Trace formula of the integro-differential operator on a quantum graph

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Abstract. In this paper we study the eigenvalue problem for integro-differential operators on a lasso graph. The trace formula of the operator is established by applying the residual technique in complex analysis.

§1 Introduction

The theory of differential operators on quantum graphs is a rapidly developing area of modern mathematical physics. Such operators can be used to describe the motion of quantum particles confined to certain low dimensional structures. Spectral properties of differential operators in such structures, especially the integro-differential operators, have attracted considerable attention during past years. Various aspects of spectral problems for integro-differential operators were studied in [1,2,7-10,14,17,26,27]. In particular, the paper [17] considered the eigenvalue problem for integro-differential operators with separated boundary conditions on the finite interval and found a trace formula for the integro-differential operator. At the same time, nonlocal and, in particular, integro-differential operators are of great interest, because they have many applications (see [11]).

In this paper, we shall study the trace problem of integro-differential operators on a lasso graph. The theory of regularized traces of Sturm-Liouville operators stems from the paper [4] of Gelfand and Levitan. Later on, a number of authors turned their attention to trace theory and obtained interesting results [6,12,13,15,16,18-25]. Trace formulas for differential operators have many applications in inverse problem theory, the numerical calculation of eigenvalues, the theory of integrable system, etc.

Consider the lasso graph $G$, represented in Figure 1, the lengths of both the edge $e_1$ and the loop $e_2$ are equal to 1. The parameter $x$ is introduced on each edge $e_i$, $i = 1, 2$, where $x \in (0, 1)$. To be convenient, we set the orientations as follows: for the edge $e_1$, the value $x = 0$ and $x = 1$.
correspond to the boundary vertex and the internal vertex, respectively; for the loop $e_2$, both ends $x = 0$ and $x = 1$ correspond to the internal vertex.

![Figure 1. Lasso graph.](image)

Let $y(x) = \{y_i(x)\}_{i=1,2}$ be a vector function on the graph $G$, where $y_i(x)$ is a function on the edge $e_i$. Consider the integro-differential expressions

$$
l y(x) := \begin{pmatrix}
-y''_1(x) + q_1(x)y_1(x) + \int_0^x M_1(x,t)y_1(t)dt \\
-y''_2(x) + q_2(x)y_2(x) + \int_0^x M_2(x,t)y_2(t)dt
\end{pmatrix}
$$
on the edges of $G$, where $q_i(x)$ and $M_i(x,t)$ are real-valued functions, $q_i \in W^1_1[0,1]$, $M_i \in W^1_1(D)$, $D := \{(x,t): 0 \leq t \leq x \leq 1\}$.

We study the boundary value problem $L(q,M)$ for the differential equations on the graph $G$:

$$
l y(x) = \lambda^2 y(x), \quad x \in (0,1),
$$

with the matching conditions

$$
y_1(1) = y_2(0) = y_2(1), \quad y'_1(1) - y'_2(0) + y'_2(1) = 0
$$
in the internal vertex, and the Dirichlet boundary condition

$$
y_1(0) = 0
$$
in the boundary vertex. In this paper, it is mainly divided into two parts: (1) the asymptotic estimations of the eigenvalues of the operator $L(q,M)$; (2) its regularized trace formula. We point out that our results are an extension to those in [5]. In particular, when $M_i(x,t) = 0$, the trace formula is consistent with that of the Sturm-Liouville operator on a lasso-graph.

§2 Characteristic function

Firstly, we calculate the characteristic function of the operator, and then we can transform the eigenvalue problem into the roots of the entire function.

We shall use the following notation

$$
[q_i] = \frac{1}{2} \int_0^1 q_i(x)dx, \quad Q^+_i = \frac{q_i(1) + q_i(0)}{4}, \quad Q^-_i = \frac{q_i(1) - q_i(0)}{4},
$$
\[ \Delta_0(\lambda) = \frac{(3 \cos \lambda - 2) \sin \lambda}{\lambda}, \]

Then, we can obtain that the zeros of \( \Delta_0(\lambda) \) are as follows:

\[ \mu_n = n\pi, n \in \mathbb{Z} \setminus \{0\}, \quad \mu_n^\pm = 2n\pi \pm \arccos \frac{2}{3}, n \in \mathbb{Z}. \]

Denote

\[ \gamma_n = \{ \lambda : |\lambda| = (\mu_n + 1/2)^2 \} \cup \{ \lambda : |\lambda| = (\mu_n^\pm + 1/2)^2 \}. \]

We know that \(|\Delta_0(\lambda)| > |\Delta(\lambda) - \Delta_0(\lambda)|, \lambda \in \gamma_n, \) for sufficiently large \( n \). Then by Rouché's
the number of zeros of $\Delta(\lambda)$ inside $\gamma_n$ coincide with the number of zeros of $\Delta_0(\lambda)$.

Denote the contours, traversed counterclockwise:

$$C_n = \{ \lambda \in \mathbb{C} : |\lambda - \mu_n| = \delta \}, \quad D_n^\pm = \{ \lambda \in \mathbb{C} : |\lambda - \mu_n^\pm| = \delta \}$$

for $\delta > 0$ and the square contour $\Gamma_n$ with vertices ($i = \sqrt{-1}$)

$$A(\mu_n^+ + \epsilon + (\mu_n^+ + \epsilon)i), \quad B(-\mu_n^- - \epsilon + (\mu_n^+ + \epsilon)i),$$

$$C(-\mu_n^- - \epsilon - (\mu_n^+ + \epsilon)i), \quad D(\mu_n^+ + \epsilon - (\mu_n^+ + \epsilon)i).$$

Therefore, on contours $\Gamma_n$ or $C_n$, $D_n^\pm$, we have

$$\frac{\Delta(\lambda)}{\Delta_0(\lambda)} = 1 + \frac{1}{\lambda(3\cos\lambda - 2)} \frac{2 - 3\cos^2\lambda}{\lambda(3\cos\lambda - 2) \sin \lambda} \left[ q_1 \right] + \frac{2 - 3\cos^2\lambda}{\lambda(3\cos\lambda - 2) \sin \lambda} \left[ q_2 \right]$$

$$- \frac{3\cos \lambda}{\lambda^2(3\cos \lambda - 2)} \left[ q_1 \right] \left[ q_2 \right] - \frac{[q_1]^2 + M_1}{2\lambda^2} + \frac{2\cos \lambda - 2}{\lambda^2(3\cos \lambda - 2)} Q_1^+$$

$$- \frac{3\cos \lambda}{\lambda^2(3\cos \lambda - 2)} \left( \frac{[q_2]^2 + M_2}{2} + \frac{Q_1^- - Q_2^+}{3} \right) + o \left( \frac{1}{|\lambda|^2} \right).$$

By applying the Taylor series expansion and simplifying, we can get

$$\ln \frac{\Delta(\lambda)}{\Delta_0(\lambda)} = \frac{1}{\lambda(3\cos\lambda - 2) \sin \lambda} \left[ q_1 \right] + \frac{2 - 3\cos^2\lambda}{\lambda(3\cos\lambda - 2) \sin \lambda} \left[ q_2 \right] - \frac{[q_1]^2}{2\lambda^2}$$

$$- \frac{(1 + 2\cos \lambda - 3\cos^2 \lambda)^2}{2\lambda^2(3\cos \lambda - 2)^2 \sin^2 \lambda} \left[ q_1 \right]^2 - \frac{3\cos \lambda}{2\lambda^2(3\cos \lambda - 2)} \left[ q_1 \right] \left[ q_2 \right]$$

$$- \frac{(2 - 3\cos^2\lambda)^2}{2\lambda^2(3\cos\lambda - 2)^2 \sin^2 \lambda} \left[ q_2 \right]^2 - \frac{(2 - 3\cos^2 \lambda)(1 + 2\cos \lambda - 3\cos^2 \lambda)}{\lambda^2(3\cos \lambda - 2)^2 \sin^2 \lambda} \left[ q_1 \right] \left[ q_2 \right] - \frac{M_1}{2\lambda^2}$$

$$+ \frac{2\cos \lambda - 2}{\lambda^2(3\cos \lambda - 2)} Q_1^+ - \frac{3\cos \lambda}{\lambda^2(3\cos \lambda - 2)} \left( \frac{M_2}{2} + \frac{Q_1^- - Q_2^+}{3} \right) + o \left( \frac{1}{|\lambda|^2} \right).$$

### §3 Eigenvalue asymptotics

In this section, the asymptotic expression of eigenvalues of $L(q, M)$ is obtained by applying the Rouché’s theorem and the residue calculation.

By restoring to the integral identity, we can get the following formula

$$\lambda_n - \mu_n = - \frac{1}{2\pi i} \oint_{C_n} \ln \frac{\Delta(\lambda)}{\Delta_0(\lambda)} d\lambda$$

$$= \frac{1}{2\pi i} \oint_{C_n} \left[ \frac{1}{\lambda(3\cos\lambda - 2)} \frac{2 - 3\cos^2\lambda}{\lambda(3\cos\lambda - 2) \sin \lambda} \left[ q_1 \right] + \frac{2 - 3\cos^2\lambda}{\lambda(3\cos\lambda - 2) \sin \lambda} \left[ q_2 \right] \right]$$

$$- \frac{[q_1]^2}{2\lambda^2} \left( \frac{1 + 2\cos \lambda - 3\cos^2 \lambda}{2\lambda^2(3\cos \lambda - 2)^2 \sin^2 \lambda} \left[ q_1 \right]^2 - \frac{3\cos \lambda}{2\lambda^2(3\cos \lambda - 2)} \left[ q_2 \right]^2 \right) + o \left( \frac{1}{|\lambda|^2} \right).$$
The eigenvalues of the operator $\text{Trace}$ logarithmic integral formula, we can get

$$\frac{(2 - 3 \cos^2 \lambda)^2}{2 \lambda^2 (3 \cos \lambda - 2)^2 \sin^2 \lambda} [q_2]^2 - \frac{3 \cos \lambda}{\lambda^2 (3 \cos \lambda - 2)} [q_1][q_2]$$

$$= \frac{(2 - 3 \cos^2 \lambda)(1 + 2 \cos \lambda - 3 \cos^2 \lambda)}{\lambda^2 (3 \cos \lambda - 2)^2 \sin^2 \lambda} [q_1][q_2] - \frac{M_1}{2 \lambda^2}$$

$$+ \frac{2 \cos \lambda - 2}{\lambda^2 (3 \cos \lambda - 2)} Q_1^+ - \frac{3 \cos \lambda}{\lambda^2 (3 \cos \lambda - 2)} \left( \frac{M_2}{2} + \frac{Q^+ - Q^+_2}{3} \right) + o \left( \frac{1}{|\lambda|^2} \right) \int d\lambda.$$  

By calculating residues, we obtain

$$\lambda_n - \mu_n = \frac{(2 - (-1)^n)[q_1]}{(3 - (-1)^n)2n\pi} + \frac{[q_2]}{(3 - (-1)^n)2n\pi} + o \left( \frac{1}{n^2} \right).$$

In the same way, we can get

$$\lambda_n^\pm - \mu_n^\pm = \frac{3[q_1]}{5\mu_n^\pm} + \frac{2[q_2]}{5\mu_n^\pm} + \frac{[q_1] - [q_2]}{25(\mu_n^\pm)^2 \sin \mu_n^\pm} - \frac{q_1(1) + 3M_2 - 2Q^+_1}{9(\mu_n^\pm)^2 \sin \mu_n^\pm} + o \left( \frac{1}{n^2} \right).$$

Therefore, we have proven the following theorem.

**Theorem 3.1.** The eigenvalues of the operator $L(q, M)$, $\{\lambda_n^2\}_{n \in \mathbb{Z} \setminus \{0\}}$ and $\{\lambda_n^2\}_{n \in \mathbb{Z}}$, have the following asymptotic expressions. For sufficiently large $|n|

$$\lambda_n = n\pi + \frac{2(1 - (-1)^n)}{(3 - (-1)^n)2n\pi}[q_1] + \frac{[q_2]}{(3 - (-1)^n)2n\pi} + o \left( \frac{1}{n^2} \right),$$

and

$$\lambda_n^\pm = \mu_n^\pm + \frac{3[q_1]}{5\mu_n^\pm} + \frac{2[q_2]}{5\mu_n^\pm} + \frac{[q_1] - [q_2]}{25(\mu_n^\pm)^2 \sin \mu_n^\pm} - \frac{q_1(1) + 3M_2 - 2Q^+_1}{9(\mu_n^\pm)^2 \sin \mu_n^\pm} + o \left( \frac{1}{n^2} \right),$$

where $[q_i]$, $M_i$, and $Q_i$ are defined by (4), (5), and (6), respectively.

§4 Trace

The section is devoted to obtaining the regularized trace formula of the operator $L(q, M)$ by applying contour integral.

First of all, using the derivative formula, we can obtain

$$\frac{d}{d\lambda} \left( \lambda^t \ln \frac{\Delta(\lambda)}{\Delta_0(\lambda)} \right) = t\lambda^{t-1} \ln \frac{\Delta(\lambda)}{\Delta_0(\lambda)} + \lambda^t \left( \frac{\Delta'(\lambda)}{\Delta(\lambda)} \right),$$

where $t \in \mathbb{N}$, namely,

$$\lambda^t \left( \frac{\Delta'(\lambda)}{\Delta(\lambda)} \right) = -t \lambda^{t-1} \ln \frac{\Delta(\lambda)}{\Delta_0(\lambda)} + \frac{d}{d\lambda} \left( \lambda^t \ln \frac{\Delta(\lambda)}{\Delta_0(\lambda)} \right).$$

Taking $t = 2$ and combining with single-valuedness of $\ln \frac{\Delta(\lambda)}{\Delta_0(\lambda)}$ along the contour $\Gamma_n$ and logarithmic integral formula, we can get

$$\sum_{k \neq 0, -2n} [\lambda_k^2 - (k\pi)^2] + \sum_{k = -2n}^{2n} [(\lambda_k^+)^2 - (\mu_k^+)^2] + \sum_{k = -2n}^{2n} [(\lambda_k^-)^2 - (\mu_k^-)^2]$$
\[
\begin{align*}
&= -\frac{2}{n} \int_{\Gamma_n} \lambda \ln \frac{\Delta(\lambda)}{\Delta_0(\lambda)} d\lambda \\
&= -\frac{2}{n} \int_{\Gamma_n} \left[ 1 + 2 \cos \lambda - 3 \cos^2 \lambda \right] [q_1] + \frac{2 - 3 \cos^2 \lambda}{(3 \cos \lambda - 2) \sin \lambda} [q_2] \\
&\quad - \frac{[q_1]^2}{2\lambda} - \frac{(1 + 2 \cos \lambda - 3 \cos^2 \lambda)^2}{2\lambda(3 \cos \lambda - 2)^2 \sin^2 \lambda} [q_1]^2 - \frac{3 \cos \lambda}{2\lambda(3 \cos \lambda - 2)} [q_2]^2 \\
&\quad - \frac{(2 - 3 \cos^2 \lambda)^2}{2\lambda(3 \cos \lambda - 2)^2 \sin^2 \lambda} [q_2]^2 - \frac{3 \cos \lambda}{\lambda(3 \cos \lambda - 2)} [q_1][q_2] \\
&\quad - \frac{(2 - 3 \cos^2 \lambda)(1 + 2 \cos \lambda - 3 \cos^2 \lambda)}{\lambda(3 \cos \lambda - 2)^2 \sin^2 \lambda} [q_1][q_2] - \frac{M_1}{2\lambda} \\
&\quad + \frac{2 \cos \lambda - 2}{\lambda(3 \cos \lambda - 2)} Q_1^+ - \frac{3 \cos \lambda}{\lambda(3 \cos \lambda - 2)} \left( \frac{M_2}{2} + \frac{Q_1 - Q_2^+}{3} \right) + o \left( \frac{1}{|\lambda|} \right) \right] d\lambda \\
&= \sum_{k=-2n}^{2n} \left( 1 - (-1)^k \right) \frac{4[q_1]}{3 - (-1)^k} + 2 \sum_{k=-2n}^{2n} \frac{6[q_1]}{5} + \frac{2}{3 - (-1)^k} \\
&\quad + 2 \sum_{k=-2n}^{2n} \frac{6\sqrt{5}}{125\mu_k} ([q_1] - [q_2])^2 - \frac{2n}{6\sqrt{5}125\mu_k} ([q_1] - [q_2])^2 \\
&\quad + [q_1]^2 + 2[q_1][q_2] + \frac{[q_2]^2}{3} + M_1 + 3M_2 - \sum_{k=-2n}^{2n} \frac{2\sqrt{5}}{5\mu_k} M_2 + \sum_{k=-2n}^{2n} \frac{2\sqrt{5}}{5\mu_k} M_2 \\
&\quad + 2Q_1^+ - \sum_{k=-2n}^{2n} \frac{2\sqrt{5}}{15\mu_k} q_1(1) + \sum_{k=-2n}^{2n} \frac{2\sqrt{5}}{15\mu_k} q_1(1) - 2Q_2^+ + \sum_{k=-2n}^{2n} \frac{4\sqrt{5}}{15\mu_k} Q_2^+ \\
&\quad - \sum_{k=-2n}^{2n} \frac{4\sqrt{5}}{15\mu_k} Q_2^+ + o(1), \\
\end{align*}
\]

that is,
\[
\begin{align*}
&\sum_{k=-2n}^{2n} \left[ \lambda_k^2 - (k\pi)^2 - (1 - (-1)^k) \right] \frac{4[q_1]}{3 - (-1)^k} + \frac{2}{3 - (-1)^k} \\
&+ \sum_{k=-2n}^{2n} \left[ (\lambda_k^+)^2 - (\mu_k^+)^2 \right] - \frac{6\sqrt{5}}{5\mu_k} ([q_1] - [q_2])^2 \\
&+ \frac{2\sqrt{5}}{5\mu_k} M_2 + \frac{2\sqrt{5}}{15\mu_k} q_1(1) - \frac{4\sqrt{5}}{15\mu_k} Q_2^+ + \sum_{k=-2n}^{2n} \left[ (\lambda_k^-)^2 - (\mu_k^-)^2 \right] \\
&- \frac{6\sqrt{5}}{5\mu_k} [q_1] - \frac{4\sqrt{5}}{5\mu_k} + \frac{6\sqrt{5}}{125\mu_k} ([q_1] - [q_2])^2 - \frac{2\sqrt{5}}{5\mu_k} M_2 - \frac{2\sqrt{5}}{15\mu_k} q_1(1) + \frac{4\sqrt{5}}{15\mu_k} Q_2^+ \\
&= 2[q_1][q_2] + [q_2]^2 + \frac{[q_1]^2}{3} + M_1 + 3M_2 + 2Q_1^+ - 2Q_2^+ + o(1).
\end{align*}
\]

Taking \(n \to +\infty\), we can get the following theorem.
Theorem 4.1. The trace formula for the eigenvalues of the operator \( L(q, M) \) is as follows:

\[
\sum_{n \in \mathbb{Z}\setminus\{0\}} \left[ \lambda_n^2 - (n\pi)^2 - (1 - (-1)^n) \frac{4[q_1]}{3 - (-1)^n} - \frac{2[q_2]}{3} \right] \\
+ \sum_{n \in \mathbb{Z}} \left[ (\lambda_n^+)^2 - (\mu_n^+)^2 - \frac{6}{5} [q_1] - \frac{4}{5} [q_2] + \frac{2\sqrt{5}}{5\mu_n^+} A \right] \\
+ \sum_{n \in \mathbb{Z}} \left[ (\lambda_n^-)^2 - (\mu_n^-)^2 - \frac{6}{5} [q_1] - \frac{4}{5} [q_2] - \frac{2\sqrt{5}}{5\mu_n^-} A \right] \\
= 2[q_1][q_2] + \frac{[q_1]^2}{3} + M_1 + 3M_2 + 2Q_1^+ - 2Q_2^+, \]

where \([q_1], M_i, \) and \( Q_i \) are defined by (4), (5), and (6), respectively, and \( A = M_2 + \frac{q_1(1)}{3} - \frac{2}{3}Q_2^+ - \frac{3}{25} ([q_1] - [q_2])^2 \).

Declarations
Conflict of interest The authors declare no conflict of interest.

References


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