t^{p-1}W(t^{3-2p}), t\ge 0\}.$$

Then we can get (4). Similarly, (3) follows from Theorem 1.1 and the law of the iterated logarithm for Brownian motion. The proof of Corollary 1.1 is completed. □

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Declarations

Conflict of interest The authors declare no conflict of interest.

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[1]International Institute of Finance, School of Management, University of Science and Technology of China, Hefei 230026, China.

[2]Center for Basic Teaching and Experiment, Nanjing University of Science and Technology, Jiangyin 214443, China.

Email: dongliang@njust.edu.cn