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On the global smooth solutions of 3D incompressible Hall-MHD equations

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Abstract. The present paper is devoted to the well-posedness issue for the 3D incompressible Hall-MHD system obtained from kinetic models. Our analysis strongly relies on the use of the Fourier analysis. We establish the global existence of smooth solutions for a class of large initial data, this result implies the initial velocity and magnetic field can be arbitrarily large.

§1 Introduction

In this paper, we study the global well-posedness of the following three-dimensional incompressible two-fluid magnetohydrodynamics system:

$$\begin{cases} \partial_t u + (u \cdot \nabla)u + \nabla p - \nu \Delta u = (\nabla \times B) \times B, \quad (t,x) \in \mathbb{R}^+ \times \mathbb{R}^3, \\ \partial_t B - \nabla \times (u \times B) + \nabla \times ((\nabla \times B) \times B) - \mu \Delta B = 0, \\ \nabla \cdot u = 0, \quad \nabla \cdot B = 0, \\ u|_{t=0} = u_0, \ B|_{t=0} = B_0, \end{cases} \tag{1.1}$$

where $u = u(t,x)$ and $B = B(t,x)$ are the fluid velocity and magnetic field, depending on the spatial position x and the time t. The scalar functions $p = p(t,x)$ denote the pressure. The positive constants ν and μ are the viscosity and the resistivity coefficients respectively.

Using vector identity, we can rewrite (1.1) as follows:

$$\begin{cases} \partial_t u + (u \cdot \nabla)u + \nabla \left(p + \frac{|B|^2}{2}\right) - \nu \Delta u = (B \cdot \nabla)B, \quad (t,x) \in \mathbb{R}^+ \times \mathbb{R}^3, \\ \partial_t B + (u \cdot \nabla)B + \nabla \times ((\nabla \times B) \times B) - \mu \Delta B = (B \cdot \nabla)u, \\ \nabla \cdot u = 0, \quad \nabla \cdot B = 0, \\ u|_{t=0} = u_0, \ B|_{t=0} = B_0. \end{cases} \tag{1.2}$$

Systems of this type can be derived from either two fluids models or kinetic models. Compared with the usual viscous incompressible MHD system, the Hall-MHD equations have the Hall term

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$\nabla \times ((\nabla \times B) \times B)$, which plays an important role in magnetic reconnection when the magnetic shear is large. We refer to [10,17] for the physical background of the magnetic reconnection and the Hall-MHD. Note that the Hall term breaks the scaling invariant of such a system, which is different from that of the classical MHD system.

There have been a lot of studies on Hall-MHD by physicists and mathematicians because of their prominent roles in modeling many phenomena in e.g., space plasmas, star formation, neutron stars and geo-dynamo, see [3,11,15,18,19] and the references cited therein. The global existence of weak solutions in the periodic domain is done in [1] by a Galerkin approximation. The global existence in the whole domain in $\mathbb{R}^3$ as well as the local well-posedness of smooth solution is proved in [4], where the global existence of smooth solution for small initial data is also established. Chae and Lee [5] proved an optimal blow-up criterion for classical solutions to the incompressible resistive Hall-magnetohydrodynamic equations and established two global-in-time existence results of the classical solutions for small initial data, the smallness conditions of which are given by the suitable Sobolev and the Besov norms, respectively. Chae, Wan and Wu proved in [8] the local wellposedness of classical solutions to the Hall-MHD equations with the magnetic diffusion given by a fractional Laplacian operator, $(-\Delta)^{\alpha}$. Temporal decay for the weak solution and smooth solution with small data to Hall-MHD are also established in [6]. Dai [9] studied the regularity problem for the 3D incompressible resistive viscous Hall-magneto-hydrodynamic system by splitting the wavenumber combined with an estimate of the energy flux. Very recently, Chae and Weng [7] proved that the incompressible Hall-MHD system without resistivity is not globally in time well-posed in any Sobolev space $H^m(\mathbb{R}^3), m > \frac{7}{2}$.

We should mention that the recent work of Lei-Lin [13] and Lin et al. [14] will play a crucial role in our work. This method has been used in the study of the global well-posedness to 3-D incompressible Navier-Stokes equations and MHD equations. Throughout this paper, C represents some "harmless" constant, which can be understood from the context.

The aim of this paper is to establish global smooth solutions to the Cauchy's problem of Hall-MHD system (1.1) with a class of large initial data in three space dimensions. Our result generalizes that of Kwak et al. [12] for the 3D incompressible Hall-MHD equations to the case of such equations with large velocity fields and magnetic fields. We first construct a stability result for the 3D Hall-MHD system which generalizes a small data global well-posedness result of Kwak and Lkhagvasuren [12] for this equation. We can now state the main result of this paper.

Theorem 1.1. *Consider the Cauchy problem* (1.2). *Suppose that*

$$u_0(x) = u_{01}(x) + u_{02}(x), \tag{1.3}$$

$$B_0(x) = B_{01}(x) + B_{02}(x), \tag{1.4}$$

with

$$\nabla \cdot u_{01}(x) = \nabla \cdot B_{01}(x) = 0, \tag{1.5}$$

$$u_{02}(x) = \alpha_1 v_0(x), \tag{1.6}$$

$$B_{02}(x) = \alpha_2 v_0(x), \tag{1.7}$$

where α_1, α_2 are two real constants and $v_0(x)$ has the following properties

$$\begin{aligned}
&(i) \quad \nabla \cdot v_0(x) = 0,\\
&(ii) \quad \nabla \times v_0(x) = \sqrt{-\Delta} v_0(x),\\
&(iii) \quad supp\hat{v_0}(\xi) \subset \{\xi : 1-\delta \le |\xi| \le 1+\delta\}, \quad 0 \le \delta \le \frac{1}{2},\\
&(iv) \quad |\hat{v_0}|_{L^1(\mathbb{R}^3)} \le M,
\end{aligned}$$

where $\hat{v_0}$ denotes the Fourier transform for v_0.

Then there exists a positive constant ε sufficiently small, and depending only on $\nu, \mu, \alpha_1, \alpha_2$ and M such that the Cauchy problem (1.2) *has a global smooth solution provided that*

$$\delta \le \varepsilon \tag{1.8}$$

and

$$\int_{\mathbb{R}^3} (1 + \frac{1}{|\xi|})|\hat{u_{01}}(\xi)|d\xi + \int_{\mathbb{R}^3} (1 + \frac{1}{|\xi|})|\hat{B_{01}}(\xi)|d\xi \le \varepsilon. \tag{1.9}$$

§2 The proof of Theorem 1.1

In order to obtain the existence of a solution of (1.1), we shall seek a solution of the form

$$u = u_F + w \quad \text{and} \quad B = B_F + b,$$

where (w, b) and (u_F, B_F) solve (2.1)-(2.2) and (2.3)-(2.4), respectively.

By substituting the above formula into (1.2), we find that w, B must satisfy

$$\begin{cases}
\partial_t w + (w \cdot \nabla)w + \nabla \left(p + \frac{|B|^2}{2}\right) - \nu\Delta w + (w \cdot \nabla)u_F + (u_F \cdot \nabla)w \\
\quad = -(u_F \cdot \nabla)u_F + (B_F \cdot \nabla)B_F + (b \cdot \nabla)b + (B_F \cdot \nabla)b + (b \cdot \nabla)B_F,\\
\nabla \cdot w = 0,\\
w|_{t=0} = u_{01} \overset{def}{=} u_0 - u_{02},
\end{cases} \tag{2.1}$$

and

$$\begin{cases}
\partial_t b + (u_F \cdot \nabla)B_F + (u_F \cdot \nabla)b + (w \cdot \nabla)B_F + (w \cdot \nabla)b - \mu\Delta b \\
\quad +\nabla \times ((\nabla \times B_F) \times b) + \nabla \times ((\nabla \times B_F) \times B_F)\\
\quad +\nabla \times ((\nabla \times b) \times B_F) + \nabla \times ((\nabla \times b) \times b)\\
\quad = (B_F \cdot \nabla)u_F + (B_F \cdot \nabla)w + (b \cdot \nabla)u_F + (b \cdot \nabla)w,\\
\nabla \cdot b = 0,\\
b|_{t=0} = B_{01} \overset{def}{=} B_0 - B_{02},
\end{cases} \tag{2.2}$$

and u_F, B_F must satisfy

$$\begin{cases}
\partial_t u_F - \nu\Delta u_F = 0,\\
\nabla \cdot u_F = 0,\\
u_F(0) = u_{02},
\end{cases} \tag{2.3}$$

and

$$\begin{cases}\partial_t B_F - \mu\Delta B_F = 0,\\ B_F(0) = B_{02}.\end{cases} \tag{2.4}$$

Since $\nabla\cdot w = 0$, by using the vector identity $a\cdot\nabla a = (\nabla\times a)\times a + \frac{1}{2}\nabla|a|^2$, we can rewrite (2.1) by projection it onto the divergence-free space. Then, we obtain

$$\begin{cases}\partial_t w - \nu\Delta w + \mathbb{P}\big[(w\cdot\nabla)w + (w\cdot\nabla)u_F + (u_F\cdot\nabla)w\\ \quad -(b\cdot\nabla)b - (B_F\cdot\nabla)b - (b\cdot\nabla)B_F\big] = \mathbb{P}\big[(\nabla\times B_F)\times B_F - (\nabla\times u_F)\times u_F\big],\\ \nabla\cdot w = 0,\\ w|_{t=0} = u_{01} \overset{def}{=} u_0 - u_{02},\end{cases} \tag{2.5}$$

and

$$\begin{cases}\partial_t b + (u_F\cdot\nabla)b + (w\cdot\nabla)B_F + (w\cdot\nabla)b - \mu\Delta b\\ \quad +\nabla\times((\nabla\times B_F)\times b) + \nabla\times((\nabla\times b)\times B_F) + \nabla\times((\nabla\times b)\times b)\\ \quad -(B_F\cdot\nabla)w - (b\cdot\nabla)u_F - (b\cdot\nabla)w = -\nabla\times(B_F\times u_F) - \nabla\times((\nabla\times B_F)\times B_F),\\ \nabla\cdot b = 0,\\ b|_{t=0} = B_{01} \overset{def}{=} B_0 - B_{02}.\end{cases} \tag{2.6}$$

We first prove a stability result which generalizes a theorem of Kwak and Lkhagvasuren [12].

Theorem 2.1. *Consider the system of equations* (2.5)-(2.6)*, suppose that* u_F *satisfies* (2.3) *and* B_F *satisfies* (2.4) *and suppose that*

$$\int_{\mathbb{R}^3}(1+\frac{1}{|\xi|^2})|\hat{u_{02}}(\xi)|d\xi + \int_{\mathbb{R}^3}(1+\frac{1}{|\xi|^2})|\hat{B_{02}}(\xi)|d\xi \le 4M. \tag{2.7}$$

Then there exists a small positive constant δ_0 *depending only on* ν, μ *and* M *such that* (2.5)-(2.6) *have a global smooth solution satisfying*

$$\begin{aligned}&\int_{\mathbb{R}^3}(1+\frac{1}{|\xi|})|\hat{w}(t,\xi)|d\xi + \nu\int_0^t\int_{\mathbb{R}^3}|\xi|(1+|\xi|)|\hat{w}(t,\xi)|d\xi d\tau\\ &\quad + \int_{\mathbb{R}^3}(1+\frac{1}{|\xi|})|\hat{b}(t,\xi)|d\xi + \mu\int_0^t\int_{\mathbb{R}^3}|\xi|(1+|\xi|)|\hat{b}(t,\xi)|d\xi d\tau \le C_*\delta_0,\end{aligned} \tag{2.8}$$

provided that

$$\begin{aligned}&\int_{\mathbb{R}^3}(1+\frac{1}{|\xi|})|\hat{u_{01}}(t,\xi)|d\xi + \int_{\mathbb{R}^3}(1+\frac{1}{|\xi|})|\hat{B_{01}}(t,\xi)|d\xi\\ &\quad + \int_0^t\int_{\mathbb{R}^3}(1+\frac{1}{|\xi|})|\hat{F}(t,\xi)|d\xi d\tau + \int_0^t\int_{\mathbb{R}^3}(1+\frac{1}{|\xi|})|\hat{G}(t,\xi)|d\xi d\tau \le \delta_0,\end{aligned} \tag{2.9}$$

where $F = (\nabla\times B_F)\times B_F - (\nabla\times u_F)\times u_F$ *and* $G = -\nabla\times(u_F\times B_F) - \nabla\times((\nabla\times B_F)\times B_F)$.

Proof. Let

$$E_0(t) = \int_{\mathbb{R}^3}(1+\frac{1}{|\xi|})|\hat{w}(t,\xi)|d\xi + \int_{\mathbb{R}^3}(1+\frac{1}{|\xi|})|\hat{b}(t,\xi)|d\xi, \tag{2.10}$$

$$E_1(t) = \nu\int_0^t\int_{\mathbb{R}^3}|\xi|(1+|\xi|)|\hat{w}(t,\xi)|d\xi d\tau + \mu\int_0^t\int_{\mathbb{R}^3}|\xi|(1+|\xi|)|\hat{b}(t,\xi)|d\xi d\tau. \tag{2.11}$$

Taking the Fourier transform of the first equation of (2.5)-(2.6), we get

$$\partial_t \hat{w} + \nu|\xi|^2 \hat{w} = \hat{H} + \hat{\mathbb{P}F} \tag{2.12}$$

and

$$\partial_t \hat{b} + \mu|\xi|^2 \hat{b} = \hat{I} + \hat{\mathbb{P}G}, \tag{2.13}$$

where

$$\begin{aligned} H =& -\mathbb{P}\big[(w\cdot\nabla)w + (w\cdot\nabla)u_F + (u_F\cdot\nabla)w - (b\cdot\nabla)b - (B_F\cdot\nabla)b - (b\cdot\nabla)B_F\big],\\ I =& -(u_F\cdot\nabla)b - (w\cdot\nabla)B_F - (w\cdot\nabla)b - \nabla\times((\nabla\times B_F)\times b)\\ &-\nabla\times((\nabla\times b)\times B_F) - \nabla\times((\nabla\times b)\times b) + (B_F\cdot\nabla)w + (b\cdot\nabla)u_F + (b\cdot\nabla)w. \end{aligned}$$

Taking an inner product of (2.12), (2.13) by $\frac{\bar{\hat{w}}}{|\hat{w}|}(1+\frac{1}{|\xi|})$, $\frac{\bar{\hat{b}}}{|\hat{b}|}(1+\frac{1}{|\xi|})$ respectively, we obtain

$$(1+\frac{1}{|\xi|})\partial_t\hat{w}\cdot\frac{\bar{\hat{w}}}{|\hat{w}|} + \nu|\xi|(1+|\xi|)|\hat{w}| = (1+\frac{1}{|\xi|})\hat{H}\cdot\frac{\bar{\hat{w}}}{|\hat{w}|} + (1+\frac{1}{|\xi|})\hat{\mathbb{P}F}\cdot\frac{\bar{\hat{w}}}{|\hat{w}|}, \tag{2.14}$$

and

$$(1+\frac{1}{|\xi|})\partial_t\hat{b}\cdot\frac{\bar{\hat{b}}}{|\hat{b}|} + \nu|\xi|(1+|\xi|)|\hat{b}| = (1+\frac{1}{|\xi|})\hat{I}\cdot\frac{\bar{\hat{b}}}{|\hat{b}|} + (1+\frac{1}{|\xi|})\hat{\mathbb{P}G}\cdot\frac{\bar{\hat{b}}}{|\hat{b}|}, \tag{2.15}$$

Note $\Re(\partial_t\hat{w}\cdot\frac{\bar{\hat{w}}}{|\hat{w}|}) = \partial_t(|\hat{w}|)$, $\Re(\partial_t\hat{b}\cdot\frac{\bar{\hat{b}}}{|\hat{b}|}) = \partial_t(|\hat{b}|)$, taking the real part of (2.14) and (2.15), we get

$$\partial_t[(1+\frac{1}{|\xi|})|\hat{w}|] + \nu|\xi|(1+|\xi|)|\hat{w}| = \Re\big[(1+\frac{1}{|\xi|})\hat{H}\cdot\frac{\bar{\hat{w}}}{|\hat{w}|} + (1+\frac{1}{|\xi|})\hat{\mathbb{P}F}\cdot\frac{\bar{\hat{w}}}{|\hat{w}|}\big], \tag{2.16}$$

and

$$\partial_t[(1+\frac{1}{|\xi|})|\hat{b}|] + \mu|\xi|(1+|\xi|)|\hat{b}| = \Re\big[(1+\frac{1}{|\xi|})\hat{I}\cdot\frac{\bar{\hat{b}}}{|\hat{b}|} + (1+\frac{1}{|\xi|})\hat{\mathbb{P}G}\cdot\frac{\bar{\hat{b}}}{|\hat{b}|}\big], \tag{2.17}$$

Using the divergence free condition, we have

$$\begin{aligned} (1+\frac{1}{|\xi|})|\hat{\mathbb{P}F}| &\le (1+\frac{1}{|\xi|})|\hat{F}|,\\ (1+\frac{1}{|\xi|})|\hat{\mathbb{P}G}| &\le (1+\frac{1}{|\xi|})|\hat{G}|,\\ (1+\frac{1}{|\xi|})|\hat{H}| &\le (1+|\xi|)\big[|\hat{w}*\hat{w}| + 2|\hat{w}*\hat{u_F}| + |\hat{b}*\hat{b}| + 2|\hat{b}*\hat{B_F}|\big],\\ (1+\frac{1}{|\xi|})|\hat{I}| &\le 2(1+|\xi|)\big[|\hat{b}*\hat{u_F}| + |\hat{w}*\hat{B_F}| + |\hat{w}*\hat{b}|\big]\\ &\quad + |\xi|(1+|\xi|)\big[2|\hat{b}*\hat{B_F}| + |\hat{b}*\hat{b}|\big]. \end{aligned}$$

Adding (2.16) and (2.17), and then integrating on $[0,t]\times\mathbb{R}^3$, according to (2.9), we have

$$\begin{aligned} &E_0(t)+E_1(t)\\ \le&\delta_0 + C\int_0^t\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}(1+|\xi|)^2(|\hat{w}(\tau,\xi-\eta)| + |\hat{b}(\tau,\xi-\eta)|)\cdot(|\hat{w}(\tau,\eta)| + |\hat{b}(\tau,\eta)|)d\xi d\eta d\tau\\ &+C\int_0^t\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}(1+|\xi|)^2(|\hat{w}(\tau,\xi-\eta)| + |\hat{b}(\tau,\xi-\eta)|)\cdot(|\hat{u_F}(\tau,\eta)| + |\hat{B_F}(\tau,\eta)|)d\xi d\eta d\tau\\ \overset{def}{=}&\delta_0 + I + II, \end{aligned} \tag{2.18}$$

where

$$I = C\int_0^t \int_{\mathbb{R}^3}\int_{\mathbb{R}^3}(1+|\xi|)^2(|\hat{w}(\tau,\xi-\eta)|+|\hat{b}(\tau,\xi-\eta)|)\cdot(|\hat{w}(\tau,\eta)|+|\hat{b}(\tau,\eta)|)d\xi d\eta d\tau,$$

$$II = C\int_0^t \int_{\mathbb{R}^3}\int_{\mathbb{R}^3}(1+|\xi|)^2(|\hat{w}(\tau,\xi-\eta)|+|\hat{b}(\tau,\xi-\eta)|)\cdot(|\hat{u_F}(\tau,\eta)|+|\hat{B_F}(\tau,\eta)|)d\xi d\eta d\tau.$$

Since

$$1 \le \frac{|\xi-\eta|}{|\eta|}+\frac{|\eta|}{|\xi-\eta|},$$
$$|\xi| \le |\eta|+|\xi-\eta|,$$
$$|\xi|^2 \le 2(|\eta|^2+|\xi-\eta|^2),$$

we get

$$I \le C\big(\sup_t E_0(t)\big)E_1(t). \tag{2.19}$$

To estimate II, we divide into the cases $|\xi-\eta| \le L$ and $|\xi-\eta| \ge L$, then

$$\begin{aligned} II \le & CL\int_0^t \Big(\int_{\mathbb{R}^3}(|\hat{u_F}(\tau,\eta)|+|\hat{B_F}(\tau,\eta)|)d\eta\Big)E_0(\tau)d\tau \\ & + \frac{C}{L}\Big(\sup_\tau \int_{\mathbb{R}^3}(|\hat{u_F}(\tau,\eta)|+|\hat{B_F}(\tau,\eta)|)d\eta\Big)E_1(t). \end{aligned} \tag{2.20}$$

By (2.7), we obtain

$$\sup_\tau \int_{\mathbb{R}^3}(|\hat{u_F}(\tau,\eta)|+|\hat{B_F}(\tau,\eta)|)d\eta \le 4M.$$

Plugging the estimates (2.19)-(2.20) into (2.18) gives

$$\begin{aligned} E_0(t)+E_1(t) \le & \delta_0 + C(\sup_t E_0(t))E_1(t)+\frac{CM}{L}E_1(t) \\ & + CL\int_0^t \Big(\int_{\mathbb{R}^3}(|\hat{u_F}(\tau,\eta)|+|\hat{B_F}(\tau,\eta)|)d\eta\Big)E_0(\tau)d\tau. \end{aligned}$$

We prove (2.8) by induction. We assume that

$$E_0(t)+E_1(t) \le 2C_*\delta_0,$$

then

$$C\big(\sup_t E_0(t)\big)E_1(t) \le 4CC_*^2\delta_0^2 \le \delta_0,$$

provided that δ_0 is sufficiently small.

We choose L such that $L = 2CM$, then we finally obtain

$$E_0(t)+\frac{1}{2}E_1(t) \le 2\delta_0 + CM\int_0^t \Big(\int_{\mathbb{R}^3}(|\hat{u_F}(\tau,\eta)|+|\hat{B_F}(\tau,\eta)|)d\eta\Big)E_0(\tau)d\tau.$$

Applying the Gronwall's inequality to the above inequality, one has

$$\begin{aligned} E_0(t)+E_1(t) \le & 8\delta_0 \exp\Big(CM\int_0^t \int_{\mathbb{R}^3}(|\hat{u_F}(\tau,\eta)|+|\hat{B_F}(\tau,\eta)|)d\eta d\tau\Big) \\ \le & 8\delta_0\exp(CM^2). \end{aligned}$$

Obviously, we complete the proof of Theorem 2.1 if we let $C_* = 8\exp(CM^2)$.

Now we prove Theorem 1.1. Since Theorem 2.1, we only need to estimate

$$\int_0^\infty \int_{\mathbb{R}^3} (1+\frac{1}{|\xi|})|\hat{F}(t,\xi)|d\xi dt + \int_0^\infty \int_{\mathbb{R}^3} (1+\frac{1}{|\xi|})|\hat{G}(t,\xi)|d\xi dt.$$

Note that

$$\begin{aligned} F =&(\nabla \times B_F) \times B_F - (\nabla \times u_F) \times u_F \\ =&(\sqrt{-\Delta} B_F) \times B_F - (\sqrt{-\Delta} u_F) \times u_F \\ =&(\sqrt{-\Delta} B_F - B_F) \times B_F - (\sqrt{-\Delta} u_F - u_F) \times u_F, \end{aligned}$$

and

$$\begin{aligned} G =& -\nabla \times (u_F \times B_F) - \nabla \times ((\nabla \times B_F) \times B_F) \\ =& -\nabla \times (u_F \times B_F) - \nabla \times \Big((\sqrt{-\Delta} B_F - B_F) \times B_F \Big), \end{aligned}$$

and the Fourier transform is supported on $|\xi| \geq \frac{1}{2}$, we have

$$\begin{aligned} &\int_0^\infty \int_{\mathbb{R}^3} (1+\frac{1}{|\xi|})|\hat{F}(t,\xi)|d\xi dt \\ &\leq 3 \int_0^\infty \int_{\mathbb{R}^3} |\hat{F}(t,\xi)|d\xi dt \\ &\leq 3 \int_0^\infty \big(|\hat{B_F}(t)|_{L^1} |(|\xi|-1)\hat{B_F}(t)|_{L^1} + |\hat{u_F}(t)|_{L^1} |(|\xi|-1)\hat{u_F}(t)|_{L^1} \big) dt \\ &\leq 3\delta \sup_t |\hat{B_F}(t)|_{L^1} \int_0^\infty |\hat{B_F}(t)|_{L^1} dt + 3\delta \sup_t |\hat{u_F}(t)|_{L^1} \int_0^\infty |\hat{u_F}(t)|_{L^1} dt \\ &\leq CM^2\delta. \end{aligned}$$

On the other hand, if $\mu = \nu$, then $u_F \times B_F = 0$. We need only to consider the case $\mu < \nu$, the case $\mu > \nu$ can be proved in the same manner.

$$\begin{aligned} & u_F \hat{\times} B_F(t,\xi) \\ =& \alpha_1\alpha_2 \int_{\mathbb{R}^3} e^{-\nu|\xi-\eta|^2 t - \mu|\eta|^2 t} \hat{v_0}(\xi-\eta) \times \hat{v_0}(\eta) d\eta \\ =& \alpha_1\alpha_2 \int_{\mathbb{R}^3} e^{-\nu|\eta|^2 t - \mu|\xi-\eta|^2 t} \hat{v_0}(\eta) \times \hat{v_0}(\xi-\eta) d\eta \\ =& \frac{1}{2}\alpha_1\alpha_2 \int_{\mathbb{R}^3} \big(e^{-\nu|\xi-\eta|^2 t - \mu|\eta|^2 t} - e^{-\nu|\eta|^2 t - \mu|\xi-\eta|^2 t} \big) \hat{v_0}(\xi-\eta) \times \hat{v_0}(\eta) d\eta. \end{aligned}$$

Since in the support of $\hat{v_0}(\xi-\eta) \times \hat{v_0}(\eta)$, we have

$$\frac{\big||\xi-\eta|^2 - |\eta|^2\big|}{|\xi-\eta|^2 + |\eta|^2} \leq 10\delta.$$

So, we have the following estimate

$$\begin{aligned} & \big| e^{-\nu|\xi-\eta|^2 t - \mu|\eta|^2 t} - e^{-\nu|\eta|^2 t - \mu|\xi-\eta|^2 t} \big| \\ &= e^{-\mu(|\xi-\eta|^2+|\eta|^2)t} \big| e^{-(\nu-\mu)|\xi-\eta|^2 t} - e^{-(\nu-\mu)|\eta|^2 t} \big| \\ &\leq C e^{-\mu(|\xi-\eta|^2+|\eta|^2)t} \big||\xi-\eta|^2 - |\eta|^2\big| t \\ &\leq C e^{-\frac{\mu}{2}(|\xi-\eta|^2+|\eta|^2)t} \frac{\big||\xi-\eta|^2 - |\eta|^2\big|}{|\xi-\eta|^2 + |\eta|^2}. \end{aligned}$$

Finally, from the above estimates we arrive at

$$\begin{aligned}
&\int_0^\infty \int_{\mathbb{R}^3} (1+\frac{1}{|\xi|})|\hat{G}(t,\xi)|d\xi dt \\
\leq &3\int_0^\infty \int_{\mathbb{R}^3} \big[|u_F \hat{\times} B_F(t,\xi)| + |\mathcal{F}\big[(\sqrt{-\Delta}B_F - B_F)\times B_F\big](t,\xi)|\big]d\xi dt \\
\leq &CM\delta^2 + 3\int_0^\infty |\hat{B_F}(t)|_{L^1}|(|\xi|-1)\hat{B_F}(t)|_{L^1}dt \\
\leq &CM\delta^2 + 3\delta \sup_t |\hat{B_F}(t)|_{L^1}\int_0^\infty |\hat{B_F}(t)|_{L^1}dt \\
\leq &CM^2\delta.
\end{aligned}$$

Thus, this concludes the proof of Theorem 1.1. □

Declarations

Conflict of interest The authors declare no conflict of interest.

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