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A generalization of Banach's lemma and its applications to perturbations of bounded linear operators

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Abstract. Let X be a Banach space and let $P : X \to X$ be a bounded linear operator. Using an algebraic inequality on the spectrum of P, we give a new sufficient condition that guarantees the existence of $(I - P)^{-1}$ as a bounded linear operator on X, and a bound on its spectral radius is also obtained. This generalizes the classic Banach lemma. We apply the result to the perturbation analysis of general bounded linear operators on X with commutative perturbations.

§1 Introduction

The celebrated Banach lemma is a fundamental tool for the perturbation theory of linear operators [6]. Let X be a Banach space and let P be a bounded linear operator from X into itself. Banach's lemma claims that if the operator norm $\|P\| < 1$, then the operator $I - P$ is one-to-one and onto, and so its inverse $(I - P)^{-1}$ is a bounded linear operator, which can be expressed as the sum of the absolutely convergent operator series

$$(I - P)^{-1} = \sum_{n=0}^{\infty} P^n, \tag{1}$$

where I is the identity operator. A consequence of (1) is the inequality

$$\|(I - P)^{-1}\| \leq \frac{1}{1 - \|P\|}, \tag{2}$$

which provides an upper bound of the inverse for a perturbation of the identity operator. Moreover, under a weaker condition that the spectral radius of P is less than 1, Banach's lemma is still true (see Problem 4.6 in [6]), including the absolute convergence of (1), but the upper bound (2) may not be satisfied.

Banach's lemma can be applied to the perturbation analysis of inverses or generalized inverses of bounded linear operators [7]. As an example, for any invertible bounded linear operator T from X onto itself so that T^{-1} is bounded on X by the Banach inverse mapping theorem, if

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we let $S = T + A$ and the bounded linear operator A satisfies the inequality $\|A\| < 1/\|T^{-1}\|$, then S^{-1} exists as a bounded linear operator on X and $\|S^{-1}\| \leq \|T^{-1}\|/(1 - \|T^{-1}\|\|A\|)$ [6].

Due to the importance of the perturbation theory in operator theory, numerical analysis and various applied areas, there have been variants and extensions of Banach's lemma in the literature. Motivated by the problems in frame theory, [2] initiated a new approach to investigating the existence of $(I - P)^{-1}$, which relates the norm of the perturbation P to that of $I - P$ via some inequality that generalizes the original assumption $\|P\| < 1$ in Banach's lemma. The result of [2] was improved and extended in [4] with the help of the concept of the approximate point spectrum, and new perturbation bounds of the generalized inverse of some classes of bounded linear operators were also obtained.

However, all such perturbation results use the operator norm as the quantity to measure the size of the perturbation, and so the condition for smallness of the perturbation in terms of the norm is too strong to be satisfied in certain applications. According to its definition, operator norms only give the maximal stretching of all unit vectors under the given operator, and consequently it may not provide the general behavior of the operator applied to all vectors. On the other hand, the spectrum of an operator provides its more intrinsic properties, so the spectral radius appears to be a more suitable measure for the size of perturbations.

In this paper, we improve the results of [4] by using the spectrum instead of the operator norm to generalize Banach's lemma. The idea behind the new approach is based on a simple algebraic inequality for all the spectral points of the operator. It turns out that the classic Banach lemma with the spectral radius condition will be a special case of the new theorem. The obtained basic perturbation result can be applied to the perturbation analysis for inverses or generalized inverses of operators.

The next section contains the generalized Banach lemma, and in Section 3 we present some applications to the perturbation analysis of bounded linear operators. We conclude in Section 4.

§2 A generalized Banach lemma

Suppose X is a Banach space with its norm $\| \ \|$ and let $B(X)$ denote the Banach space of all bounded linear operators $T : X \to X$ with the corresponding operator norm $\|T\| = \sup\{\|Tx\| : x \in X, \|x\| = 1\}$.

Let $T \in B(X)$ be given. A complex number α is said to be a spectral point of T if $T - \alpha I$ is not one-to-one or not onto. The collection of all the spectral points of T is called the *spectrum* of T and is denoted as $\sigma(T)$. It is well-known that the spectrum of T is closed and bounded, and the maximum absolute value of the points of $\sigma(T)$, denoted as $r(T)$, is called the *spectral radius* of T. It is well-known [6] that $r(T) \leq \|T\|$ and $r(T) = \lim_{n\to\infty} \|T^n\|^{1/n} = \inf_{n\geq 1} \|T^n\|^{1/n}$. Furthermore, if $T, S \in B(X)$ satisfy $TS = ST$, then there are the following multiplication and addition inequalities

$$r(TS) \leq r(T)r(S) \ \text{ and } \ r(T + S) \leq r(T) + r(S). \tag{3}$$

The following result may be called a generalized Banach's lemma, and thus we state it as a lemma instead of theorem, even if it is the main result of the paper.

Lemma 2.1 Let $P \in B(X)$ satisfy the condition that there exist two real numbers $\lambda_1 < 1$ and $\lambda_2 < 1$ such that for all $\alpha \in \sigma(P)$,

$$|\alpha| \le \lambda_1 + \lambda_2 |1-\alpha|. \tag{4}$$

Then $\lambda_1 \in (-1,1)$, $\lambda_2 \in (-1,1)$, and $I-P$ is one-to-one and onto so that $(I-P)^{-1} \in B(X)$. Moreover, the following inequalities are valid:

$$\frac{1-\lambda_1}{1+\lambda_2} \le r(I-P) \le \frac{1+\lambda_1}{1-\lambda_2}; \tag{5}$$

$$\frac{1-\lambda_2}{1+\lambda_1} \le r((I-P)^{-1}) \le \frac{1+\lambda_2}{1-\lambda_1}; \tag{6}$$

$$r((I-P)^{-1}-I) \le \frac{1+\lambda_2}{1-\lambda_1} r(P). \tag{7}$$

Proof. Let $\lambda \in \sigma(I-P)$. Then $\lambda = 1-\alpha$ for some $\alpha \in \sigma(P)$. By (4),

$$|\lambda| = |1-\alpha| \ge 1-|\alpha| \ge 1-\lambda_1-\lambda_2|1-\alpha|.$$

Thus $(1+\lambda_2)|1-\alpha| \ge 1-\lambda_1 > 0$, from which $1+\lambda_2 > 0$, and it follows that $(1-\lambda_1)/(1+\lambda_2) \le |1-\alpha| \le r(I-P)$, which gives the left side inequality of (5). On the other hand, (4) also implies that

$$|1-\alpha| \le 1+|\alpha| \le 1+\lambda_1+\lambda_2|1-\alpha|.$$

Consequently $(1-\lambda_2)|1-\alpha| \le 1+\lambda_1$. Hence $\lambda_1 > -1$, and $|1-\alpha| \le (1+\lambda_1)/(1-\lambda_2)$. Since $\lambda = 1-\alpha \in \sigma(I-P)$ is arbitrary, the right side inequality of (5) is valid.

If $0 \in \sigma(I-P)$, then $1 \in \sigma(P)$. Substituting 1 for α in (4) gives that $1 \le \lambda_1+\lambda_2|1-1| = \lambda_1$, which contradicts the assumption that $\lambda_1 < 1$. Therefore, $1 \notin \sigma(P)$. Namely $(I-P)^{-1}$ exists and $(I-P)^{-1} \in B(X)$.

Now we verify the two inequalities in (6). Since $(I-P)^{-1}(I-P) = (I-P)(I-P)^{-1} = I$, we see that $r((I-P)^{-1})r(I-P) \ge 1$ from the multiplication inequality of the spectral radius in (3). It follows from the right side of (5) that

$$r((I-P)^{-1}) \ge \frac{1}{r(I-P)} \ge \frac{1-\lambda_2}{1+\lambda_1},$$

which gives the left side inequality of (6). For the other inequality of (6), recall that $\sigma((I-P)^{-1}) = \{1/(1-\alpha) : \alpha \in \sigma(P)\}$ [6]. Let $\beta \in \sigma((I-P)^{-1})$. Then $0 \ne \beta = 1/(1-\alpha)$ for some $\alpha \in \sigma(P)$. So $\alpha = 1-1/\beta$. By (4),

$$1-\frac{1}{|\beta|} \le \left|1-\frac{1}{\beta}\right| \le \lambda_1+\lambda_2\left|1-1+\frac{1}{\beta}\right| = \lambda_1+\lambda_2\cdot\frac{1}{|\beta|},$$

from which $1-\lambda_1 \le (1+\lambda_2)/|\beta|$. Hence $|\beta| \le (1+\lambda_2)/(1-\lambda_1)$. Since $\beta \in \sigma((I-P)^{-1})$ is arbitrary, we obtain the right side inequality of (6).

Finally, since $(I-P)^{-1}$ and P commute, the inequality (7) comes from

$$\begin{aligned} r((I-P)^{-1}-I) &= r((I-P)^{-1}(I-I+P)) = r((I-P)^{-1}P) \\ &\le r((I-P)^{-1})r(P) \le \frac{1+\lambda_2}{1-\lambda_1} r(P). \square \end{aligned}$$

We illustrate the above lemma for its usefulness with a simple example. Consider the 2×2

matrix

$$P = \begin{bmatrix} -1 & 1 \\ 0 & \frac{2}{3} \end{bmatrix}.$$

Then the matrix 1-norm $\|P\|_1 = 1 + 2/3 = 5/3$, the eigenvalues of P are -1 and $1/2$, so $r(P) = 1$. Thus, Banach's lemma cannot be used to show that $I - P$ is invertible. However, it is easy to see that with the choice of $\lambda_1 = \lambda_2 = 1/2$, the condition (4) of Lemma 2.1 is satisfied, so the bounds (5)-(7) in the lemma are $1/3 \le r(I-P) \le 3, 1/3 \le r((I-P)^{-1}) \le 3$, and $r((I-P)^{-1} - I) \le 3$, respectively.

Remark 2.1 If $\lambda_2 = 0$ in Lemma 2.1, then the condition (4) is nothing but $r(P) \le \lambda_1 < 1$, and this special case gives exactly the classic Banach lemma.

Remark 2.2 If the condition (4) is replaced with the weaker one $r(P) \le \lambda_1 + \lambda_2 r(I-P)$, then (5) is still true. However, $(I-P)^{-1}$ may not exist. For example, let P be the orthogonal projection from two-dimensional complex space $\mathbb{C}^2$ onto a one-dimensional subspace. Then $r(P) = r(I-P) = 1$, so any choice of $\lambda_1, \lambda_2 < 1$ with $\lambda_1 + \lambda_2 \ge 1$ satisfies the above weakened inequality, and $I - P$ is singular.

§3 Perturbation of bounded linear operators

With the help of the generalized Banach lemma in the previous section, we are able to study the perturbation problem for inverses or generalized inverses of bounded linear operators of Banach spaces. First, we consider the perturbation of inverses for bounded invertible operators.

Theorem 3.1 Let $T \in B(X)$ be such that T^{-1} exists with $T^{-1} \in B(X)$, and let $A \in B(X)$ commute with T. Suppose there exist two constants $\lambda_1 < 1$ and $\lambda_2 < 1$ such that for all $\alpha \in \sigma(T^{-1}A)$,

$$|\alpha| \le \lambda_1 + \lambda_2 |1 + \alpha|. \tag{8}$$

Then $(T+A)^{-1}$ exists with $(T+A)^{-1} \in B(X)$. Moreover,

$$r((T+A)^{-1}) \le \frac{1+\lambda_2}{1-\lambda_1} r(T^{-1}) \tag{9}$$

and

$$r((T+A)^{-1} - T^{-1}) \le \frac{1+\lambda_2}{1-\lambda_1} r(T^{-1})^2 r(A). \tag{10}$$

Proof. From Lemma 2.1, the condition (8) implies that $(I + T^{-1}A)^{-1}$ exists and belongs to $B(X)$. Now $(T+A)^{-1} = (T(I+T^{-1}A))^{-1} = (I+T^{-1}A)^{-1}T^{-1}$. Since T and A commute, $T(I+T^{-1}A)^{-1} = (I+T^{-1}A)^{-1}T$. So by the multiplication inequality in (3) and formulas (6) and (7),

$$r((T+A)^{-1}) \le r((I+T^{-1}A)^{-1}) r(T^{-1}) \le \frac{1+\lambda_2}{1-\lambda_1} r(T^{-1}),$$

$$r((T+A)^{-1} - T^{-1}) \le r(((I+T^{-1}A)^{-1} - I)T^{-1})$$

$$\le \frac{1+\lambda_2}{1-\lambda_1} r(T^{-1}A) r(T^{-1}) \le \frac{1+\lambda_2}{1-\lambda_1} r(T^{-1})^2 r(A). \square$$

We give a direct application of the above theorem to the frame theory [2]. A frame in a separable Hilbert space H with inner product $\langle\cdot,\cdot\rangle$ is a sequence $\{x_n\}$ of elements in H such that

$$c\|x\|^2 \le \sum_{n=1}^{\infty} |\langle x, x_n\rangle|^2 \le d\|x\|^2,\ \forall\, x \in H$$

with two positive constants c and d. A particular frame of H is an orthonormal basis $\{e_n\}$ of H, for which Parseval's identity

$$\|x\|^2 = \sum_{n=1}^{\infty} |\langle x, e_n\rangle|^2,\ \forall\, f \in H$$

is satisfied. Given any frame $\{x_n\}$ of H, it is well known [2] that the frame operator $T : H \to H$ defined by

$$Tx = \sum_{n=1}^{\infty} \langle x, x_n\rangle x_n$$

is invertible. Then, Theorem 3.1 immediately implies the following result.

Corollary 3.2 Let T be the frame operator with respect to a frame $\{x_n\}$ of the Hilbert space H. Suppose that $A \in B(H)$ commutes with T and there exist two constants $\lambda_1 < 1$ and $\lambda_2 < 1$ such that for all $\alpha \in \sigma(T^{-1}A)$, inequality (8) is satisfied. Then $T + A$ is invertible and inequalities (9) and (10) are valid.

We extend Theorem 3.1 to generalized inverses. First we recall the concept of generalized inverses for linear operators from a Banach space into itself. Let $T \in B(X)$ be such that its range $R(T)$ is closed. We assume that $R(T)$ and the null space $N(T)$ are complemented by closed subspaces $R(T)^c$ and $N(T)^c$, respectively. In other words, $X = R(T) \oplus R(T)^c = N(T) \oplus N(T)^c$. Then the restriction of the operator $T : N(T)^c \to R(T)$ is one-to-one and onto. We can define the generalized inverse $T^+ \in B(X)$ of T with respect to $R(T)^c$ and $N(T)^c$ as follows: $T^+y = x$ for all $y \in R(T)$, where $x \in N(T)^c$ satisfies $Tx = y$, and $T^+y = 0$ for all $y \in R(T)^c$; and extend T^+ to X linearly via the topological decomposition of $X = R(T) \oplus R(T)^c$. The generalized inverse T^+ is characterized by the two equalities $TT^+T = T$ and $T^+TT^+ = T^+$ and the property that the operators T^+T and TT^+ are the projections from X onto $N(T)^c$ and $R(T)$ along $N(T)$ and $R(T)^c$, respectively [1].

It is well-known that the inverse operation is continuous for all $T \in B(X)$ such that $T^{-1} \in B(X)$, namely the perturbation of T^{-1} is small when a perturbation of T is small in terms of the operator norm. Theorem 3.1 shows that when the perturbation commutes with T, the continuity is still true under the spectral radius. However, the operation of generalized inverses is not continuous in either the operator norm or spectral radius, so the perturbation should satisfy some kind of stability condition. A classic one is the so-called Nashed condition [7]: The operator $(I + AT^+)^{-1}(T + A)$ maps $N(T)$ to $R(T)$. This condition has been shown to be equivalent to the simpler one $R((I + AT^+)^{-1}(T + A)) \subset R(T)$ in [5]. Another stability condition that $R(T + A) \cap N(T^+) = \{0\}$ was first obtained in [3], and such a perturbation A is said to be stable for T. It was shown [5] that A satisfies Nashed's condition if and only if it is a stable perturbation for T.

Theorem 3.3 Let $T \in B(X)$ and $R(T)$ is closed, let T^+ exist with respect to $R(T)^c$ and $N(T)^c$, and let $A \in B(X)$ commute with T^+. Suppose there exist constants $\lambda_1 < 1$ and $\lambda_2 < 1$

such that (8) is valid for all $\alpha \in \sigma(T^+A)$. If A is a stable perturbation for T, then $(T+A)^+$ exists with $(T+A)^+ \in B(X)$. Moreover, $(T+A)^+ = (I+T^+A)^{-1}T^+ = T^+(I+AT^+)^{-1}$ and the following upper bounds hold:

$$r((T+A)^+) \leq \frac{1+\lambda_2}{1-\lambda_1} r(T^+); \tag{11}$$

$$r((T+A)^+ - T^+) \leq \frac{1+\lambda_2}{1-\lambda_1} r(T^+)^2 r(A). \tag{12}$$

Proof. The condition (8) guarantees that $(I+T^+A)^{-1}$ exists and belongs to $B(X)$. Since $T^+A = AT^+$, we see that $(I+T^+A)^{-1}T^+ = T^+(I+AT^+)^{-1}$.

Denote $S = T^+(I+AT^+)^{-1} = (I+T^+A)^{-1}T^+$. Then

$$\begin{aligned} & T+A-(T+A)S(T+A) \\ = \; & T+A-(T+A)T^+(I+AT^+)^{-1}(T+A) \\ = \; & [I-(T+A)T^+(I+AT^+)^{-1}](T+A) \\ = \; & [(I+AT^+)-(T+A)T^+](I+AT^+)^{-1}(T+A) \\ = \; & (I-TT^+)(I+AT^+)^{-1}(T+A) = 0, \end{aligned}$$

where the last equality comes from $R((I+AT^+)^{-1}(T+A)) \subset R(T)$ since A satisfies Nashed's condition. Thus, $T+A = (T+A)S(T+A)$. Since

$$\begin{aligned} & S-S(T+A)S \\ = \; & S(I-(T+A)S) \\ = \; & S[I-(T+A)T^+(I+AT^+)^{-1}] \\ = \; & S[I+AT^+-(T+A)T^+](I+AT^+)^{-1} \\ = \; & S(I-TT^+)(I+AT^+)^{-1} = 0, \end{aligned}$$

where the last equality follows from the fact that $N(S) = N(T^+) = R(I-TT^+)$. So $S = S(T+A)S$. Therefore, $(T+A)^+ = S$ is the generalized inverse of $T+A$ with respect to $R(T+A)^c = N((T+A)^+) = N(T^+)$ and $N(T+A)^c = R((T+A)^+) = R(T^+)$. The proof of (11) and (12) is exactly the same as that for (9) and (10) with T^+ in place of T^{-1}. □

Remark 3.1 If one adds the condition that $N(T+A) = N(T)$, then the stable perturbation assumption in Theorem 3.2 is redundant since $T+A = (I+AT^+)T = T(I+T^+A)$ from which $(T+A)^+ = T^+(I+AT^+)^{-1} = (I+T^+A)^{-1}T^+$. In addition, if T is onto, then $T+A = (I+AT^+)T$ is onto and so $R(T+A) = R(T) = X$.

§4 Conclusions

Using a simple algebraic inequality for the spectrum of a bounded linear operator P on a Banach space, under a weakened assumption, we have generalized the classic Banach lemma and one of its extensions in terms of spectral points. Such a new approach to the invertibility of the operator $I-P$ made it possible to improve some well-known perturbation results for inverses or generalized inverses of bounded linear operators when the perturbation commutes with the generalized inverse of the unperturbed operator. The obtained new upper bounds can be applied to error estimates for least squares solutions of linear operator equations with weaker assumptions.

Declarations

Conflict of interest The authors declare no conflict of interest.

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