Appl. Math. J. Chinese Univ.
2024, 39(2): 311-318

Trace formula of the integro-differential operator on a quantum graph

WEI Li-xiao   YANG Chuan-fu

Abstract. In this paper we study the eigenvalue problem for integro-differential operators on a lasso graph. The trace formula of the operator is established by applying the residual technique in complex analysis.

§1 Introduction

The theory of differential operators on quantum graphs is a rapidly developing area of modern mathematical physics. Such operators can be used to describe the motion of quantum particles confined to certain low dimensional structures. Spectral properties of differential operators in such structures, especially the integro-differential operators, have attracted considerable attention during past years. Various aspects of spectral problems for integro-differential operators were studied in [1,2,7-10,14,17,26,27]. In particular, the paper [17] considered the eigenvalue problem for integro-differential operators with separated boundary conditions on the finite interval and found a trace formula for the integro-differential operator. At the same time, nonlocal and, in particular, integro-differential operators are of great interest, because they have many applications (see [11]).

In this paper, we shall study the trace problem of integro-differential operators on a lasso graph. The theory of regularized traces of Sturm-Liouville operators stems from the paper [4] of Gelfand and Levitan. Later on, a number of authors turned their attention to trace theory and obtained interesting results [6,12,13,15,16,18-25]. Trace formulas for differential operators have many applications in inverse problem theory, the numerical calculation of eigenvalues, the theory of integrable system, etc.

Consider the lasso graph G, represented in Figure 1, the lengths of both the edge e_1 and the loop e_2 are equal to 1. The parameter x is introduced on each edge e_i, $i = 1, 2$, where $x \in (0, 1)$. To be convenient, we set the orientations as follows: for the edge e_1, the value $x = 0$ and $x = 1$

Received: 2021-07-14. Revised: 2021-10-29.
MR Subject Classification: 34A55, 34B05, 34B45, 47E05.
Keywords: integro-differential operators, eigenvalue, trace formula.
Digital Object Identifier(DOI): https://doi.org/10.1007/s11766-024-4508-7.
Supported by the National Natural Science Foundation of China(11871031) and the National Natural Science Foundation of Jiang Su(BK20201303).

correspond to the boundary vertex and the internal vertex, respectively; for the loop e_2, both ends $x = 0$ and $x = 1$ correspond to the internal vertex.

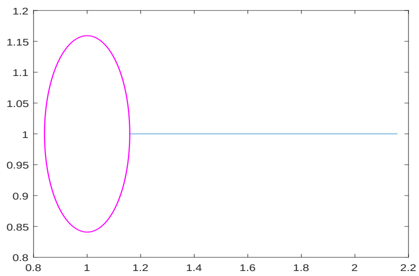

Figure 1. Lasso graph.

Let $y(x) = \{y_i(x)\}_{i=1,2}$ be a vector function on the graph G, where $y_i(x)$ is a function on the edge e_i. Consider the integro-differential expressions

$$ly(x) := \begin{pmatrix} -y_1''(x) + q_1(x)y_1(x) + \int_0^x M_1(x,t)y_1(t)dt \\ -y_2''(x) + q_2(x)y_2(x) + \int_0^x M_2(x,t)y_2(t)dt \end{pmatrix}$$

on the edges of G, where $q_i(x)$ and $M_i(x,t)$ are real-valued functions, $q_i \in W_1^1[0,1]$, $M_i \in W_1^1(D)$, $D := \{(x,t)\colon 0 \le t \le x \le 1\}$.

We study the boundary value problem $L(q,M)$ for the differential equations on the graph G:

$$ly(x) = \lambda^2 y(x), \qquad x \in (0,1), \tag{1}$$

with the matching conditions

$$y_1(1) = y_2(0) = y_2(1), \qquad y_1'(1) - y_2'(0) + y_2'(1) = 0 \tag{2}$$

in the internal vertex, and the Dirichlet boundary condition

$$y_1(0) = 0 \tag{3}$$

in the boundary vertex. In this paper, it is mainly divided into two parts: (1) the asymptotic estimations of the eigenvalues of the operator $L(q,M)$; (2) its regularized trace formula. We point out that our results are an extension to those in [5]. In particular, when $M_i(x,t) = 0$, the trace formula is consistent with that of the Sturm-Liouville operator on a lasso-graph.

§2 Characteristic function

Firstly, we calculate the characteristic function of the operator, and then we can transform the eigenvalue problem into the roots of the entire function.

We shall use the following notation

$$[q_i] = \frac{1}{2}\int_0^1 q_i(x)dx, \tag{4}$$

$$Q_i^+ = \frac{q_i(1) + q_i(0)}{4}, \quad Q_i^- = \frac{q_i(1) - q_i(0)}{4}, \tag{5}$$

$$M_i = \int_0^1 M_i(x,x)dx, \quad i = 1, 2. \tag{6}$$

Suppose the solutions $S_i(x,\lambda)$ and $C_i(x,\lambda)$ satisfy the corresponding equation (1) with $S_i(0,\lambda) = 0 = S_i'(0,\lambda) - 1$ and $C_i(0,\lambda) - 1 = 0 = C_i'(0,\lambda)$, $i = 1, 2$, respectively. Then we can get (ref.[3])

$$\begin{aligned} S_i(x,\lambda) =& \frac{\sin(\lambda x)}{\lambda} - \frac{\cos(\lambda x)}{2\lambda^2}\int_0^x q_i(\tau)d\tau - \frac{\sin(\lambda x)}{8\lambda^3}\left(\int_0^x q_i(\tau)d\tau\right)^2 \\ &+ \frac{\sin(\lambda x)}{\lambda^3}\left[\frac{q_i(x)+q_i(0)}{4} - \frac{1}{2}\int_0^x M_i(\tau,\tau)d\tau\right] + o\left(\frac{e^{|\mathrm{Im}\lambda|x}}{|\lambda|^3}\right), \end{aligned} \tag{7}$$

$$\begin{aligned} C_i(x,\lambda) =& \cos(\lambda x) + \frac{\sin(\lambda x)}{2\lambda}\int_0^x q_i(\tau)d\tau - \frac{\cos(\lambda x)}{8\lambda^2}\left(\int_0^x q_i(\tau)d\tau\right)^2 \\ &+ \frac{\cos(\lambda x)}{\lambda^2}\left[\frac{q_i(x)-q_i(0)}{4} - \frac{1}{2}\int_0^x M_i(\tau,\tau)d\tau\right] + o\left(\frac{e^{|\mathrm{Im}\lambda|x}}{|\lambda|^2}\right), \end{aligned} \tag{8}$$

and

$$\begin{aligned} S_i'(x,\lambda) =& \cos(\lambda x) + \frac{\sin(\lambda x)}{2\lambda}\int_0^x q_i(\tau)d\tau - \frac{\cos(\lambda x)}{8\lambda^2}\left(\int_0^x q_i(\tau)d\tau\right)^2 \\ &- \frac{\cos(\lambda x)}{\lambda^2}\left[\frac{q_i(x)-q_i(0)}{4} + \frac{1}{2}\int_0^x M_i(\tau,\tau)d\tau\right] + o\left(\frac{e^{|\mathrm{Im}\lambda|x}}{|\lambda|^2}\right), \end{aligned} \tag{9}$$

$$\begin{aligned} C_i'(x,\lambda) =& -\lambda\sin(\lambda x) + \frac{\cos(\lambda x)}{2}\int_0^x q_i(\tau)d\tau + \frac{\sin(\lambda x)}{8\lambda}\left(\int_0^x q_i(\tau)d\tau\right)^2 \\ &+ \frac{\sin(\lambda x)}{\lambda}\left[\frac{q_i(x)+q_i(0)}{4} + \frac{1}{2}\int_0^x M_i(\tau,\tau)d\tau\right] + o\left(\frac{e^{|\mathrm{Im}\lambda|x}}{|\lambda|}\right). \end{aligned} \tag{10}$$

The following characteristic function can be obtained from the conditions satisfied by the internal vertex (2) and the boundary vertex (3)

$$\Delta(\lambda) = S_1'(1,\lambda)S_2(1,\lambda) + S_1(1,\lambda)(S_2'(1,\lambda) + C_2(1,\lambda) - 2).$$

Using (7), (8) and (9), we obtain

$$\begin{aligned} \Delta(\lambda) =& \frac{(3\cos\lambda - 2)\sin\lambda}{\lambda} + \frac{1 + 2\cos\lambda - 3\cos^2\lambda}{\lambda^2}[q_1] + \frac{2 - 3\cos^2\lambda}{\lambda^2}[q_2] \\ &- \frac{(3\cos\lambda - 2)\sin\lambda}{2\lambda^3}\left([q_1]^2 + M_1\right) + \frac{(2\cos\lambda - 2)\sin\lambda}{\lambda^3}Q_1^+ \\ &- \frac{3\cos\lambda\sin\lambda}{\lambda^3}\left([q_1][q_2] + \frac{[q_2]^2 + M_2}{2} + \frac{Q_1^- - Q_2^+}{3}\right) + o\left(\frac{e^{2|\mathrm{Im}\lambda|}}{|\lambda|^3}\right). \end{aligned}$$

Define

$$\Delta_0(\lambda) = \frac{(3\cos\lambda - 2)\sin\lambda}{\lambda}.$$

Then, we can obtain that the zeros of $\Delta_0(\lambda)$ are as follows:

$$\mu_n = n\pi, n \in \mathbb{Z}\backslash\{0\}, \quad \mu_n^{\pm} = 2n\pi \pm \arccos\frac{2}{3}, n \in \mathbb{Z}.$$

Denote

$$\gamma_n = \left\{\lambda : |\lambda| = (\mu_n + 1/2)^2\right\} \cup \left\{\lambda : |\lambda| = (\mu_n^{\pm} + 1/2)^2\right\}.$$

We know that $|\Delta_0(\lambda)| > |\Delta(\lambda) - \Delta_0(\lambda)|$, $\lambda \in \gamma_n$, for sufficiently large n. Then by Rouchè's theorem, the number of zeros of $\Delta(\lambda)$ inside γ_n coincide with the number of zeros of $\Delta_0(\lambda)$. Denote the contours, traversed counterclockwise:

$$C_n = \{\lambda \in \mathbb{C}\colon |\lambda - \mu_n| = \delta\}, \quad D_n^{\pm} = \left\{\lambda \in \mathbb{C}\colon |\lambda - \mu_n^{\pm}| = \delta\right\}$$

for $\delta > 0$ and the square contour Γ_n with vertices $(i = \sqrt{-1})$

$$A(\mu_n^+ + \varepsilon + (\mu_n^+ + \varepsilon)i), \quad B(-\mu_n^+ - \varepsilon + (\mu_n^+ + \varepsilon)i),$$
$$C(-\mu_n^+ - \varepsilon - (\mu_n^+ + \varepsilon)i), \quad D(\mu_n^+ + \varepsilon - (\mu_n^+ + \varepsilon)i).$$

Therefore, on contours Γ_n or C_n, $D_n^{\pm}$, we have

$$\begin{aligned}\frac{\Delta(\lambda)}{\Delta_0(\lambda)} =& 1 + \frac{1 + 2\cos\lambda - 3\cos^2\lambda}{\lambda(3\cos\lambda - 2)\sin\lambda}[q_1] + \frac{2 - 3\cos^2\lambda}{\lambda(3\cos\lambda - 2)\sin\lambda}[q_2] \\ &- \frac{3\cos\lambda}{\lambda^2(3\cos\lambda - 2)}[q_1][q_2] - \frac{[q_1]^2 + M_1}{2\lambda^2} + \frac{2\cos\lambda - 2}{\lambda^2(3\cos\lambda - 2)}Q_1^+ \\ &- \frac{3\cos\lambda}{\lambda^2(3\cos\lambda - 2)}\left(\frac{[q_2]^2 + M_2}{2} + \frac{Q_1^- - Q_2^+}{3}\right) + o\left(\frac{1}{|\lambda|^2}\right).\end{aligned}$$

By applying the Taylor series expansion and simplifying, we can get

$$\begin{aligned}\ln\frac{\Delta(\lambda)}{\Delta_0(\lambda)} =& \frac{1 + 2\cos\lambda - 3\cos^2\lambda}{\lambda(3\cos\lambda - 2)\sin\lambda}[q_1] + \frac{2 - 3\cos^2\lambda}{\lambda(3\cos\lambda - 2)\sin\lambda}[q_2] - \frac{[q_1]^2}{2\lambda^2} \\ &- \frac{(1 + 2\cos\lambda - 3\cos^2\lambda)^2}{2\lambda^2(3\cos\lambda - 2)^2\sin^2\lambda}[q_1]^2 - \frac{3\cos\lambda}{2\lambda^2(3\cos\lambda - 2)}[q_2]^2 \\ &- \frac{(2 - 3\cos^2\lambda)^2}{2\lambda^2(3\cos\lambda - 2)^2\sin^2\lambda}[q_2]^2 - \frac{3\cos\lambda}{\lambda^2(3\cos\lambda - 2)}[q_1][q_2] \\ &- \frac{(2 - 3\cos^2\lambda)(1 + 2\cos\lambda - 3\cos^2\lambda)}{\lambda^2(3\cos\lambda - 2)^2\sin^2\lambda}[q_1][q_2] - \frac{M_1}{2\lambda^2} \\ &+ \frac{2\cos\lambda - 2}{\lambda^2(3\cos\lambda - 2)}Q_1^+ - \frac{3\cos\lambda}{\lambda^2(3\cos\lambda - 2)}\left(\frac{M_2}{2} + \frac{Q_1^- - Q_2^+}{3}\right) + o\left(\frac{1}{|\lambda|^2}\right).\end{aligned}$$

§3 Eigenvalue asymptotics

In this section, the asymptotic expression of eigenvalues of $L(q, M)$ is obtained by applying the Rouché's theorem and the residue calculation.

By restoring to the integral identity, we can get the following formula

$$\begin{aligned}\lambda_n - \mu_n =& -\frac{1}{2\pi i}\oint_{C_n} \ln\frac{\Delta(\lambda)}{\Delta_0(\lambda)}d\lambda \\ =& -\frac{1}{2\pi i}\oint_{C_n}\left[\frac{1 + 2\cos\lambda - 3\cos^2\lambda}{\lambda(3\cos\lambda - 2)\sin\lambda}[q_1] + \frac{2 - 3\cos^2\lambda}{\lambda(3\cos\lambda - 2)\sin\lambda}[q_2]\right. \\ &- \frac{[q_1]^2}{2\lambda^2} - \frac{(1 + 2\cos\lambda - 3\cos^2\lambda)^2}{2\lambda^2(3\cos\lambda - 2)^2\sin^2\lambda}[q_1]^2 - \frac{3\cos\lambda}{2\lambda^2(3\cos\lambda - 2)}[q_2]^2\end{aligned}$$

$$
\begin{aligned}
&-\frac{(2-3\cos^2\lambda)^2}{2\lambda^2(3\cos\lambda-2)^2\sin^2\lambda}[q_2]^2-\frac{3\cos\lambda}{\lambda^2(3\cos\lambda-2)}[q_1][q_2]\\
&-\frac{(2-3\cos^2\lambda)(1+2\cos\lambda-3\cos^2\lambda)}{\lambda^2(3\cos\lambda-2)^2\sin^2\lambda}[q_1][q_2]-\frac{M_1}{2\lambda^2}\\
&+\frac{2\cos\lambda-2}{\lambda^2(3\cos\lambda-2)}Q_1^+-\frac{3\cos\lambda}{\lambda^2(3\cos\lambda-2)}\left(\frac{M_2}{2}+\frac{Q_1^--Q_2^+}{3}\right)+o\left(\frac{1}{|\lambda|^2}\right)\Bigg]d\lambda.
\end{aligned}
$$

By calculating residues, we obtain

$$\lambda_n-\mu_n=\frac{(2-(-1)^n2)[q_1]}{(3-(-1)^n2)n\pi}+\frac{[q_2]}{(3-(-1)^n2)n\pi}+o\left(\frac{1}{n^2}\right).$$

In the same way, we can get

$$\lambda_n^\pm-\mu_n^\pm=\frac{3[q_1]}{5\mu_n^\pm}+\frac{2[q_2]}{5\mu_n^\pm}+\frac{([q_1]-[q_2])^2}{25(\mu_n^\pm)^2\sin\mu_n^\pm}-\frac{q_1(1)+3M_2-2Q_2^+}{9(\mu_n^\pm)^2\sin\mu_n^\pm}+o\left(\frac{1}{n^2}\right).$$

Therefore, we have proven the following theorem.

Theorem 3.1. *The eigenvalues of the operator* $L(q,M)$, $\{\lambda_n^2\}_{n\in\mathbb{Z}\setminus\{0\}}$ *and* $\{(\lambda_n^\pm)^2\}_{n\in\mathbb{Z}}$, *have the following asymptotic expressions. For sufficiently large* $|n|$

$$\lambda_n=n\pi+\frac{2(1-(-1)^n)}{(3-(-1)^n2)n\pi}[q_1]+\frac{[q_2]}{(3-(-1)^n2)n\pi}+o\left(\frac{1}{n^2}\right),$$

and

$$\lambda_n^\pm=\mu_n^\pm+\frac{3[q_1]}{5\mu_n^\pm}+\frac{2[q_2]}{5\mu_n^\pm}+\frac{([q_1]-[q_2])^2}{25(\mu_n^\pm)^2\sin\mu_n^\pm}-\frac{q_1(1)+3M_2-2Q_2^+}{9(\mu_n^\pm)^2\sin\mu_n^\pm}+o\left(\frac{1}{n^2}\right),$$

where $[q_i]$, M_i *and* Q_i *are defined by* (4), (5), *and* (6), *respectively.*

§4 Trace

The section is devoted to obtaining the regularized trace formula of the operator $L(q,M)$ by applying contour integral.

First of all, using the derivative formula, we can obtain

$$\frac{d}{d\lambda}\left(\lambda^t\ln\frac{\Delta(\lambda)}{\Delta_0(\lambda)}\right)=t\lambda^{t-1}\ln\frac{\Delta(\lambda)}{\Delta_0(\lambda)}+\lambda^t\left(\frac{\Delta'(\lambda)}{\Delta(\lambda)}-\frac{\Delta_0'(\lambda)}{\Delta_0(\lambda)}\right),$$

where $t\in\mathbb{N}$, namely,

$$\lambda^t\left(\frac{\Delta'(\lambda)}{\Delta(\lambda)}-\frac{\Delta_0'(\lambda)}{\Delta_0(\lambda)}\right)=-t\lambda^{t-1}\ln\frac{\Delta(\lambda)}{\Delta_0(\lambda)}+\frac{d}{d\lambda}\left(\lambda^t\ln\frac{\Delta(\lambda)}{\Delta_0(\lambda)}\right).$$

Taking $t=2$ and combining with single-valuedness of $\ln\dfrac{\Delta(\lambda)}{\Delta_0(\lambda)}$ along the contour Γ_n and logarithmic integral formula, we can get

$$\sum_{k\neq0,-2n}^{2n}[\lambda_k^2-(k\pi)^2]+\sum_{k=-2n}^{2n}[(\lambda_k^+)^2-(\mu_k^+)^2]+\sum_{k=-2n}^{2n}[(\lambda_k^-)^2-(\mu_k^-)^2]$$

$$
\begin{aligned}
&= -\frac{2}{2\pi i}\oint_{\Gamma_n} \lambda \ln \frac{\Delta(\lambda)}{\Delta_0(\lambda)} d\lambda \\
&= -\frac{2}{2\pi i}\oint_{\Gamma_n} \left[\frac{1+2\cos\lambda-3\cos^2\lambda}{(3\cos\lambda-2)\sin\lambda}[q_1] + \frac{2-3\cos^2\lambda}{(3\cos\lambda-2)\sin\lambda}[q_2] \right. \\
&- \frac{[q_1]^2}{2\lambda} - \frac{(1+2\cos\lambda-3\cos^2\lambda)^2}{2\lambda(3\cos\lambda-2)^2\sin^2\lambda}[q_1]^2 - \frac{3\cos\lambda}{2\lambda(3\cos\lambda-2)}[q_2]^2 \\
&- \frac{(2-3\cos^2\lambda)^2}{2\lambda(3\cos\lambda-2)^2\sin^2\lambda}[q_2]^2 - \frac{3\cos\lambda}{\lambda(3\cos\lambda-2)}[q_1][q_2] \\
&- \frac{(2-3\cos^2\lambda)(1+2\cos\lambda-3\cos^2\lambda)}{\lambda(3\cos\lambda-2)^2\sin^2\lambda}[q_1][q_2] - \frac{M_1}{2\lambda} \\
&\left. + \frac{2\cos\lambda-2}{\lambda(3\cos\lambda-2)}Q_1^+ - \frac{3\cos\lambda}{\lambda(3\cos\lambda-2)}\left(\frac{M_2}{2} + \frac{Q_1^- - Q_2^+}{3}\right) + o\left(\frac{1}{|\lambda|}\right)\right] d\lambda \\
&= \sum_{k\neq 0,-2n}^{2n} (1-(-1)^k)\frac{4[q_1]}{3-(-1)^k 2} + 2\sum_{k=-2n}^{2n}\frac{6[q_1]}{5} + \sum_{k\neq 0,-2n}^{2n}\frac{2[q_2]}{3-(-1)^k 2} \\
&+ 2\sum_{k=-2n}^{2n}\frac{4}{5}[q_2] + \sum_{k=-2n}^{2n}\frac{6\sqrt{5}}{125\mu_k^+}([q_1]-[q_2])^2 - \sum_{k=-2n}^{2n}\frac{6\sqrt{5}}{125\mu_k^-}([q_1]-[q_2])^2 \\
&+ [q_1]^2 + 2[q_1][q_2] + \frac{[q_2]^2}{3} + M_1 + 3M_2 - \sum_{k=-2n}^{2n}\frac{2\sqrt{5}}{5\mu_k^+}M_2 + \sum_{k=-2n}^{2n}\frac{2\sqrt{5}}{5\mu_k^-}M_2 \\
&+ 2Q_1^- - \sum_{k=-2n}^{2n}\frac{2\sqrt{5}}{15\mu_k^+}q_1(1) + \sum_{k=-2n}^{2n}\frac{2\sqrt{5}}{15\mu_k^-}q_1(1) - 2Q_2^+ + \sum_{k=-2n}^{2n}\frac{4\sqrt{5}}{15\mu_k^+}Q_2^+ \\
&- \sum_{k=-2n}^{2n}\frac{4\sqrt{5}}{15\mu_k^-}Q_2^+ + o(1),
\end{aligned}
$$

that is,

$$
\begin{aligned}
&\sum_{k\neq 0,-2n}^{2n}\left[\lambda_k^2 - (k\pi)^2 - (1-(-1)^k)\frac{4[q_1]}{3-(-1)^k 2} - \frac{2[q_2]}{3-(-1)^k 2}\right] \\
&+ \sum_{k=-2n}^{2n}\left[(\lambda_k^+)^2 - (\mu_k^+)^2 - \frac{6}{5}[q_1] - \frac{4}{5}[q_2] - \frac{6\sqrt{5}}{125\mu_k^+}([q_1]-[q_2])^2\right. \\
&\left. + \frac{2\sqrt{5}}{5\mu_k^+}M_2 + \frac{2\sqrt{5}}{15\mu_k^+}q_1(1) - \frac{4\sqrt{5}}{15\mu_k^+}Q_2^+\right] + \sum_{k=-2n}^{2n}\left[(\lambda_k^-)^2 - (\mu_k^-)^2\right. \\
&\left. - \frac{6}{5}[q_1] - \frac{4}{5}[q_2] + \frac{6\sqrt{5}}{125\mu_k^-}([q_1]-[q_2])^2 - \frac{2\sqrt{5}}{5\mu_k^-}M_2 - \frac{2\sqrt{5}}{15\mu_k^-}q_1(1) + \frac{4\sqrt{5}}{15\mu_k^-}Q_2^+\right] \\
&= 2[q_1][q_2] + [q_1]^2 + \frac{[q_2]^2}{3} + M_1 + 3M_2 + 2Q_1^- - 2Q_2^+ + o(1).
\end{aligned}
$$

Taking $n \to +\infty$, we can get the following theorem.

Theorem 4.1. *The trace formula for the eigenvalues of the operator $L(q, M)$ is as follows:*

$$\sum_{n\in\mathbb{Z}\setminus\{0\}}\left[\lambda_n^2-(n\pi)^2-(1-(-1)^n)\frac{4[q_1]}{3-(-1)^n 2}-\frac{2[q_2]}{3-(-1)^n 2}\right]$$
$$+\sum_{n\in\mathbb{Z}}\left[(\lambda_n^+)^2-(\mu_n^+)^2-\frac{6}{5}[q_1]-\frac{4}{5}[q_2]+\frac{2\sqrt{5}}{5\mu_n^+}A\right]$$
$$+\sum_{n\in\mathbb{Z}}\left[(\lambda_n^-)^2-(\mu_n^-)^2-\frac{6}{5}[q_1]-\frac{4}{5}[q_2]-\frac{2\sqrt{5}}{5\mu_n^-}A\right]$$
$$=2[q_1][q_2]+[q_1]^2+\frac{[q_2]^2}{3}+M_1+3M_2+2Q_1^- -2Q_2^+,$$

where $[q_i]$, M_i *and* Q_i *are defined by* (4), (5), *and* (6), *respectively, and* $A=M_2+\dfrac{q_1(1)}{3}$ $-\dfrac{2}{3}Q_2^+-\dfrac{3}{25}\left([q_1]-[q_2]\right)^2$.

Declarations

Conflict of interest The authors declare no conflict of interest.

References

[1] N P Bondarenko. *An inverse problem for an integro-differential operator on a star-shaped graph*, Mathematical Methods in the Applied Sciences, 2018, 41(4): 1697-1702.

[2] S A Buterin. *On an inverse spectral problem for a convolution integro-differential operator*, Results in Mathematics, 2007, 50(3-4): 173-181.

[3] S R Chen. *A trace formula for integro-differential operators*, Acta Mathematica Scientia, 2019, 39A(2): 244-252.

[4] I M Gelfand, B M Levitan. *On a simple identity for eigenvalues of the differential operator of second order*, Doklady Akademii Nauk SSSR (Soviet Math Dokl), 1953, 88(4): 593-596.

[5] S Y Guan, C F Yang. *New trace formulae for Sturm-Liouville operators on the lasso-graph*, Results in Mathematics, 2020, 75, DOI: 10.1007/s00025-020-01212-5.

[6] N J Guliyev. *The regularized trace formula for the Sturm-Liouville equation with spectral parameter in the boundary conditions*, Proceedings of Institute of Mathematics and Mechanics, 2005, 22: 99-102.

[7] Y T Hu, N P Bondarenko, C T Shieh, C F Yang. *Traces and inverse nodal problems for Dirac-type integro-differential operators on a graph*, Applied Mathematics and Computation, 2019, 363, https://doi.org/10.1016/j.amc.2019.124606.

[8] B Keskin, A S Ozkan. *Inverse nodal problems for Dirac-type integro-differential operators*, Journal of Differential Equations, 2017, 263(12): 8838-8847.

[9] Y V Kuryshova. *Inverse spectral problem for integro-differential operators*, Mathematical Notes, 2007, 81(5-6): 767-777.

[10] Y V Kuryshova, C T Shieh. *An inverse nodal problem for integro-differential operators*, Journal of Inverse and Ill-posed Problems, 2010, 18(4): 357-369.

[11] V Lakshmikantham, R Rao. *Theory of integro-differential equations*, Singapure: Gordon and Breach, 1995.

[12] V G Papanicolaou. *Trace formulas and the behavior of large eigenvalues*, SIAM Journal on Mathematical Analysis, 1995, 26(1): 218-237.

[13] V A Sadovnichii, V E Podol'skii. *Traces of differential operators*, Differential Equations, 2009, 45(4): 477-493.

[14] M Sat, C T Shieh. *Inverse nodal problems for integro-differential operators with a constant delay*, Journal of Inverse and Ill-Posed Problems, 2019, 27(4): 501-509.

[15] A M Savchuk. *First-order regularised trace of the Sturm-Liouville operator with δ-potential*, Russian Mathematical Surveys, 2000, 55(6): 2217-2228.

[16] V A Vinokurov, V A Sadovnichiĭ. *The eigenvalue and trace of the Sturm-Liouville operator as differentiable functions of a summable potential*, Doklady Mathematics, 1999, 365(3): 295-297.

[17] Y P Wang, H Koyunbakan, C F Yang. *A trace formula for integro-differential operators on the finite interval*, Acta Mathematicae Applicatae Sinica, 2017, 33(1): 141-146.

[18] C F Yang. *Regularized trace for Sturm-Liouville differential operator on a star-shaped graph*, Complex Analysis and Operator Theory, 2013, 7(4): 1185-1196.

[19] C F Yang. *Trace and inverse problem of a discontinuous Sturm-Liouville operator with retarded argument*, Journal of Mathematical Analysis and Applications, 2012, 395(1): 30-41.

[20] C F Yang. *Trace for differential pencils on a star-type graph*, Zeitschrift für Naturforschung A, 2013, 68(6-7): 421-426.

[21] C F Yang. *Trace formulae for matrix integro-differential operators*, Zeitschrift für Naturforschung A, 2012, 67(3-4): 180-184.

[22] C F Yang. *Trace formulae for the matrix Schrödinger equation with energy-dependent potential*, Journal of Mathematical Analysis and Applications, 2012, 393(2): 526-533.

[23] C F Yang. *Traces of Sturm-Liouville operators with discontinuities*, Inverse Problems in Science and Engineering, 2014, 22(5): 803-813.

[24] C F Yang, Z Y Huang. *Spectral asymptotics and regularized traces for Dirac operators on a star-shaped graph*, Applicable Analysis, 2012, 91(9): 1717-1730.

[25] C F Yang, Z Y Huang, Y P Wang. *Trace formulae for the Schrödinger equation with energy-dependent potential*, Journal of Physics A Mathematical and Theoretical, 2010, 43(41), DOI: 10.1088/1751-8113/43/41/415207.

[26] V A Yurko. *An inverse spectral problem for integro-differential operators*, Far East Journal of Mathematical Sciences, 2014, 92(2): 247-261.

[27] V A Yurko. *Inverse problem for integro-differential operators*, Mathematical Notes, 1991, 50(5): 1188-1197.

School of Mathematics and Statistics, Nanjing University of Science and Technology, Nanjing 210094, China.

Emails: 119113011120@njust.edu.cn, chuanfuyang@njust.edu.cn