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Characterizations of semihoops based on derivations

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Abstract. In this paper, we discuss the related properties of some particular derivations in semihoops and give some characterizations of them. Then, we prove that every Heyting algebra is isomorphic to the algebra of all multiplicative derivations and show that every Boolean algebra is isomorphic to the algebra of all implicative derivations. Finally, we show that the sets of multiplicative and implicative derivations on bounded regular idempotent semihoops are in one-to-one correspondence.

§1 Introduction

BL-algebras were defined by Hájek as the equivalent algebraic semantics of Basic Logic in 1996 [13]. Although these algebras were introduced in the late 1990s, it was proved in [12] that their $\{0\}$-free subreducts determine a class of algebraic structures that were already introduced by Büchi and Owens during the 1970s in the unpublished manuscript [5]. These structures were called hoops and BL-algebras turned out to be equivalent to bounded basic hoops [12]. Indeed, every hoop is a meet-semilattice ordered residuated integral divisible and commutative monoid [2]. A semihoop is a hoop without the divisibility equation, and this algebraic structure covers all the mathematical structures that appear in a fuzzy logic framework. Therefore, semihoops play an important role in studying fuzzy logic and the related algebraic structures.

The notion of derivations, introduced from the analytic theory, is helpful for studying algebraic structures and properties in algebraic systems. In 1957, Posner [1] introduced the notion of derivations in a prime ring $(R, +, \cdot)$, which is a map $d : R \to R$ satisfying the two conditions:

$$\text{(i) } d(\sigma + \delta) = d(\sigma) + d(\delta), \text{ (ii) } d(\sigma \cdot \delta) = d(\sigma) \cdot \delta + \sigma \cdot d(\delta)$$

for all $\sigma, \delta \in R$. Inspired by derivations on rings, Y B Jun, et al. [16,22] applied the notion of derivations to BCI-algebras and gave some characterizations of p-semisimple BCI-algebras. In 2008, Xin [20,21] introduced the concept of derivations in a lattice $(\mathcal{L}, \wedge, \vee)$, which is a map $d : \mathcal{L} \to \mathcal{L}$ satisfies the following condition:

$$d(\sigma \wedge \delta) = (d(\sigma) \wedge \delta) \vee (\sigma \wedge d(\delta))$$

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for all $\sigma, \delta \in \mathcal{L}$. They also characterized modular lattices and distributive lattices by isotone derivations. Subsequently, Alshehri et al. [11,18] applied the notions of derivations to an MV-algebra $(\mathcal{L}, \oplus, ^*, 0, 1)$, which is a map $d : \mathcal{L} \to \mathcal{L}$ satisfying the following condition:

$$d(\sigma \odot \delta) = (d(\sigma) \odot \delta) \oplus (\sigma \odot d(\delta))$$

for any $\sigma, \delta \in \mathcal{L}$. Recently, Wang et al. [6,8] further explored derivations of MV-algebras and obtained some interesting and meaningful results. In particular, they proved in [6] that the fixed point set of Boolean additive derivations and that of their adjoint derivations are isomorphic, and showed that every MV-algebra is isomorphic to the direct product of the fixed point set of Boolean additive derivations and that of their adjoint derivations, which provide a new representation of MV-algebras based on these derivations. In 2016, He [15] investigated the notion of multiplicative derivation is a map $d : \mathcal{L} \to \mathcal{L}$ satisfying the following condition:

$$d(\sigma \odot \delta) = (d(\sigma) \odot \delta) \vee (\sigma \odot d(\delta))$$

and implicative derivation is a map $g : \mathcal{L} \to \mathcal{L}$ satisfying the following condition:

$$g(\sigma \to \delta) = (g(\sigma) \to \delta) \vee (\sigma \to g(\delta))$$

in residuated lattices, and characterized Heyting algebras in terms of the above derivations.

The paper is motivated by the following considerations:

(1) As we have mentioned in the above paragraph, derivations have been studied on MV-algebras, BL-algebras and residuated lattices, etc, we have observed that although they are essentially different logical algebras, they are all particular types of bounded semihoops. Then it is meaningful to establish the derivation theory of semihoops for studying the common properties of derivations in the above-mentioned logical algebras.

(2) The previous research about derivations on logical algebras mainly involves its basic algebraic properties, without using it to characterize the algebraic structures. Then, it is interesting to characterize the algebraic structure of logical algebras by kinds of derivations.

(3) It has always been known that Galois connections play a central role in studying logical algebras, and so the relation between derivations and Galois connections is an important research topic to study. However, there are few researches about the relation between derivations and Galois connections on logical algebras so far. Then, it is necessary for us to study the relation between derivations and Galois connections on logical algebras.

Over these considerations, we introduce and study the derivations of semihoops. Indeed, we will obtain the following main results:

(1) We introduce the concept of derivations on semihoops, and show that is the natural generalization of derivations on residuated lattices (See Definitions 3.1 and 4.1, Remarks 3.10 and 4.).

(2) Every Heyting algebra is isomorphic to the algebra of all multiplicative derivations on Heyting algebras (See Theorem 3.17).

(3) Every Boolean algebra is isomorphic to the algebra of all implicative derivations on Boolean algebras (See Theorem 4.16).

(4) The sets of multiplicative and implicative derivations on bounded regular idempotent semihoops are in one-to-one correspondence (See Theorem 4.18).

The paper is organized as follows: In Section 2, we review some basic definitions and results about semihoops. In Section 3, we introduce the notion of multiplicative derivations on semihoops and give a representation of Heyting algebras by them. In Section 4, we introduce the concept of implicative derivations on semihoops and discuss the relation between multiplicative and implicative derivations.

§2 Preliminaries

In this section, we recall some fundamental definitions and notations which are necessary for the reader to follow the paper.

Definition 2.1. ([3])An algebra $(A, \wedge)$ is called a $\wedge$-semilattice if it satisfies the following conditions: for any $\sigma, \delta, \gamma \in A$,

(1) $\sigma \wedge \sigma = \sigma$,

(2) $\sigma \wedge \delta = \delta \wedge \sigma$,

(3) $(\sigma \wedge \delta) \wedge \gamma = \sigma \wedge (\delta \wedge \gamma)$.

Given a $\wedge$-semilattice A, we can define a binary relation $\leq$ on A by

$$\sigma \leq \delta \text{ iff } \sigma \wedge \delta = \sigma.$$

Then $\leq$ is a partial order relation on A.

Definition 2.2. ([7,6,17])An algebra $(\mathcal{L}, \odot, \rightarrow, \wedge, 1)$ of type $(2, 2, 2, 0)$ is called a semihoop if it satisfies the following conditions:

(1) $(\mathcal{L}, \wedge, 1)$ is a $\wedge$-semilattice with upper bound 1,

(2) $(\mathcal{L}, \odot, 1)$ is a commutative monoid,

(3) $(\sigma \odot \delta) \rightarrow \gamma = \sigma \rightarrow (\delta \rightarrow \gamma)$, for any $\sigma, \delta, \gamma \in \mathcal{L}$.

In what follows, by $\mathcal{L}$ we denote the universe of a semihoop $(\mathcal{L}, \odot, \rightarrow, \wedge, 1)$. A semihoop $\mathcal{L}$ is a bounded semihoop if there exists an element $0 \in \mathcal{L}$ such that $0 \leq \sigma$ for all $\sigma \in \mathcal{L}$. In a bounded semihoop, we define the negation $\neg$: $\neg\sigma = \sigma \rightarrow 0$ for all $\sigma \in \mathcal{L}$. If $\neg\neg\sigma = \sigma$, that is, $\neg(\neg\sigma) = \sigma$ for any $\sigma \in \mathcal{L}$, then the bounded semihoop is said to have a double negation property (DNP, for short). If $\sigma \odot \sigma = \sigma$ for any $\sigma \in \mathcal{L}$, then the semihoop $\mathcal{L}$ is said to be idempotent. We denote the set of all idempotent element of $\mathcal{L}$ by $I(\mathcal{L})$.

Definition 2.3. ([2])Let $\mathcal{L}$ be a semihoop. Then

(1) $\mathcal{L}$ is said to be regular if it satisfies the double negation property: $\neg\neg\sigma = \sigma$ for any $\sigma \in \mathcal{L}$.

(2) $\mathcal{L}$ is said to be a hoop if it satisfies the divisibility equation: $\sigma \wedge \delta = \sigma \odot (\sigma \rightarrow \delta)$ for any $\sigma, \delta \in \mathcal{L}$.

(3) $\mathcal{L}$ is said to be a Brouwerian semilattice if it satisfies the equation: $\sigma \odot \sigma = \sigma$ for any $\sigma \in \mathcal{L}$.

It must be pointed out here that not all semihoops (hoops, Brouwerian semilattices) have a lattice reduct, the ones that do are exactly the join free reducts of residuated lattice (Rℓ-monoid, Heyting algebra). Moreover, in any regular bounded semihoop, we define further operation in the following manner:

$$\sigma \oplus \delta = \neg(\neg\sigma \odot \neg\delta),$$

and check that

$$\sigma \odot \delta = \neg(\neg\sigma \oplus \neg\delta),\ \sigma \to \delta = \neg\sigma \oplus \delta.$$

Proposition 2.4. ([6,12,14,17])*Let $\mathcal{L}$ be a semihoop. Then the following properties hold: for all $\sigma, \delta, \gamma \in \mathcal{L}$,*

(1) $\sigma \leq \delta$ *iff* $\sigma \to \delta = 1$,
(2) $\sigma \odot \delta \leq \gamma$ *iff* $\sigma \leq \delta \to \gamma$,
(3) $\sigma \odot \delta \leq \sigma \wedge \delta$,
(4) $1 \to \sigma = \sigma, \sigma \to 1 = 1$,
(5) $\sigma \odot (\sigma \to \delta) \leq \delta$,
(6) *if* $\sigma \leq \delta$, *then* $\delta \to \gamma \leq \sigma \to \gamma$, $\gamma \to \sigma \leq \gamma \to \delta$ *and* $\sigma \odot \gamma \leq \delta \odot \gamma$,
(7) $\sigma \leq (\sigma \to \delta) \to \delta$,
(8) $\sigma \to (\sigma \wedge \delta) = \sigma \to \delta$,
(9) $(\sigma \to \delta) \odot \gamma \leq (\sigma \odot \gamma) \to (\delta \odot \gamma)$,
(10) $\sigma \leq \delta \to \sigma$,
(11) $\sigma \to (\delta \wedge \gamma) \leq (\sigma \to \delta) \wedge (\sigma \to \gamma)$,
(12) $\sigma \to \delta \leq (\gamma \to \sigma) \to (\gamma \to \delta)$,
(13) $\sigma \to \delta \leq (\delta \to \gamma) \to (\sigma \to \gamma)$,
(14) *if $\mathcal{L}$ is a hoop, then* $\sigma \odot (\delta \wedge \gamma) = (\sigma \odot \delta) \wedge (\sigma \odot \gamma)$,
(15) *if* $\sigma \in I(\mathcal{L})$ *and* $\delta \in \mathcal{L}$, *then*
(i) $\sigma \odot \delta = \sigma \wedge \delta = \sigma \odot (\sigma \to \delta)$,
(ii) $\sigma \to (\delta \to \gamma) = (\sigma \to \delta) \to (\sigma \to \gamma)$.

Definition 2.5. ([10])A residuated lattice is a algebraic structure $(\mathcal{L}, \wedge, \vee, \odot, \to, 0, 1)$ of type $(2, 2, 2, 2, 0, 0)$ satisfying the following conditions:

(1) $(\mathcal{L}, \wedge, \vee, 0, 1)$ is a bounded lattice,
(2) $(\mathcal{L}, \odot, 1)$ is a commutative semigroup (with the unit element 1),
(3) $\sigma \odot \delta \leq \gamma$ iff $\sigma \leq \delta \to \gamma$, for any $\sigma, \delta, \gamma \in \mathcal{L}$.

Proposition 2.6. ([10])*Let $\mathcal{L}$ be a residuated lattice. Then the following properties hold: for all $\sigma, \delta, \gamma \in \mathcal{L}$,*

(1) $\sigma \to \delta \leq (\sigma \vee \gamma) \to (\delta \vee \gamma)$,
(2) $\sigma \to \delta = (\sigma \vee \delta) \to \delta$,
(3) $\sigma \odot (\delta \vee \gamma) = (\sigma \odot \delta) \vee (\sigma \odot \gamma)$.

Definition 2.7. ([19])A bounded commutative Rℓ-monoid is a algebraic structure $(\mathcal{L}, \wedge, \vee, \odot, \to, 0, 1)$ of type $(2, 2, 2, 2, 0, 0)$ satisfying the following conditions:

(1) $(\mathcal{L}, \wedge, \vee, 0, 1)$ is a bounded lattice,
(2) $(\mathcal{L}, \odot, 1)$ is a commutative monoid,
(3) $\sigma \odot \delta \leq \gamma$ iff $\sigma \leq \delta \to \gamma$, for any $\sigma, \delta, \gamma \in \mathcal{L}$,
(4) $\sigma \wedge \delta = \sigma \odot (\sigma \to \delta)$, for any $\sigma, \delta \in \mathcal{L}$.

Definition 2.8. ([3])A Heyting algebra is an algebra $(\mathcal{L}, \wedge, \vee, \to, 0, 1)$ such that $(\mathcal{L}, \wedge, \vee, 0, 1)$ is a bounded distributive lattice that satisfies the condition:

$$\sigma \wedge \delta \leq \gamma \text{ iff } \sigma \leq \delta \to \gamma,$$

for any $\sigma, \delta, \gamma \in \mathcal{L}$.

Definition 2.9. A semihoop $\mathcal{L}$ is said to be prelinear if for any $\sigma, \delta \in \mathcal{L}$, 1 is the unique upper bound in $\mathcal{L}$ of the set

$$\{(\sigma \to \delta), (\delta \to \sigma)\}.$$

Example 2.10. Let $\mathcal{L} = \{0, \alpha, \beta, \gamma, \theta, 1\}$, where $0 \leq \alpha, \beta, \gamma, \theta \leq 1$. Defining operations $\odot$ and $\to$ as follows:

$\odot$	0	α	β	γ	θ	1
0	0	0	0	0	0	0
α	0	α	β	θ	θ	α
β	0	β	β	0	0	β
γ	0	θ	0	θ	θ	γ
θ	0	θ	0	θ	θ	θ
1	0	α	β	γ	θ	1

$\to$	0	α	β	γ	θ	1
0	1	1	1	1	1	1
α	0	1	β	γ	γ	1
β	γ	α	1	γ	γ	1
γ	β	α	β	1	α	1
θ	β	α	β	α	1	1
1	0	α	β	γ	θ	1

Then $(\mathcal{L}, \wedge, \odot, \to, 0, 1)$ is a semihoop. However, it is not a prelinear semihoop since 1 is not the unique upper bound of the set

$$\{(\gamma \to \theta), (\theta \to \gamma)\}.$$

Example 2.11. Let $\mathcal{L} = \{0, \alpha, \beta, \theta, 1\}$, where $0 \leq \alpha \leq \beta, \theta \leq 1$. Defining operations $\odot$ and $\to$ as follows:

$\odot$	0	α	β	θ	1
0	0	0	0	0	0
α	0	α	α	α	α
β	0	α	β	α	β
θ	0	α	α	θ	θ
1	0	α	β	θ	1

$\to$	0	α	β	θ	1
0	1	1	1	1	1
α	0	1	1	1	1
β	0	θ	1	θ	1
θ	0	β	β	1	1
1	0	α	β	θ	1

Then $(\mathcal{L}, \wedge, \odot, \to, 0, 1)$ is a prelinear semihoop.

Notice that the prelinearity does not necessitate the presence of a join operator in a semihoop $\mathcal{L}$. However, in the following, we will show that every prelinear semihoop has a lattice reduct whereby the join operation is definable in terms of the meet and the implication.

Proposition 2.12. *Let $\mathcal{L}$ be a prelinear semihoop. Then we have: for any $\sigma, \delta, \gamma \in \mathcal{L}$,*
(1) $\sigma \to (\delta \wedge \gamma) = (\sigma \to \delta) \wedge (\sigma \to \gamma)$,
(2) *$\mathcal{L}$ has a lattice reduct whereby* $\sigma \vee \delta = ((\sigma \to \delta) \to \delta) \wedge ((\delta \to \sigma) \to \sigma)$.

Proof. (1) By Proposition 2.4(6),(8) and (12), we have

$$\begin{aligned} \delta \to \gamma &= \delta \to (\delta \wedge \gamma) \\ &\leq (\sigma \to \delta) \to (\sigma \to (\delta \wedge \gamma)) \\ &\leq ((\sigma \to \delta) \wedge (\sigma \to \gamma)) \to (\sigma \to (\delta \wedge \gamma)). \end{aligned}$$

By similarity, we have

$$\gamma \to \delta \leq ((\sigma \to \delta) \wedge (\sigma \to \gamma)) \to (\sigma \to (\delta \wedge \gamma)).$$

Further by prelinearity, we have

$$((\sigma \to \delta) \wedge (\sigma \to \gamma)) \to (\sigma \to (\delta \wedge \gamma)) = 1,$$

which implies

$$(\sigma \to \delta) \wedge (\sigma \to \gamma) \leq \sigma \to (\delta \wedge \gamma).$$

The opposite inequality follows from Proposition 2.4(11).

(2) It is well known in [17] that every prelinear residuated $\wedge$-semilattice is a lattice, whereby the join operation is given by

$$\sigma \vee \delta = ((\sigma \to \delta) \to \delta) \wedge ((\delta \to \sigma) \to \sigma).$$

Notice that every prelinear semihoop is a prelinear residuated $\wedge$-semilattice. Hence the above result also holds in semihoops. $\square$

Theorem 2.13. *Let $\mathcal{L}$ be a semihoop. Then the following statements are equivalent: for any $\sigma, \delta, \gamma \in \mathcal{L}$,*

(1) *$\mathcal{L}$ is prelinear,*

(2) $\sigma \to (\delta \vee \gamma) = (\sigma \to \delta) \vee (\sigma \to \gamma)$,

(3) $\sigma \to \gamma \leq (\sigma \to \delta) \vee (\delta \to \gamma)$.

Proof. (1) $\Rightarrow$ (2) By Proposition 2.12(2), we get that every prelinear semihoop has a lattice structure, then it follows from Proposition 2.6(2), Proposition 2.4(6) and (12), we have

$$\begin{aligned} \delta \to \gamma &= (\delta \vee \gamma) \to \gamma \\ &\leq (\sigma \to (\delta \vee \gamma)) \to (\sigma \to \gamma) \\ &\leq (\sigma \to (\delta \vee \gamma)) \to ((\sigma \to \gamma) \vee (\sigma \to \delta)). \end{aligned}$$

By similarity, we have

$$\gamma \to \delta = (\gamma \vee \delta) \to \delta \leq (\sigma \to (\delta \vee \gamma)) \to ((\sigma \to \gamma) \vee (\sigma \to \delta)).$$

Hence by prelinearity, we have

$$(\sigma \to (\delta \vee \gamma)) \to ((\sigma \to \gamma) \vee (\sigma \to \delta)) = 1,$$

that is,

$$\sigma \to (\delta \vee \gamma) \leq (\sigma \to \delta) \vee (\sigma \to \gamma).$$

On the other hand, using Proposition 2.4(6),

$$\sigma \to \delta \leq \sigma \to (\delta \vee \gamma), \ \sigma \to \gamma \leq \sigma \to (\delta \vee \gamma).$$

Thus,

$$(\sigma \to \delta) \vee (\sigma \to \gamma) \leq \sigma \to (\delta \vee \gamma).$$

(2) $\Rightarrow$ (3) By Proposition 2.6(1), we have

$$\sigma \to \gamma \leq (\delta \vee \sigma) \to (\delta \vee \gamma) = ((\delta \vee \sigma) \to \delta) \vee ((\delta \vee \sigma) \to \gamma) \leq (\sigma \to \delta) \vee (\delta \to \gamma).$$

(3) $\Rightarrow$ (1) Taking $\sigma = \gamma$ in (3), we have

$$1 = \gamma \to \gamma \leq (\gamma \to \delta) \vee (\delta \to \gamma).$$

□

Proposition 2.14. *Every prelinear semihoop is a distributive lattice.*

Proof. By Proposition 2.6(2), we have

$$\delta \to \gamma = (\delta \vee \gamma) \to \gamma \leq (\sigma \wedge (\delta \vee \gamma)) \to (\sigma \wedge \gamma).$$

By similarity, we have

$$\gamma \to \delta = (\delta \vee \gamma) \to \delta \leq ((\sigma \wedge (\delta \vee \gamma)) \to (\sigma \wedge \delta).$$

Hence by prelinearity, we have

$$\begin{aligned} 1 &= (\delta \to \gamma) \vee (\gamma \to \delta) \\ &\leq ((\sigma \wedge (\delta \vee \gamma)) \to (\sigma \wedge \gamma)) \vee ((\sigma \wedge (\delta \vee \gamma)) \to (\sigma \wedge \delta)) \\ &= (\sigma \wedge (\delta \vee \gamma)) \to ((\sigma \wedge \delta) \vee (\sigma \wedge \gamma)), \end{aligned}$$

which implies

$$\sigma \wedge (\delta \vee \gamma) \leq (\sigma \wedge \delta) \vee (\sigma \wedge \gamma).$$

This proves distributivity, since the opposite inequality is always valid. □

Definition 2.15. ([3])Given ordered sets E, F and order-preserving maps $f : E \longrightarrow F$ and $g : F \longrightarrow E$, we say that the pair (f, g) establishes a Galois connection between E and F if $fg \geq id_F$ and $gf \leq id_E$.

§3 Multiplicative derivations of semihoops

In this section, we introduce some derivations in semihoops and give some characterizations of them. Then, we show that every Heyting algebra is isomorphic to the algebraic structure the set of all multiplicative derivations on Heyting algebras.

Definition 3.1. Let $\mathcal{L}$ be a semihoop. A map $d : \mathcal{L} \longrightarrow \mathcal{L}$ is called a multiplicative derivation on $\mathcal{L}$ if it satisfies the following condition: for any $\sigma, \delta \in \mathcal{L}$,

$$d(\sigma \odot \delta) = \sigma \odot d(\delta).$$

We denote by $D(\mathcal{L})$ the set of all multiplicative derivations of $\mathcal{L}$.

Now, we will present some examples for multiplicative derivations on semihoops.

Example 3.2. Let $\mathcal{L}$ be a bounded semihoop. Defining a map $d_0 : \mathcal{L} \to \mathcal{L}$ by $d_0(\sigma) = 0$ for all $\sigma \in \mathcal{L}$. Then d_0 is a multiplicative derivation on $\mathcal{L}$, which is called the zero multiplicative derivation. Moreover, we define a map $d_1 : \mathcal{L} \to \mathcal{L}$ by $d_1(\sigma) = \sigma$ for all $\sigma \in \mathcal{L}$. Then d_1 is also a multiplicative derivation on $\mathcal{L}$, which is called the identity multiplicative derivation.

Example 3.3. Let $\mathcal{L} = \{0, \alpha, \beta, \gamma, 1\}$, where $0 \leq \alpha \leq \beta \leq \gamma \leq 1$. Defining operations $\odot$ and $\to$ as follows:

$\odot$	0	α	β	γ	1
0	0	0	0	0	0
α	0	α	0	α	α
β	0	0	β	β	β
γ	0	α	β	γ	γ
1	0	α	β	γ	1

$\to$	0	α	β	γ	1
0	1	1	1	1	1
α	β	1	1	1	1
β	α	α	1	1	1
γ	0	α	β	1	1
1	0	α	β	γ	1

Then $(\mathcal{L}, \wedge, \odot, \to, 0, 1)$ is a bounded semihoop. Now we define $d : \mathcal{L} \to \mathcal{L}$ as follows:

$$d(\sigma) = \begin{cases} 0, & \sigma = 0, \beta \\ \alpha, & \sigma = \alpha, \gamma, 1 \end{cases}$$

It is verified that d is a multiplicative derivation on $\mathcal{L}$.

Example 3.4. Let $(G, +, -, \wedge, \vee)$ be an arbitrary ℓ-group. For an arbitrary element $u \in G$, $u \geq 0$ defined on the set $G[u] = [0, u]$ the following operations:

$$\alpha \odot \beta = (\alpha - u + \beta) \vee 0, \ \alpha \to \beta = (\beta - \alpha + u) \wedge u.$$

Then $(G[u], \wedge, \odot, \to, 0, u)$ is a bounded semihoop. Now, we define a map $d : G[u] \to G[u]$ as follows: for all $\sigma \in [0, u]$, and $\beta \in [0, u]$

$$d(\sigma) = \begin{cases} \beta, & \sigma = u \\ \beta \odot \sigma, & \sigma \neq u \end{cases}$$

It is easily verified that d is a multiplicative derivation on $G[u]$.

Proposition 3.5. *Let $\mathcal{L}$ be a semihoop and d be a multiplicative derivation on $\mathcal{L}$. Then we have: for any $\sigma, \delta \in \mathcal{L}$,*

(1) *if $\mathcal{L}$ is bounded, then $d(0) = 0$,*
(2) *$d(\sigma) = \sigma \odot d(1)$,*
(3) *$d(\sigma) \leq \sigma$,*
(4) *$d(\sigma) \odot \delta = \sigma \odot d(\delta)$,*
(5) *if $\sigma \leq \delta$, then $d(\sigma) \leq d(\delta)$,*
(6) *$d(\sigma) \odot d(\delta) \leq d(\sigma \odot \delta)$,*
(7) *$d(\sigma \to \delta) \leq d(\sigma) \to d(\delta)$,*
(8) *if $\mathcal{L}$ is a hoop, then $d(\sigma \wedge \delta) = d(\sigma) \wedge d(\delta)$.*

Proof. (1) Taking $\sigma = \delta = 0$ in Definition 3.1, we have

$$d(0) = d(0 \odot 0) = 0 \odot d(0) = 0.$$

(2) Taking $\delta = 1$ in Definition 3.1, we have

$$d(\sigma) = d(1 \odot \sigma) = \sigma \odot d(1).$$

(3) It follows from (2) that $d(\sigma) = \sigma \odot d(1) \leq \sigma$.
(4) From (2), we have

$$d(\sigma) \odot \delta = \sigma \odot d(1) \odot \delta = \sigma \odot (\delta \odot d(1)) = \sigma \odot d(\delta).$$

(5) If $\sigma \leq \delta$, then it follows from (2) that

$$d(\sigma) = \sigma \odot d(1) \leq \delta \odot d(1) = d(\delta).$$

(6) From (2), we have

$$d(\sigma) \odot d(\delta) = \sigma \odot d(1) \odot \delta \odot d(1) \leq (\sigma \odot \delta) \odot d(1) = d(\sigma \odot \delta).$$

(7) From Proposition 2.4(9), we have

$$d(\sigma \to \delta) = (\sigma \to \delta) \odot d(1) \leq (\sigma \odot d(1)) \to (\delta \odot d(1)) = d(\sigma) \to d(\delta).$$

(8) If $\mathcal{L}$ is a hoop, then it follows from (2) and Proposition 2.4(14) that

$$d(\sigma \wedge \delta) = (\sigma \wedge \delta) \odot d(1) = (\sigma \odot d(1)) \wedge (\delta \odot d(1)) = d(\sigma) \wedge d(\delta).$$

The following example shows that the converse of Proposition 3.5(6) and (7) are not true in general.

Example 3.6. Let $\mathcal{L}$ be the bounded semihoop in Example 2.10. Now we define $d : \mathcal{L} \to \mathcal{L}$ as follows:

$$d(\sigma) = \begin{cases} 0, & \sigma = 0, \beta \\ \theta, & \sigma = \alpha, \gamma, \theta \\ \gamma, & \sigma = 1 \end{cases}$$

It is verified that d is a multiplicative derivation on $\mathcal{L}$. Since

$$d(1) \odot d(1) = \gamma \odot \gamma = \theta \ngeq \gamma = d(1) = d(1 \odot 1),$$

which shows that the converse of Proposition 3.5(6) does not hold in general.

Example 3.7. Let $\mathcal{L}$ be the bounded semihoop and d be the multiplicative derivation on $\mathcal{L}$ in Example 3.3. Since

$$d(\alpha \to \gamma) = d(1) = \alpha \ngeq 1 = \alpha \to \alpha = d(\alpha) \to d(\gamma),$$

which shows that the converse of Proposition 3.5(7) does not hold in general.

The following example shows that the condition in Proposition 3.5(8) is necessary.

Example 3.8. Let $\mathcal{L}$ be the bounded semihoop and d be a multiplicative derivation on $\mathcal{L}$ in Example 3.3. Since

$$\alpha = d(\alpha) = d(\alpha \wedge \beta) \neq d(\alpha) \wedge d(\beta) = \alpha \wedge 0 = 0,$$

which shows that Proposition 3.5(8) does not hold in semihoops in general.

Proposition 3.9. *Let $\mathcal{L}$ be a semihoop and $d : \mathcal{L} \to \mathcal{L}$ be a map on $\mathcal{L}$. Then the following statements are equivalent: for any $\sigma, \delta \in \mathcal{L}$,*

(1) *d is a multiplicative derivation on $\mathcal{L}$,*

(2) *$d(\sigma \odot \delta) = d(\sigma) \odot \delta$,*

(3) *$d(\sigma) = d(1) \odot \sigma$.*

Proof. (1) $\Rightarrow$ (2) By Definition 3.1 and Proposition 3.5(4), we have $d(\sigma\odot\delta) = \sigma\odot d(\delta) = d(\sigma)\odot\delta$.

(2) $\Rightarrow$ (3) Taking $\delta = \sigma$ and $\sigma = 1$ in (2).

(3) $\Rightarrow$ (1) From (3), we have

$$d(\sigma \odot \delta) = d(1) \odot (\sigma \odot \delta) = \sigma \odot (d(1) \odot \delta) = \sigma \odot d(\delta).$$

Remark 3.10. He introduced in [13] that a multiplicative derivation on a residuated lattice as a map $d : \mathcal{L} \to \mathcal{L}$ satisfies the following condition:

$$\text{(DR)} \quad d(\sigma \odot \delta) = (d(\sigma) \odot \delta) \vee (\sigma \odot d(\delta)).$$

Proposition 3.9 shows that a map $d : \mathcal{L} \to \mathcal{L}$ is a multiplicative derivation if and only if

$$\text{(DS)} \quad d(\sigma \odot \delta) = \sigma \odot d(\delta) = d(\sigma) \odot \delta.$$

In the case of residuated lattice, the condition (DS) implies the validity of (DR). Hence the notion of a multiplicative derivation on semihoops essentially generalized that of a multiplicative derivation on residuated lattices.

The following example shows that the condition (DR) does not imply (DS).

Example 3.11. Let $\mathcal{L} = \{0, \alpha, \beta, 1\}$ be a chain and operations $\odot$ and $\to$ be defined as follows:

$\odot$	0	α	β	1
0	0	0	0	0
α	0	0	α	α
β	0	α	β	β
1	0	α	β	1

$\to$	0	α	β	1
0	1	1	1	1
α	α	1	1	1
β	0	α	1	1
1	0	α	β	1

Then $(\mathcal{L}, \wedge, \vee, \odot, \to, 0, 1)$ is a residuated lattice, where $\sigma \wedge \delta = min\{\sigma, \delta\}$ and $\sigma \vee \delta = max\{\sigma, \delta\}$ for all $\sigma, \delta \in \mathcal{L}$. Now, we define a map $d : \mathcal{L} \to \mathcal{L}$ as follows:

$$d(\sigma) = \begin{cases} 0, & \sigma = 0, \\ \alpha, & \sigma = \alpha, \beta, 1 \end{cases}$$

One can check that d is a derivation on $(\mathcal{L}, \wedge, \vee, \odot, \to, 0, 1)$. However, it is not a derivations on semihoops, since $\alpha = d(\alpha) = d(\alpha \odot \beta) \neq \alpha \odot d(\beta) = \alpha \odot \alpha = 0$.

Then we introduce the notion of idempotent multiplicative derivation in semihoops and give some characterization of Brouwerian semilattices by them.

Definition 3.12. A multiplicative derivation d on a semihoop is called an idempotent multiplicative derivation provided that $d(1) \in I(\mathcal{L})$, that is $d(1) \odot d(1) = d(1)$.

Example 3.13. The multiplicative derivation d in Example 3.3 is idempotent. However, the multiplicative derivation d in Example 3.4 is not idempotent since $d(u) = \beta \notin I(\mathcal{L})$.

Proposition 3.14. *Let $\mathcal{L}$ be a semihoop and d be an idempotent multiplicative derivation on $\mathcal{L}$. Then we have: for any $\sigma, \delta \in \mathcal{L}$,*

(1) $d(d(\sigma)) = d(\sigma)$,

(2) $d(\sigma \wedge \delta) = d(\sigma) \wedge d(\delta)$,

(3) $d(\sigma \odot \delta) = d(\sigma) \odot d(\delta)$,

(4) $Fix_d(\mathcal{L}) = d(\mathcal{L})$, *where* $Fix_d(\mathcal{L}) = \{\sigma \in \mathcal{L} | d(\sigma) = \sigma\}$,

(5) $d(\mathcal{L}) = \mathcal{L}$ iff $d = d_1$, where d_1 is the multiplicative derivation in Example 3.2,
(6) $d(\sigma) \leq \delta$ iff $d(\sigma) \leq d(\delta)$.

Proof. (1) From Proposition 3.5(2) and $d(1) \in I(\mathcal{L})$, we have

$$d(d(\sigma)) = d(d(1) \odot \sigma) = d(1) \odot d(1) \odot \sigma = d(1) \odot \sigma = d(\sigma).$$

(2) From Proposition 3.5(2), we have

$$\begin{aligned} d(\sigma \wedge \delta) &= d(1) \odot (\sigma \wedge \delta) \\ &= d(1) \wedge (\sigma \wedge \delta) \\ &= (d(1) \wedge \sigma) \wedge (d(1) \wedge \delta) \\ &= (d(1) \odot \sigma) \wedge (d(1) \odot \delta) \\ &= d(\sigma) \wedge d(\delta). \end{aligned}$$

(3) From Proposition 3.5(2), we have

$$\begin{aligned} d(\sigma \odot \delta) &= d(1) \odot (\sigma \odot \delta) \\ &= d(1) \odot d(1) \odot (\sigma \odot \delta) \\ &= (d(1) \odot \sigma) \odot (d(1) \odot \delta) \\ &= d(\sigma) \odot d(\delta). \end{aligned}$$

(4) Let $\delta \in d(\mathcal{L})$. Then there exists $\sigma \in \mathcal{L}$ such that $\delta = d(\sigma)$. Hence by (1), $d(\delta) = d(d(\sigma)) = d(\sigma) = \delta$. It follows that $\delta \in Fix_d(\mathcal{L})$. Conversely, if $\delta \in Fix_d(\mathcal{L})$, we have $\delta \in d(\mathcal{L})$. Therefore, $d(\mathcal{L}) = Fix_d(\mathcal{L})$.

(5) ($\Rightarrow$) Suppose that $d(\mathcal{L}) = \mathcal{L}$, then for any $\sigma \in \mathcal{L}$, there exists a $\delta \in \mathcal{L}$ such that $d(\delta) = \sigma$. Hence $d(\sigma) = d(d(\delta)) = d(\delta) = \sigma$, that is $d(\sigma) = \sigma$. Therefore, $d = d_1$.

($\Leftarrow$) Suppose that $d = d_1$, that is for any $\sigma \in \mathcal{L}$, $d(\sigma) = \sigma$. Hence $d(\mathcal{L}) = \mathcal{L}$.

(6) By Proposition 3.9(3) and d be an idempotent multiplicative derivation , we have $d(\sigma) \leq \delta$ iff $d(1) \odot \sigma \leq \delta$ iff $d(1) \odot d(1) \odot \sigma \leq d(1) \odot \delta$ iff $d(1) \odot \sigma \leq d(1) \odot \delta$ iff $d(\sigma) \leq d(\delta)$. □

Theorem 3.15. *Let $\mathcal{L}$ be a semihoop and d be a multiplicative derivation on $\mathcal{L}$. Then the following statements are equivalent:*

(1) *$\mathcal{L}$ is a Brouwerian semilattice,*
(2) *every multiplicative derivation d on $\mathcal{L}$ satisfies $d(\sigma) \odot d(\sigma) = d(\sigma)$ for any $\sigma \in \mathcal{L}$,*
(3) *every multiplicative derivation d on $\mathcal{L}$ satisfies*

$$d(\sigma \wedge \delta) = d(\sigma) \odot d(\delta) = d(\sigma) \odot (d(\sigma) \to d(\delta))$$

for any $\sigma, \delta \in \mathcal{L}$.

Proof. (1) $\Rightarrow$ (2) It is noted that any Brouwerian semilattice is equivalent to the idempotent semihoop. In this case, by Proposition 3.5(2), we have $d(\sigma) \odot d(\sigma) = d(1) \odot \sigma \odot d(1) \odot \sigma = d(1) \odot \sigma = d(\sigma)$.

(2) $\Rightarrow$ (3) By Propositions 2.4(15)(i), 3.14(2) and (3), we can get

$$d(\sigma \wedge \delta) = d(\sigma \odot \delta) = d(\sigma) \odot d(\delta) = d(\sigma) \odot (d(\sigma) \to d(\delta)).$$

(3) $\Rightarrow$ (1) If every multiplicative derivation d on $\mathcal{L}$ satisfies $d(\sigma \wedge \delta) = d(\sigma) \odot d(\delta)$ for any $\sigma, \delta \in \mathcal{L}$, then taking $d = d_1$, we have $\sigma \wedge \delta = \sigma \odot \delta$ for any $\sigma, \delta \in \mathcal{L}$, which implies that $\mathcal{L}$ is a Brouwerian semilattice. □

In what follows, we focus on algebraic structure of the set of all multiplicative derivations.

Proposition 3.16. *Let $\mathcal{L}$ be an $R\ell$-monoid. Then $(D(\mathcal{L}), \sqcap, \sqcup, d_0, d_1)$ is a Heyting algebras, where*

$$(d_i \sqcap d_j)(\sigma) = d_i(\sigma) \wedge d_j(\sigma),$$
$$(d_i \sqcup d_j)(\sigma) = d_i(\sigma) \vee d_j(\sigma),$$
$$(d_i \mapsto d_j)(\sigma) = \sqcup\{d | d_i \sqcap d \leq d_j\}(\sigma),$$

for any $d_i, d_j \in D(\mathcal{L})$ and $\sigma \in \mathcal{L}$.

Proof. For any d_i, $d_j \in D(\mathcal{L})$ and $\sigma \in \mathcal{L}$, by Proposition 2.4(14), we have

$$\begin{aligned}(d_i \sqcap d_j)(\sigma \odot \delta) &= d_i(\sigma \odot \delta) \wedge d_j(\sigma \odot \delta) \\ &= (\sigma \odot d_i(\delta)) \wedge (\sigma \odot d_j(\delta)) \\ &= \sigma \odot (d_i(\delta) \wedge d_j(\delta)) \\ &= \sigma \odot (d_i \sqcap d_j)(\delta),\end{aligned}$$

and by Proposition 2.6(3), we have

$$\begin{aligned}(d_i \sqcup d_j)(\sigma \odot \delta) &= d_i(\sigma \odot \delta) \vee d_j(\sigma \odot \delta) \\ &= (\sigma \odot d_i(\delta)) \vee (\sigma \odot d_j(\delta)) \\ &= \sigma \odot (d_i(\delta) \vee d_j(\delta)) \\ &= \sigma \odot (d_i \sqcup d_j)(\delta),\end{aligned}$$

which implies $d_i \sqcap d_j, d_i \sqcup d_j \in D(\mathcal{L})$.

Also, for any $d_i \in D(\mathcal{L})$ and $\sigma \in \mathcal{L}$, we have

$$\begin{aligned}(d_i \sqcap d_0)(\sigma) &= d_i(\sigma) \wedge d_0(\sigma) \\ &= d_i(\sigma) \wedge 0 \\ &= d_0(\sigma), \\ (d_i \sqcup d_1)(\sigma) &= d_i(\sigma) \vee d_1(\sigma) \\ &= d_i(\sigma) \vee 1 \\ &= d_1(\sigma),\end{aligned}$$

which implies $d_i \sqcap d_0 = d_0$ and $d_i \sqcup d_1 = d_1$.

Then we will prove that $(D(\mathcal{L}), \sqcap, \sqcup, d_0, d_1)$ is a bounded distributive lattice. In particular, for any $d_i, d_j, d_k \in D(\mathcal{L})$ and $\sigma \in \mathcal{L}$, by the distributivity of Rℓ-monoid, we have

$$\begin{aligned}(d_i \sqcap (d_j \sqcup d_k))(\sigma) &= d_i(\sigma) \wedge (d_j(\sigma) \vee d_k(\sigma)) \\ &= (d_i(\sigma) \wedge d_j(\sigma)) \vee (d_i(\sigma) \wedge d_k(\sigma)) \\ &= ((d_i \sqcap d_j) \sqcup (d_i \sqcap d_k))(\sigma)\end{aligned}$$

which implies $d_i \sqcap (d_j \sqcup d_k) = (d_i \sqcap d_j) \sqcup (d_i \sqcap d_k)$. Similaly, one can prove $d_i \sqcup (d_j \sqcap d_k) = (d_i \sqcup d_j) \sqcap (d_i \sqcup d_k)$.

Moreover, by $D(\mathcal{L})$ is closed under $\sqcap, \sqcup$, we obtain that $d_i \mapsto d_j$ is well defined. Then we will show that

$$d_i \sqcap d \leq d_j \text{ if and only if } d_i \leq d \mapsto d_j.$$

In particular, if $d_i \sqcap d \leq d_j$, that is $d_i(\sigma) \wedge d(\sigma) \leq d_j(\sigma)$ for any $\sigma \in L$, which implies that $d \in \{f | d_i \sqcap f \leq d_j\}$. So $d \leq \sqcup\{f | d_i \sqcap f \leq d_j\}$, that is, $d_i \leq d \mapsto d_j$. Conversely, if $d_i \leq d \mapsto d_j$, then $d_i(\sigma) \leq d \mapsto d_j(\sigma)$ for any $\sigma \in L$. So $d_i \leq \sqcup\{f | d \sqcap f \leq d_j\}$, which implies $d_i \sqcap d \leq d_j$.

Thus $(D(\mathcal{L}), \sqcap, \sqcup, \mapsto, d_0, d_1)$ is a Heyting algebra. □

Theorem 3.17. *Let $(\mathcal{L}, \wedge, \vee, \rightarrow, 0, 1)$ be a Heyting algebra. Then Heyting algebra $(\mathcal{L}, \wedge, \vee, \rightarrow, 0, 1)$ is isomorphic to $(D(\mathcal{L}), \sqcap, \sqcup, \mapsto, d_0, d_1)$.*

Proof. Notice that each Heyting algebra is an idempotent Rℓ-monoid [4]. Then it follows from Proposition 3.16 that $(D(\mathcal{L}), \sqcap, \sqcup, \mapsto, d_0, d_1)$ is a Heyting algebra.

Let $\lambda : \mathcal{L} \rightarrow D(\mathcal{L})$ be defined by

$$\lambda(a)(\sigma) = a \wedge \sigma,$$

for a given $a \in \mathcal{L}$ and any $\sigma \in \mathcal{L}$. Then it follows from Proposition 3.9 (1) $\Leftrightarrow$ (3) that λ is well defined.

(1) If $\lambda(a) = \lambda(b)$, then $\lambda(a)(\sigma) = \lambda(b)(\sigma)$, and hence $a \wedge \sigma = b \wedge \sigma$ for all $\sigma \in \mathcal{L}$. Now, if $\sigma = a$, then $a = a \wedge a = a \wedge b$, that is, $a \leq b$. If $\sigma = b$, then $a \wedge b = b \wedge b = b$, and hence $a \wedge b = b$, that is, $b \leq a$. So $a = b$, which shows that λ is an injective function.

(2) For any $d \in D(\mathcal{L})$, there exists a $d(1) \in \mathcal{L}$ such that $d = \lambda(d(1))$, which implies that λ is a surjective function. Indeed, by Proposition 3.9(3), we have

$$d(\sigma) = d(1) \odot \sigma = d(1) \wedge \sigma = \lambda(d(1))(\sigma)$$

for any $\sigma \in \mathcal{L}$.

(3) For any $a, b \in \mathcal{L}$, we have

$$\begin{aligned}\lambda(a \wedge b)(\sigma) &= (a \wedge b) \wedge \sigma = (a \wedge \sigma) \wedge (b \wedge \sigma) = (\lambda(a) \sqcap \lambda(b))(\sigma),\\ \lambda(a \vee b)(\sigma) &= (a \vee b) \wedge \sigma = (a \wedge \sigma) \vee (b \wedge \sigma) = (\lambda(a) \sqcup \lambda(b))(\sigma),\\ \lambda(a \rightarrow b)(\sigma) &= (a \rightarrow b) \wedge \sigma = \vee\{\lambda(\sigma) | \lambda_a(\sigma) \wedge \lambda(\sigma) \leq \lambda_b(\sigma)\} = (\lambda_a \mapsto \lambda_b)(\sigma),\end{aligned}$$

which implies that λ is a homomorphism.

Therefore $(\mathcal{L}, \wedge, \vee, \rightarrow, 0, 1)$ is isomorphic to $(D(\mathcal{L}), \sqcap, \sqcup, \mapsto, d_0, d_1)$. □

§4 Implicative derivations of semihoops

In this section, we introduce implicative derivations in semihoops and give some characterizations of them. Then, we show that every Boolean algebra is isomorphic to the algebraic structure the set of all implicative derivations on Boolean algebra and discuss the relation between multiplicative and implicative derivations on semihoops.

Definition 4.1. Let $\mathcal{L}$ be a semihoop. A map $g : \mathcal{L} \longrightarrow \mathcal{L}$ is called an implicative derivation on $\mathcal{L}$ if it satisfies the following condition: for any $\sigma, \delta \in \mathcal{L}$,

$$g(\sigma \rightarrow \delta) = \sigma \rightarrow g(\delta).$$

We will denote $G(\mathcal{L})$ as the set of all implicative derivations of $\mathcal{L}$.

Now, we present some examples of implicative derivations on semihoops.

Example 4.2. Let $\mathcal{L}$ be a semihoop. Defining a map $1_g : \mathcal{L} \to \mathcal{L}$ by $1_g(\sigma) = 1$ for all $\sigma \in \mathcal{L}$. Then 1_g is an implicative derivation on $\mathcal{L}$, which is called the one implicative derivation. Moreover, we define a map $g_1 : \mathcal{L} \to \mathcal{L}$ by $g_1(\sigma) = \sigma$ for all $\sigma \in \mathcal{L}$. Then g_1 is also an implicative derivation on $\mathcal{L}$, which is called the identity implicative derivation.

Example 4.3. Let $\mathcal{L}$ be the bounded semihoop in Example 3.3. Now, we define $g : \mathcal{L} \to \mathcal{L}$ as follows:

$$g(\sigma) = \begin{cases} 0, & \sigma = 0, \\ \alpha, & \sigma = \alpha, \\ \beta, & \sigma = \beta, \\ 1, & \sigma = \gamma, 1 \end{cases}$$

It is verified that g is an implicative derivation on $\mathcal{L}$.

Example 4.4. Let $\mathcal{L}$ be the bounded semihoop in Example 3.4. Now, we define a map $g : G[u] \to G[u]$ as follows: for all $\sigma \in [0, u]$, and $\beta \in [0, u]$

$$g(\sigma) = \begin{cases} u, & \sigma = u \\ \beta \to \sigma, & \sigma \neq u \end{cases}$$

It is easily verified that g is an implicative derivation on $\mathcal{L}$.

Proposition 4.5. *Let $\mathcal{L}$ be a bounded semihoop and g be an implicative derivation on $\mathcal{L}$. Then we have: for any $\sigma, \delta \in \mathcal{L}$,*

(1) $g(1) = 1$,
(2) $\sigma \leq g(\sigma)$,
(3) *if* $\sigma \leq \delta$, *then* $\sigma \leq g(\delta)$,
(4) $g(\sigma) \to \delta \leq \sigma \to g(\delta)$,
(5) *if $\mathcal{L}$ is a residuated lattice, then* $g(\sigma \to \delta) = (g(\sigma) \to \delta) \vee (\sigma \to g(\delta))$.

Proof. (1) Taking $\sigma = 0$ in Definition 4.1, we have

$$g(1) = g(0 \to \delta) = 0 \to g(\delta) = 1.$$

(2) Taking $\delta = \sigma$ in Definition 4.1, we have

$$1 = g(1) = g(\sigma \to \sigma) = \sigma \to g(\sigma),$$

which implies $\sigma \leq g(\sigma)$.

(3) If $\sigma \leq \delta$, then

$$1 = g(1) = g(\sigma \to \delta) = \sigma \to g(\delta)$$

which implies $\sigma \leq g(\delta)$.

(4) It follows from (2) and Proposition 2.4(6).

(5) From (4), we have

$$g(\sigma \to \delta) = \sigma \to g(\delta) = (g(\sigma) \to \delta) \vee (\sigma \to g(\delta)).$$

□

The following example shows that the converse of Proposition 4.5(4) is not true in general.

Example 4.6. Let $\mathcal{L}$ be the bounded semihoop and g be the implication derivation on $\mathcal{L}$ in Example 4.3. Since

$$g(\gamma) \to \gamma = 1 \to \gamma = \gamma \not\geq 1 = \gamma \to 1 = \gamma \to g(\gamma),$$

which shows that the converse of Proposition 4.5(4) does not hold in general.

Remark 4.7. He introduced in [13] that an implicative derivation on a residuated lattice as a map $g : L \to L$ satisfies the Proposition 4.5(5). Proposition 4.5(5) shows that the notion of an implicative derivation on semihoops essentially generalized that of an implicative derivation on residuated lattices.

Then we introduce the notion of regular implicative derivation in bounded semihoops and give some characterization of them.

Definition 4.8. An implicative derivation g on a bounded semihoop is called a regular implicative derivation provided that $g(\sigma) = g(\neg\neg\sigma)$ for any $\sigma \in \mathcal{L}$.

Example 4.9. The implicative derivation g in Example 4.3 is regular.

Example 4.10. Let $\mathcal{L}$ be the bounded semihoop in Example 3.3. Now, we define $g : \mathcal{L} \to \mathcal{L}$ as follow:

$$g(\sigma) = \begin{cases} 0, & \sigma = 0 \\ \alpha, & \sigma = \alpha \\ \beta, & \sigma = \beta, \gamma \\ 1, & \sigma = 1. \end{cases}$$

It is verified that g is an implicative derivation on $\mathcal{L}$, but it is not regular, since $g(\gamma) = \beta \neq 1 = \neg\gamma \to g(0)$.

Proposition 4.11. *Let $\mathcal{L}$ be a bounded semihoop and g be a regular implicative derivation on $\mathcal{L}$. Then we have: for any $\sigma, \delta \in \mathcal{L}$,*

(1) $g(\sigma) = \neg\sigma \to g(0)$,

(2) *if $\sigma \leq \delta$, then $g(\sigma) \leq g(\delta)$.*

Proof. (1) For any $\sigma \in \mathcal{L}$, we have

$$g(\sigma) = g(\neg\neg\sigma) = g(\neg\sigma \to 0) = \neg\sigma \to g(0).$$

(2) If $\sigma \leq \delta$, by Proposition 2.4(6), then $g(\sigma) = \neg\sigma \to g(0) \leq \neg\delta \to g(0) = g(\delta)$. □

Theorem 4.12. *Let $\mathcal{L}$ be a bounded semihoop and g be an implicative derivation on $\mathcal{L}$. Then the following statements are equivalent:*

(1) *$\mathcal{L}$ is regular,*

(2) *every implicative derivation is regular,*

(3) *$g(\sigma) = \neg\sigma \to g(0)$ for any $\sigma \in \mathcal{L}$.*

Proof. (1) ⇒ (2) Clearly. □

(2) $\Rightarrow$ (3) It follows from Proposition 4.11(1).

(3) $\Rightarrow$ (1) Taking $g = g_1$ in (3), we have

$$\sigma = g_1(\sigma) = \neg\sigma \to g_1(0) = \neg\sigma \to 0 = \neg\neg\sigma,$$

which implies $\sigma = \neg\neg\sigma$ for any $\sigma \in \mathcal{L}$.

Remark 4.13. Theorem 4.12 shows that every implicative derivation g is completely determined by the element $g(0)$ on the regular bounded semihoops. However, it does not hold in bounded semihoops in general, see Example 4.10.

We also focus on algebraic structure of the set of all implicative derivations.

Theorem 4.14. *Let $\mathcal{L}$ be a bounded prelinear semihoop. Then $(G(\mathcal{L}), \cap, \cup, g_1, 1_g)$ is a bounded distributive lattice, where*

$$(g_i \cap g_j)(\sigma) = g_i(\sigma) \wedge g_j(\sigma),$$
$$(g_i \cup g_j)(\sigma) = g_i(\sigma) \vee g_j(\sigma).$$

for any $g_i, g_j \in G(\mathcal{L})$, and $\sigma \in \mathcal{L}$.

Proof. For any $g_i, g_j \in G(\mathcal{L})$ and $\sigma \in \mathcal{L}$, by Propositions 2.12(1) and Theorem 2.13(2), we have

$$\begin{aligned}(g_i \cap g_j)(\sigma \to \delta) &= g_i(\sigma \to \delta) \wedge g_j(\sigma \to \delta)\\ &= (\sigma \to g_i(\delta)) \wedge (\sigma \to g_j(\delta))\\ &= \sigma \to (g_i(\delta) \wedge g_j(\delta))\\ &= \sigma \to (g_i \cap g_j)(\delta),\end{aligned}$$

and

$$\begin{aligned}(g_i \cup g_j)(\sigma \to \delta) &= g_i(\sigma \to \delta) \vee g_j(\sigma \to \delta)\\ &= (\sigma \to g_i(\delta)) \vee (\sigma \to g_j(\delta))\\ &= \sigma \to (g_i(\delta) \vee g_j(\delta))\\ &= \sigma \to (g_i \cup g_j)(\delta),\end{aligned}$$

which implies $g_i \cap g_j, g_i \cup g_j \in G(\mathcal{L})$.

Also, for any $g_i \in G(\mathcal{L})$ and $\sigma \in \mathcal{L}$, we have

$$(g_i \cap 1_g)(\sigma) = g_i(\sigma) \wedge 1_g(\sigma) = g_i(\sigma) \wedge 1 = g_i(\sigma),$$
$$(g_i \cup 1_g)(\sigma) = g_i(\sigma) \vee 1_g(\sigma) = g_i(\sigma) \vee 1 = 1_g(\sigma)$$

which implies $g_i \cap 1_g = g_i, g_i \cup 1_g = 1_g$.

It is easily verified that $(G(\mathcal{L}), \cap, \cup, g_1, 1_g)$ is a bounded distributive lattice by Proposition 2.14. $\square$

Theorem 4.15. *Let $(\mathcal{L}, \wedge, \odot, \to, 0, 1)$ be a bounded idempotent prelinear semihoop. Then $(G(\mathcal{L}), \cap, \cup, \Rightarrow, g_1, 1_g)$ is also a bounded idempotent prelinear semihoop, where*

$$(g_i \cap g_j)(\sigma) = g_i(\sigma) \wedge g_j(\sigma),$$
$$(g_i \cup g_j)(\sigma) = g_i(\sigma) \vee g_j(\sigma),$$
$$(g_i \Rightarrow g_j)(\sigma) = g_i(\sigma) \to g_j(\sigma).$$

for any $g_i, g_j \in G(\mathcal{L})$, and $\sigma \in \mathcal{L}$.

Proof. Theorem 4.14 shows that $(G(\mathcal{L}), \cap, \cup, g_1, 1_g)$ is a bounded distributive lattice if $\mathcal{L}$ is a bounded prelinear semihoop. Now, we prove that $(G(\mathcal{L}), \cap, \cup, \Rightarrow, g_1, 1_g)$ is a bounded idempotent prelinear semihoop if $\mathcal{L}$ is a bounded idempotent prelinear semihoop. Indeed, for any $g_i, g_j \in G(\mathcal{L})$ and $\sigma \in \mathcal{L}$, by Proposition 2.4(15)(ii), we have

$$\begin{aligned}(g_i \Rightarrow g_j)(\sigma \to \delta) &= g_i(\sigma \to \delta) \to g_j(\sigma \to \delta)\\ &= (\sigma \to g_i(\delta)) \to (\sigma \to g_j(\delta))\\ &= \sigma \to (g_i(\delta) \to g_j(\delta))\\ &= \sigma \to (g_i \Rightarrow g_j)(\delta),\end{aligned}$$

which implies $g_i \Rightarrow g_j \in G(\mathcal{L})$.

Therefore $(G(\mathcal{L}), \cap, \cup, \Rightarrow, g_1, 1_g)$ is a bounded idempotent prelinear semihoop. □

Inspired by Theorem 4.15, it is natural to ask that whether there exists a function such that bounded idempotent prelinear semihoops $(\mathcal{L}, \wedge, \vee, \to, 0, 1)$ and $(G(\mathcal{L}), \cap, \cup, \Rightarrow, g_1, 1_g)$ are isomorphism. For this question, we give the positive answer under certain conditions in Boolean algebras.

Theorem 4.16. *Let $(\mathcal{L}, \wedge, \vee, \neg, 0, 1)$ be a Boolean algebra. Then Boolean algebras $(\mathcal{L}, \wedge, \vee, \neg, 0, 1)$ and $(G(\mathcal{L}), \cap, \cup, \star, g_1, 1_g)$ are isomorphism, where $(g_i)^\star(\sigma) = (g_i \Rightarrow g_1)(\sigma)$ for any $g_i \in G(\mathcal{L})$ and $\sigma \in \mathcal{L}$.*

Proof. Notice that each Boolean algebra is a bounded idempotent prelinear semihoop. Then it follows from Theorem 4.15 that $(G(\mathcal{L}), \cap, \cup, \Rightarrow, g_1, 1_g)$ is a bounded idempotent prelinear semihoop, and hence is a bounded distributive lattice. Now, we further prove that is a Boolean algebra. Indeed, for any $g_i \in G(\mathcal{L})$, and $\sigma \in \mathcal{L}$, we have

$$(g_i \cap (g_i)^\star)(\sigma) = g_i(\sigma) \wedge (g_i(\sigma))^\star = g_i(\sigma) \wedge (g_i(\sigma) \to g_1(\sigma)) = g_i(\sigma) \wedge g_1(\sigma) = g_1(\sigma),$$
$$(g_i \cup (g_i)^\star)(\sigma) = g_i(\sigma) \vee (g_i(\sigma) \to g_1(\sigma)) = g_i(\sigma) \vee (g_i(\sigma))^\star \vee g_1(\sigma) = 1 \vee g_1(\sigma) = 1 = 1_g(\sigma),$$

which implies $g_i \cap (g_i)^\star = g_1$ and $g_i \cup (g_i)^\star = 1_g$.

Let $\chi : \mathcal{L} \to G(\mathcal{L})$ be defined by

$$\chi(a)(\sigma) = a \vee \sigma,$$

for a given $a \in \mathcal{L}$ and any $\sigma \in \mathcal{L}$. Then it follows from Theorem 4.12 (1) $\Leftrightarrow$ (3) that χ is well defined.

(1) If $\chi(a) = \chi(b)$, then $\chi(a)(\sigma) = \chi(b)(\sigma)$, and hence $a \vee \sigma = b \vee \sigma$ for all $\sigma \in \mathcal{L}$. Now, if $\sigma = a$, then $a = a \vee a = a \vee b$, that is, $b \leq a$. If $\sigma = b$, then $a \vee b = b \vee b = b$, and hence $a \vee b = b$, that is, $a \leq b$. So $a = b$, which shows that χ is an injective function.

(2) For any $g \in G(\mathcal{L})$, there exists a $g(0) \in L$ such that $g = \chi(g(0))$, which implies that χ is a surjective function. Indeed, by Theorem 4.12(3), we have

$$g(\sigma) = \neg\sigma \to g(0) = \sigma \oplus g(0) = \sigma \vee g(0) = \chi(g(0))(\sigma)$$

for any $\sigma \in \mathcal{L}$.

(3) For any $a, b \in \mathcal{L}$, we have

$$\chi(a \wedge b)(\sigma) = (a \wedge b) \vee \sigma = (a \vee \sigma) \wedge (b \vee \sigma) = (\chi(a) \sqcap \chi(b))(\sigma),$$
$$\chi(a \vee b)(\sigma) = (a \vee b) \vee \sigma = (a \vee \sigma) \vee (b \vee \sigma) = (\chi(a) \sqcup \chi(b))(\sigma),$$
$$\chi(\neg a)(\sigma) = \neg a \vee \sigma = a \to \sigma = (a \to \sigma) \wedge (\sigma \to \sigma) = (a \vee \sigma) \to \sigma = (\chi(a)(\sigma))^\star.$$

which implies that χ is a homomorphism.

Therefore $(\mathcal{L}, \wedge, \vee, \neg, 0, 1)$ is isomorphic to $(G(\mathcal{L}), \cap, \cup, \star, g_1, 1_g)$. $\square$

Then, we discuss the relations between $D(\mathcal{L})$ and $G(\mathcal{L})$.

Let $\varphi : D(\mathcal{L}) \to G(\mathcal{L})$ be the map such that

$$\varphi(d)(\sigma) = \neg(d(\neg\sigma))$$

for any $d \in D(\mathcal{L})$ and $\sigma \in \mathcal{L}$, and $\psi : G(\mathcal{L}) \to D(\mathcal{L})$ be the map such that

$$\psi(g)(\sigma) = \neg(g(\neg\sigma))$$

for any $g \in G(\mathcal{L})$, and $\sigma \in \mathcal{L}$.

Theorem 4.17. *Let L be a regular bounded semihoop. Then φ and ψ form an isotone Galois connection between $D(\mathcal{L})$ and $G(\mathcal{L})$. Namely,*

$$d \le \psi(g) \text{ if and only if } g \le \varphi(d)$$

for any $d \in D(\mathcal{L})$ and $g \in G(\mathcal{L})$.

Proof. (1) It follows from Propositions 3.5(5) and 4.11(2) that φ and ψ are isotone.

(2) If $d \le \psi(g)$, then $d(\sigma) \le \psi(g)(\sigma) = \neg(g(\neg\sigma))$, and hence $g(\neg\sigma) \le \neg(d(\sigma))$ for any $\sigma \in \mathcal{L}$. So $g(\sigma) \le \neg(d(\neg\sigma))$, which implies $g(\sigma) \le \varphi(d)(\sigma)$ for any $\sigma \in \mathcal{L}$. Thus $g \le \varphi(d)$. Conversely, if $g \le \varphi(d)$, then $g(\sigma) \le \varphi(d)(\sigma) = \neg(d(\neg\sigma))$, and hence $d(\neg\sigma) \le \neg(g(\sigma))$, which implies $d(\sigma) \le \neg(g(\neg\sigma)) = \psi(g)(\sigma)$ for any $\sigma \in \mathcal{L}$. Thus $d \le \psi(g)$. $\square$

Theorem 4.18. *Let $\mathcal{L}$ be a bounded regular idempotent semihoop. Then there exists a one to one correspondence between $G(\mathcal{L})$ and $D(\mathcal{L})$. Namely,*

(1) *if $d \in D(\mathcal{L})$, then $\varphi(d) \in G(\mathcal{L})$,*

(2) *if $g \in G(\mathcal{L})$, then $\psi(g) \in D(\mathcal{L})$,*

(3) *$\psi\varphi(d) = d$ and $\varphi\psi(g) = g$.*

Proof. If d is a multiplicative derivation on $\mathcal{L}$, then

$$\begin{aligned} \varphi(d)(\sigma \to \delta) &= \neg(d\neg(\sigma \to \delta)) \\ &= \neg(d(\sigma \odot \neg\delta)) \\ &= \neg(\sigma \odot d(\neg\delta)) \\ &= \sigma \to \varphi(d)(\delta), \end{aligned}$$

for any $\sigma, \delta \in \mathcal{L}$, which implies that $\varphi(d)$ is an implicative derivation on $\mathcal{L}$.

Conversely, if g is an implicative derivation on $\mathcal{L}$, then

$$\begin{aligned} \psi(g)(\sigma \odot \delta) &= \neg(g(\neg(\sigma \odot \delta))) \\ &= \neg(g(\sigma \to \neg\delta)) \\ &= \neg(\sigma \to g(\neg\delta)) \\ &= \sigma \odot \neg(g(\neg\delta)) \\ &= \sigma \odot \psi(g)(\delta), \end{aligned}$$

for any $\sigma, \delta \in \mathcal{L}$, which implies that $\psi(g)$ is a multiplicative derivation on $\mathcal{L}$.

Moreover, we have

$$\begin{aligned}\psi\varphi(d)(\sigma) &= d(\neg(d(\neg\sigma)))\\ &= d(1)\odot((d(1))\to\sigma)\\ &= d(1)\odot\sigma\\ &= d(\sigma)\end{aligned}$$

for any $\sigma\in\mathcal{L}$, and so $\psi\varphi(d)=d$. Similarly, we also have

$$\begin{aligned}\varphi\psi(g)(\sigma) &= g(\neg(g(\neg\sigma)))\\ &= g(\neg(\sigma\to g(0)))\\ &= (\sigma\to g(0))\to g(0)\\ &= g(\sigma)\end{aligned}$$

for any $\sigma\in\mathcal{L}$, and so $\varphi\psi(g)=g$. □

§5 Conclusions

The notion of derivations is helpful for studying structures and properties in algebraic systems. In this paper, we study some particular derivations on semihoops and give some characterizations of them. We also characterize Heyting algebras and Boolean algebras by derivations and show that the relations between derivations and Galois connection on semihoops. But there are still some issues to consider. For example, for any bounded idempotent prelinear semihoop $(\mathcal{L},\wedge,\vee,\to,0,1)$, whether there exists a function such that bounded idempotent prelinear semihoops $(\mathcal{L},\wedge,\vee,\to,0,1)$ and $(G(\mathcal{L}),\cap,\cup,\Rightarrow,g_1,1_g)$ are isomorphism. In our future work, we will consider these problems.

Declarations

Conflict of interest The authors declare no conflict of interest.

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