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A novel fractional case study of nonlinear dynamics via analytical approach

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Abstract. The present work describes the fractional view analysis of Newell-Whitehead-Segal equations, using an innovative technique. The work is carried with the help of the Caputo operator of fractional derivative. The analytical solutions of some numerical examples are presented to confirm the reliability of the proposed method. The derived results are very consistent with the actual solutions to the problems. A graphical representation has been done for the solution of the problems at various fractional-order derivatives. Moreover, the solution in series form has the desired rate of convergence and provides the closed-form solutions. It is noted that the procedure can be modified in other directions for fractional order problems.

§1 Introduction

In recent decades, several authors have found that non-integer-order derivatives and integrals are very useful for describing materials and processes having various important properties. The greater effect of the fractional derivative is observed when the past history of the results is required as compared to the classical derivative and therefore the researchers are more interested in the topic of fractional differential equations (FDEs) [1-9]. Leibniz and L'Hospital were the first to introduce the idea of fractional calculus but later on, the concept was used by many researchers and has shown many applications in different research areas [4]. With the implementation of fractional differential equations [FDEs] to model different physical systems and processes, FDEs have gained much attention from scientists and provide significant fractional modeling of various phenomena in nature [10-17]. Many physical problems are governed by FDEs, and several investigators have been interested in seeking the solution to these equations. In various physical and engineering processes, FDEs have shown fundamental importance over the last few decades. It has been investigated that fractional nonlinear FDEs are intensively used in natural phenomena in various branches of applied natural sciences [18-29].

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It is almost very difficult to find the exact solutions to nonlinear FPDEs, so it is a research field of greater importance and motivation. The nonlinear wave phenomena such as viscoelasticity chemical processes, acoustics, electrochemistry, electromagnetic, material science, biology, engineering and physics [30-38] are considered for its numerical and analytical investigations. More effective procedures and techniques are needed for the solution of mathematical models that can best analyze the real-world phenomena.

In this regard, several methods have been used to obtain the explicit solutions for the nonlinear equations of integer order. However, very limited approaches are applied to solve F-PDEs. such as Laplace Transform method (LTM) [39], Adomian Decomposition method (ADM) [40], Elzaki Transform Decomposition method (ETDM) [41], Reduced Differential Transform method (RDTM) [42], Natural Transform Decomposition method (NTDM) [43-44], Iterative Laplace Transform method (ILTM) [45], Fourier Transform method (FTM) [46], solution by ADM and Variational iteration method (VIM) with Shehu transformation is presented in [47] Approximate-analytical method (AAM) [48], Homotopy Perturbation method(HAM) [49], and Perturbation Iteration transform method (PITM) [50].

In the present research work, our focus point is to discuss the fractional view Newell-Whitehead-Segal equation, using the coupling of Shehu transformation with Homotopy perturbation method (HPM). This new hybrid technique is known as the homotopy perturbation transform method (HPTM). The HPTM solutions are calculated for some numerical examples related to the problem of NWSEs. The series form solution with a higher rate of convergence is seen by using HPTM. The nonlinearity of the problem is handled by using the present technique and we obtain the solutions in a sophisticated manner. Using the HPM, four case study problems are solved of non-linear Newell-Whitehead-Segel equations. As compared to the exact solution, the rapid convergence towards the exact solution is seen.

The Newell-Whitehead-Segel equation model is the relationship of the diffusion term effect with the reaction term nonlinear effect.

The Newell-Whitehead-Segel equation is written as:

$$D_\tau^\beta \psi = k\psi_{\mu\mu} + a\psi - b\psi^{(q)} \tag{1}$$

where $k > 0$ and is real whereas ψ represents the unknown functions in variables μ and τ such that $\tau > 0$. Also a, b are real numbers and q is a positive integer. The unknown function ψ either denotes the distribution of temperature along the infinite thin tube or the fluid velocity along the small diameter pipe with infinite length. The fractional derivative in equation (1) is represented by the Caputo derivative operator and on the right-hand side, the second term $a\psi - b\psi^{(q)}$ defines the source term.

In 1998, He introduced the homotopy perturbation technique [49-54]. Later, the nonlinear non-homogeneous partial differential equations are solved using the HPM, which is a semi-analytical technique [49-54]. The solution is assumed to be the sum of an infinite sequence that converges rapidly to the exact results. This approach was used to solve both linear and nonlinear equations. In the present research article, we proposed a new approximate analytical technique which is known as (HPSTM). The newly developed technique is the mixed form of Shehu transform and HPM. It is investigated that the present technique is very effective in finding the analytical solution of fractional NWSEs. The HPSTM results are very convincing with the exact solutions to the targeted problems. The fractional problem results by using the current method are also devoted to the fractional view analysis of the problems. It supports the improved physical analysis of the models in terms of their experimental data. It is suggested that the current technique can be modified to solve other fractional PDEs and their systems.

The rest of the article is structured as follows: In Section 2, we recall several basic properties and define the Shehu transform and fractional calculus. In Section 3, the idea of Homotopy Perturbation Shehu Transform Method is discussed. In Section 4, we explain many problems to maintain the accuracy and efficiency of the proposed method, and Section 5 is devoted to the conclusion.

§2 Preliminaries

This section is related to some important definitions regarding fractional calculus and about some of the Shehu theory. These preliminary concepts are mandatory to complete the present research work.

2.1 Definition

The Rieman-Liouville fractional integral is defined by [6-8]

$$I_0^{\beta} h(\tau) = \frac{1}{\Gamma(\beta)} \int_0^{\tau} (\tau - s)^{\beta-1} h(s) ds,$$

showing that the integral on the right side converges.

2.2 Definition

Fractional derivative in Caputo's sense is given as [6-8]

$$D_{\tau}^{\beta}\psi(\tau) = \begin{cases} I^{n-\beta} f^{n}, \ n-1 < \beta \leq n, n \in \mathbb{N} \\ \frac{d^{n}}{d\tau_{n}}\psi(\tau), \ \beta = n, n \in \mathbb{N}. \end{cases} \tag{2}$$

2.3 Definition

Mittag Leffler function having two-parameter is defined as [6-8]:

$$E_{\alpha,\beta}(\tau) = \sum_{k=0}^{\infty} \frac{\tau^{k}}{\Gamma(k\alpha+\beta)}. \tag{3}$$

For $\alpha = \beta = 1, E_{1,1}(\tau) = e^{\tau}$ and $E_{1,1}(-\tau) = e^{-\tau}$

2.4 Definition

The Shehu transformation which is defined by S(.) for a function $\nu(\tau)$ is expressed as [55-57]

$$S\{\nu(\tau)\} = V(s,u) = \int_0^\infty \nu(\tau) e^{(\frac{-s\tau}{u})}(\tau) d\tau, \quad \tau > 0 \quad s > 0. \tag{4}$$

The Shehu transformation of a function $\nu(\tau)$ is V(s,u): then $\nu(\tau)$ is called the inverse of V(s,u) which is defined as $S^{-1}\{V(s,u)\} = \nu(\tau), \quad for \quad \tau \geq 0, S^{-1}$ is inverse Shehu transformation.

2.5 Definition

Shehu transform for fractional order derivatives [55-57]. The Shehu transformation for the fractional order derivatives is expressed as

$$S\left\{\nu^{(\beta)}(\tau)\right\} = \frac{s^\beta}{u^\beta} V(s,u) - \sum_{k=0}^{n-1} \left(\frac{s}{u}\right)^{\beta-k-1} \nu^{(k)}(0), \quad n-1 < \beta \leq n, \tag{5}$$

§3 Homotopy Perturbation Shehu Transform Method

To explain the basic ideas of this approach, the following equation is considered [49-54]:

$$\begin{aligned} &D_\tau^\beta \psi(\mu,\tau) + M[\mu]\psi(\mu,\tau) + N[\mu]\psi(\mu,\tau) = h(\mu,\tau), \quad \tau > 0, \quad 0 < \beta \leq 1, \\ &\psi(\mu,0) = g(\mu), \quad \mu \in \Re. \end{aligned} \tag{6}$$

Where $D_\tau^\beta = \frac{\partial^\beta}{\partial \tau^\beta}$ Caputo's derivative, $M[\mu]$, $N[\mu]$ are the linear and nonlinear operators respectively and $h(\mu,\tau)$ is source function.

Using Shehu transformation [55-57] to (6), we have

$$\begin{aligned} &S[D_\tau^\beta \psi(\mu,\tau) + M[\mu]\psi(\mu,\tau) + N[\mu]\psi(\mu,\tau)] = S[h(\mu,\tau)], \quad \tau > 0, 0 < \beta \leq 1, \\ &R(\mu,s,u) = \frac{g(\mu)}{s} + \frac{u^\beta}{s^\beta} S[h(\mu,\tau)] - \frac{u^\beta}{s^\beta} S[M[\mu]\psi(\mu,\tau) + N[\mu]\psi(\mu,\tau)]. \end{aligned} \tag{7}$$

Now, by taking inverse Shehu transform [55-57], we get

$$\psi(\mu,\tau) = F(\mu,\tau) - S^{-1}\left(\frac{u^\beta}{s^\beta} S[M[\mu]\psi(\mu,\tau) + N[\mu]\psi(\mu,\tau)]\right), \tag{8}$$

where

$$F(\mu,\tau) = S^{-1}[\frac{g(\mu)}{s} + \frac{u^\beta}{s^\beta} S[h(\mu,\tau)]] = g(\mu) + S^{-1}\left[\frac{u^\beta}{s^\beta} S[h(\mu,\tau)]\right]. \tag{9}$$

Now, perturbation technique having parameter ϵ is given as

$$\psi(\mu,\tau) = \sum_{k=0}^\infty \epsilon^k \psi_k(\mu,\tau), \tag{10}$$

where ϵ is perturbation parameter and $\epsilon \in [0,1]$.

The nonlinear term can be decomposed as

$$N\psi(\mu,\tau) = \sum_{k=0}^\infty \epsilon^k H_n(\psi), \tag{11}$$

where H_n are He's polynomials of the form $\psi_0, \psi_1, \psi_2 \cdots, \psi_n$, and can be determined as

$$H_n(\psi_0, \psi_1, ..., \psi_n) = \frac{1}{\gamma(n+1)} D_\epsilon^k \left[N \left(\sum_{k=0}^{\infty} \epsilon^i \psi_i \right) \right]_{\epsilon=0}, \tag{12}$$

where $D_\epsilon^k = \frac{\partial^k}{\partial \epsilon^k}$. Using relation (10) and (11) in (7) and constructing the Homotopy , we get

$$\sum_{k=0}^{\infty} \epsilon^k \psi_k(\mu, \tau) = F(\mu, \tau) - \epsilon \times \left(S^{-1} \left[\frac{u^\beta}{s^\beta} S\{ M \sum_{k=0}^{\infty} \epsilon^k \psi_k(\mu, \tau) + \sum_{k=0}^{\infty} \epsilon^k H_k(\psi) \} \right] \right). \tag{13}$$

On comparing coefficient of ϵ on both sides, we obtain

$$\begin{aligned}
&\epsilon^0 : \psi_0(\mu, \tau) = F(\mu, \tau), \\
&\epsilon^1 : \psi_1(\mu, \tau) = S^{-1} \left[\frac{u^\beta}{s^\beta} S(M[\mu]\psi_0(\mu, \tau) + H_0(\psi)) \right], \\
&\epsilon^2 : \psi_2(\mu, \tau) = S^{-1} \left[\frac{u^\beta}{s^\beta} S(M[\mu]\psi_1(\mu, \tau) + H_1(\psi)) \right], \\
&\vdots \\
&\epsilon^k : \psi_n(\mu, \tau) = S^{-1} \left[\frac{u^\beta}{s^\beta} S(M[\mu]\psi_{k-1}(\mu, \tau) + H_{k-1}(\psi)) \right],
\end{aligned} \tag{14}$$

$k > 0, k \in N$.

The component $\psi_k(\mu, \tau)$ can be calculated easily, which leads us to the convergent series rapidly. By taking $\epsilon \to 1$, we obtain

$$\psi(\mu, \tau) = \lim_{M \to \infty} \sum_{k=1}^{M} \psi_k(\mu, \tau). \tag{15}$$

The obtained result is in series form and converges quickly to the exact solution of the problem.

§4 Test Problems

Four cases of nonlinear diffusion equations are presented to demonstrate the capability and the reliability of the suggested technique.

4.1 Case: I

The Newell-Whitehead-Segel equation for $a = 2, b = 3, k = 1$ and $q = 2$ becomes:

$$D_\tau^\beta \psi = \psi_{\mu\mu} + 2\psi - 3\psi^2, \quad 0 < \beta \leq 1, \tag{16}$$

with initial conditions

$$\psi(\mu, 0) = \lambda. \tag{17}$$

Taking Shehu Transform of (16), we have

$$\frac{s^\beta}{u^\beta} S[\psi(\mu, \tau)] = \psi^{(0)}(\mu, 0) \frac{s^{\beta-1}}{u^\beta} + S\left(\psi_{\mu\mu} + 2\psi - 3\psi^2\right). \tag{18}$$

$$S[\psi(\mu, \tau)] = \frac{1}{s}\lambda + \frac{u^\beta}{s^\beta} \left[S\left(\psi_{\mu\mu} + 2\psi - 3\psi^2\right) \right]. \tag{19}$$

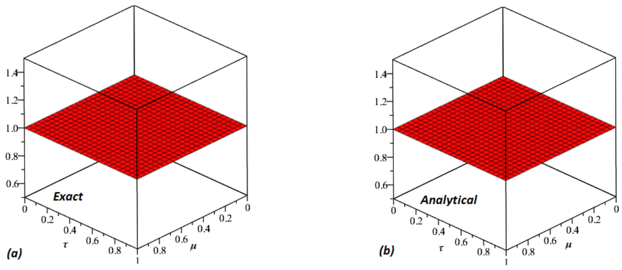

Figure 1. Graph of exact and approximate solutions of Problem 3.4.

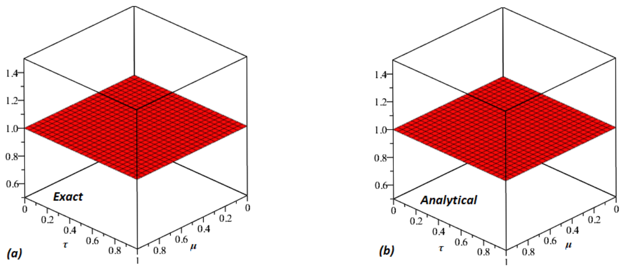

Figure 2. The solution graph of example 1, (a) Exact solution and (b) HPSTM solution at $\beta = 1$.

Taking Inverse Shehu Transform, we obtain

$$\psi(\mu,\tau) = \lambda + S^{-1}\left[\frac{u^\beta}{s^\beta}\left\{S\left(\psi_{\mu\mu} + 2\psi - 3\psi^2\right)\right\}\right]. \tag{20}$$

Now, applying the above-mentioned homotopy perturbation technique as in (13), we get

$$\sum_{k=0}^{\infty}\epsilon^k\psi_k(\mu,\tau) \tag{21}$$

$$= \lambda + \epsilon\left(S^{-1}\left[\frac{u^\beta}{s^\beta}S\left[\left(\sum_{k=0}^{\infty}\epsilon^k\psi_k(\mu,\tau)\right)_{\mu\mu} + 2\sum_{k=0}^{\infty}\epsilon^k\psi_k(\mu,\tau) - 3\left(\sum_{k=0}^{\infty}\epsilon^k\psi_k(\mu,\tau)\right)^2\right]\right]\right). \tag{22}$$

Comparing the same power coefficient of ϵ, we get

$$\begin{aligned}
&\epsilon^0 : \psi_0(\mu,\tau) = \lambda,\\
&\epsilon^1 : \psi_1(\mu,\tau) = S^{-1}\left(\frac{u^\beta}{s^\beta}S[\psi_{0\mu\mu} + \psi_0 - \psi_0^2]\right) = \lambda(2-3\lambda)\frac{\tau^\beta}{\Gamma(\beta+1)},\\
&\epsilon^2 : \psi_2(\mu,\tau) = S^{-1}\left(\frac{u^\beta}{s^\beta}S[\psi_{1\mu\mu} + \psi_1 - 2\psi_0\psi_1]\right) = 2\lambda(2-3\lambda)(1-3\lambda)\frac{\tau^{2\beta}}{\Gamma(2\beta+1)},\\
&\vdots
\end{aligned} \tag{23}$$

Now, by taking $\epsilon \to 1$ we obtain convergent series form solution as

$$\psi(\mu,\tau) = \psi_0 + \psi_1 + \psi_2 + \cdots$$

$$= \lambda + \lambda(2-3\lambda)\frac{\tau^\beta}{\Gamma(\beta+1)} + 2\lambda(2-3\lambda)(1-3\lambda)\frac{\tau^{2\beta}}{\Gamma(2\beta+1)} + \cdots. \tag{24}$$

Particularly, putting $\beta = 1$, we get the exact solution

$$\psi(\mu,\tau) = \frac{\frac{-2}{3}\lambda \exp^{2\tau}}{\frac{-2}{3} + \lambda - \lambda\exp^{2\tau}}. \tag{25}$$

4.2 Case: II

The Newell-Whitehead-Segel equation for $a = 1, b = 1, k = 1$ and $q = 2$ becomes:

$$D_\tau^\beta \psi = \psi_{\mu\mu} + \psi(1-\psi), \quad 0 < \beta \le 1, \tag{26}$$

with initial conditions

$$\psi(\mu,0) = \frac{1}{(1+\exp^{\frac{\mu}{\sqrt{6}}})^2}. \tag{27}$$

Taking Shehu Transform of (25), we have

$$\frac{s^\beta}{u^\beta} S[\psi(\mu,\tau)] = \psi^{(0)}(\mu,0)\frac{s^{\beta-1}}{u^\beta} + S\left(\psi_{\mu\mu} + \psi(1-\psi)\right). \tag{28}$$

$$S[\psi(\mu,\tau)] = \frac{1}{s}\frac{1}{(1+\exp^{\frac{\mu}{\sqrt{6}}})^2} + \frac{u^\beta}{s^\beta}\left[S\left(\psi_{\mu\mu} + \psi(1-\psi)\right)\right]. \tag{29}$$

Taking Inverse Shehu Transform, we obtain

$$\psi(\mu,\tau) = \frac{1}{(1+\exp^{\frac{\mu}{\sqrt{6}}})^2} + S^{-1}\left[\frac{u^\beta}{s^\beta}\left\{S\left(\psi_{\mu\mu} + \psi(1-\psi)\right)\right\}\right]. \tag{30}$$

Now, applying the above-mentioned homotopy perturbation technique as in (13), we get

$$\sum_{k=0}^{\infty}\epsilon^k\psi_k(\mu,\tau) \tag{31}$$

$$= \frac{1}{(1+\exp^{\frac{\mu}{\sqrt{6}}})^2} + \epsilon\left(S^{-1}\left[\frac{u^\beta}{s^\beta}S\left[(\sum_{k=0}^{\infty}\epsilon^k\psi_k(\mu,\tau))_{\mu\mu} + \sum_{k=0}^{\infty}\epsilon^k\psi_k(\mu,\tau)(1-\sum_{k=0}^{\infty}\epsilon^k\psi_k(\mu,\tau))\right]\right]\right). \tag{32}$$

Comparing the same power coefficient of ϵ, we get

$$\begin{aligned}
&\epsilon^0 : \psi_0(\mu,\tau) = \frac{1}{(1+\exp^{\frac{\mu}{\sqrt{6}}})^2},\\
&\epsilon^1 : \psi_1(\mu,\tau) = S^{-1}\left(\frac{u^\beta}{s^\beta}S[\psi_{0\mu\mu} + \psi_0 - \psi_0^2]\right) = \frac{5}{3}\frac{\exp^{\frac{\mu}{\sqrt{6}}}}{(1+\exp^{\frac{\mu}{\sqrt{6}}})^3}\frac{\tau^\beta}{\Gamma(\beta+1)},\\
&\epsilon^2 : \psi_2(\mu,\tau) = S^{-1}\left(\frac{u^\beta}{s^\beta}S[\psi_{1\mu\mu} + \psi_1 - 2\psi_0\psi_1]\right) = \frac{25}{18}\left(\frac{\exp^{\frac{\mu}{\sqrt{6}}}(-1+2\exp^{\frac{\mu}{\sqrt{6}}})}{(1+\exp^{\frac{\mu}{\sqrt{6}}})^4}\right)\frac{\tau^{2\beta}}{\Gamma(2\beta+1)},\\
&\vdots
\end{aligned} \tag{33}$$

Now, by taking $\epsilon \to 1$ we obtain convergent series form solution as

$$\begin{aligned}
\psi(\mu,\tau) &= \psi_0 + \psi_1 + \psi_2 + \cdots\\
&= \frac{1}{(1+\exp^{\frac{\mu}{\sqrt{6}}})^2} + \frac{5}{3}\frac{\exp^{\frac{\mu}{\sqrt{6}}}}{(1+\exp^{\frac{\mu}{\sqrt{6}}})^3}\frac{\tau^\beta}{\Gamma(\beta+1)} + \frac{25}{18}\left(\frac{\exp^{\frac{\mu}{\sqrt{6}}}(-1+2\exp^{\frac{\mu}{\sqrt{6}}})}{(1+\exp^{\frac{\mu}{\sqrt{6}}})^4}\right)\frac{\tau^{2\beta}}{\Gamma(2\beta+1)} + \cdots
\end{aligned} \tag{34}$$

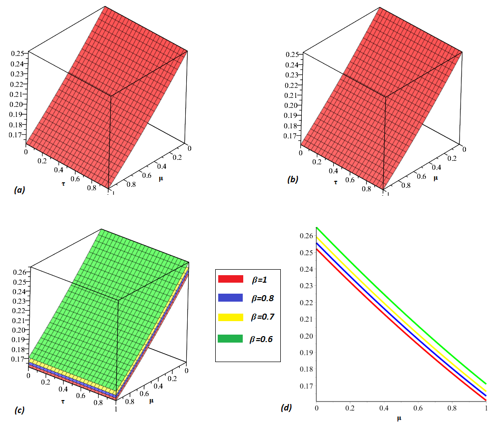

Figure 3. The (a) Exact solution and (b) HPSTM solution (c)Different fractional order (d)$y=0.5$ solution graph of example 2 at $\beta=1$.

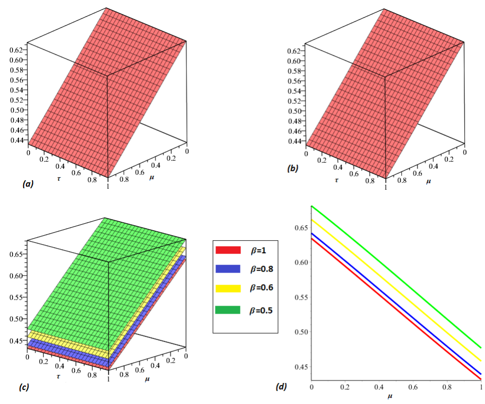

Figure 4. The (a)Exact solution and (b) HPSTM solution (c)Different fractional order (d)$y=0.5$ solution graph of example 3 at $\beta=1$.

Particularly, putting $\beta=1$, we get the exact solution

$$\psi(\mu,\tau)=\left(\frac{1}{1+\exp^{\frac{\mu}{\sqrt{6}}-\frac{5}{6}\tau}}\right)^2. \tag{35}$$

4.3 Case: III

The Newell-Whitehead-Segel equation for $a=1, b=1, k=1$ and $q=4$ becomes:

$$D_\tau^\beta\psi=\psi_{\mu\mu}+\psi-\psi^4, \quad 0<\beta\leq 1, \tag{36}$$

with initial conditions

$$\psi(\mu,0)=\frac{1}{(1+\exp^{\frac{3\mu}{\sqrt{10}}})^{\frac{2}{3}}}. \tag{37}$$

Taking Shehu Transform of (34), we have

$$\frac{s^\beta}{u^\beta}S[\psi(\mu,\tau)] = \psi^{(0)}(\mu,0)\frac{s^{\beta-1}}{u^\beta} + S\left(\psi_{\mu\mu}+\psi-\psi^4\right). \tag{38}$$

$$S[\psi(\mu,\tau)] = \frac{1}{s}\frac{1}{(1+\exp^{\frac{3\mu}{\sqrt{10}}})^{\frac{2}{3}}} + \frac{u^\beta}{s^\beta}\left[S\left(\psi_{\mu\mu}+\psi-\psi^4\right)\right]. \tag{39}$$

Taking Inverse Shehu Transform, we get

$$\psi(\mu,\tau) = \frac{1}{(1+\exp^{\frac{3\mu}{\sqrt{10}}})^{\frac{2}{3}}} + S^{-1}\left[\frac{u^\beta}{s^\beta}\left\{S\left(\psi_{\mu\mu}+\psi-\psi^4\right)\right\}\right]. \tag{40}$$

Now, applying the above-mentioned homotopy perturbation technique as in (13), we get

$$\sum_{k=0}^{\infty}\epsilon^k\psi_k(\mu,\tau) \tag{41}$$

$$= \frac{1}{(1+\exp^{\frac{3\mu}{\sqrt{10}}})^{\frac{2}{3}}} + \epsilon\left(S^{-1}\left[\frac{u^\beta}{s^\beta}S\left[(\sum_{k=0}^{\infty}\epsilon^k\psi_k(\mu,\tau))_{\mu\mu} + \sum_{k=0}^{\infty}\epsilon^k\psi_k(\mu,\tau) - (\sum_{k=0}^{\infty}\epsilon^k\psi_k(\mu,\tau))^4\right]\right]\right). \tag{42}$$

Comparing the same power coefficient of ϵ, we get

$$\begin{aligned}
&\epsilon^0 : \psi_0(\mu,\tau) = \frac{1}{(1+\exp^{\frac{3\mu}{\sqrt{10}}})^{\frac{2}{3}}},\\
&\epsilon^1 : \psi_1(\mu,\tau) = S^{-1}\left(\frac{u^\beta}{s^\beta}S[\psi_{0\mu\mu}+\psi_0-\psi_0^4]\right) = \frac{7}{5}\frac{\exp^{\frac{3}{\sqrt{10}}\mu}}{(1+\exp^{\frac{3\mu}{\sqrt{10}}})^{\frac{5}{3}}}\frac{\tau^\beta}{\Gamma(\beta+1)},\\
&\epsilon^2 : \psi_2(\mu,\tau) = S^{-1}\left(\frac{u^\beta}{s^\beta}S[\psi_{1\mu\mu}+\psi_1-4\psi_0^3\psi_1]\right) = \frac{49}{50}\frac{(2\exp^{\frac{3}{\sqrt{10}}\mu}-3)\exp^{\frac{3}{\sqrt{10}}\mu}}{(1+\exp^{\frac{3\mu}{\sqrt{10}}})^{\frac{8}{3}}}\frac{\tau^{2\beta}}{\Gamma(2\beta+1)}.\\
&\vdots
\end{aligned} \tag{43}$$

Now, by taking $\epsilon \to 1$ we obtain convergent series form solution as

$$\begin{aligned}
\psi(\mu,\tau) &= \psi_0+\psi_1+\psi_2+\cdots\\
&= \frac{1}{(1+\exp^{\frac{3\mu}{\sqrt{10}}})^{\frac{2}{3}}} + \frac{7}{5}\frac{\exp^{\frac{3}{\sqrt{10}}\mu}}{(1+\exp^{\frac{3\mu}{\sqrt{10}}})^{\frac{5}{3}}}\frac{\tau^\beta}{\Gamma(\beta+1)} + \frac{49}{50}\frac{(2\exp^{\frac{3}{\sqrt{10}}\mu}-3)\exp^{\frac{3}{\sqrt{10}}\mu}}{(1+\exp^{\frac{3\mu}{\sqrt{10}}})^{\frac{8}{3}}}\frac{\tau^{2\beta}}{\Gamma(2\beta+1)} + \cdots.
\end{aligned} \tag{44}$$

Particularly, putting $\beta = 1$, we get the exact solution

$$\psi(\mu,\tau) = \left(\frac{1}{2}\tanh\left(-\frac{3}{2\sqrt{10}}\left(\mu-\frac{7}{\sqrt{10}}\tau\right)\right)\right). \tag{45}$$

4.4 Case:IV

The Newell-Whitehead-Segel equation for $a = 3, b = 4, k = 1$ and $q = 3$ becomes:

$$D_\tau^\beta\psi = \psi_{\mu\mu} + 3\psi - 4\psi^3, \quad 0 < \beta \le 1, \tag{46}$$

with initial conditions

$$\psi(\mu,0) = \sqrt{\frac{3}{4}}\frac{\exp^{\sqrt{6}\mu}}{\exp^{\sqrt{6}\mu}+\exp^{\frac{\sqrt{6}}{2}\mu}}. \tag{47}$$

Taking Shehu Transform of (43), we have

$$\frac{s^\beta}{u^\beta}S[\psi(\mu,\tau)]=\psi^{(0)}(\mu,0)\frac{s^{\beta-1}}{u^\beta}+S\left(\psi_{\mu\mu}+3\psi-4\psi^3\right). \tag{48}$$

$$S[\psi(\mu,\tau)]=\frac{1}{s}\sqrt{\frac{3}{4}}\frac{\exp^{\sqrt{6}\mu}}{\exp^{\sqrt{6}\mu}+\exp^{\frac{\sqrt{6}}{2}\mu}}+\frac{u^\beta}{s^\beta}\left[S\left(\psi_{\mu\mu}+3\psi-4\psi^3\right)\right]. \tag{49}$$

Taking Inverse Shehu Transform, we get

$$\psi(\mu,\tau)=\sqrt{\frac{3}{4}}\frac{\exp^{\sqrt{6}\mu}}{\exp^{\sqrt{6}\mu}+\exp^{\frac{\sqrt{6}}{2}\mu}}+S^{-1}\left[\frac{u^\beta}{s^\beta}\left\{S\left(\psi_{\mu\mu}+3\psi-4\psi^3\right)\right\}\right]. \tag{50}$$

Now, applying the above-mentioned homotopy perturbation technique as in (13), we get

$$\begin{aligned}\sum_{k=0}^{\infty}\epsilon^k\psi_k(\mu,\tau)=&\sqrt{\frac{3}{4}}\frac{\exp^{\sqrt{6}\mu}}{\exp^{\sqrt{6}\mu}+\exp^{\frac{\sqrt{6}}{2}\mu}}+\epsilon\left(S^{-1}\left[\frac{u^\beta}{s^\beta}S\left[(\sum_{k=0}^{\infty}\epsilon^k\psi_k(\mu,\tau))_{\mu\mu}+3\sum_{k=0}^{\infty}\epsilon^k\psi_k(\mu,\tau)\right.\right.\right.\\&\left.\left.\left.-4(\sum_{k=0}^{\infty}\epsilon^k\psi_k(\mu,\tau))^3\right]\right]\right).\end{aligned} \tag{51}$$

Comparing the same power coefficient of ϵ, we get

$$\begin{aligned}\epsilon^0:\psi_0(\mu,\tau)&=\sqrt{\frac{3}{4}}\frac{\exp^{\sqrt{6}\mu}}{\exp^{\sqrt{6}\mu}+\exp^{\frac{\sqrt{6}}{2}\mu}},\\ \epsilon^1:\psi_1(\mu,\tau)&=S^{-1}\left(\frac{u^\beta}{s^\beta}S[\psi_{0\mu\mu}+\psi_0-\psi_0^4]\right)=\frac{9}{2}\sqrt{\frac{3}{4}}\frac{\exp^{\sqrt{6}\mu}\exp^{\frac{\sqrt{6}}{2}\mu}}{(\exp^{\sqrt{6}\mu}+\exp^{\frac{\sqrt{6}}{2}\mu})^2}\frac{\tau^\beta}{\Gamma(\beta+1)},\end{aligned} \tag{52}$$

$$\begin{aligned}\epsilon^2:\psi_2(\mu,\tau)&=S^{-1}\left(\frac{u^\beta}{s^\beta}S[\psi_{1\mu\mu}+\psi_1-4\psi_0^3\psi_1]\right)\\&=\frac{81}{4}\sqrt{\frac{3}{4}}\frac{\exp^{\sqrt{6}\mu}\exp^{\frac{\sqrt{6}}{2}\mu}(-\exp^{\sqrt{6}\mu}+\exp^{\frac{\sqrt{6}}{2}\mu})}{(\exp^{\sqrt{6}\mu}+\exp^{\frac{\sqrt{6}}{2}\mu})^3}\frac{\tau^{2\beta}}{\Gamma(2\beta+1)}.\end{aligned} \tag{53}$$

$$\vdots$$

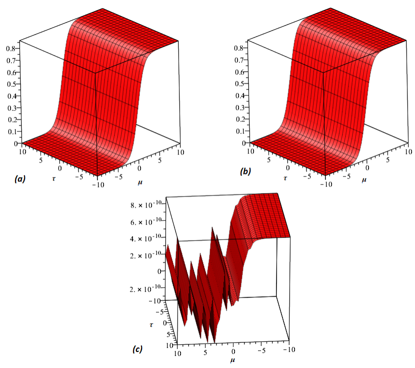

Figure 5. The (a)Exact solution and (b) HPSTM solution (c)Error graph of example 4 at $\beta=1$.

Now, by taking $\epsilon \to 1$ we obtain convergent series form solution as

$$\begin{aligned}\psi(\mu,\tau) &= \psi_0 + \psi_1 + \psi_2 + \cdots . \\ &= \sqrt{\frac{3}{4}}\frac{\exp^{\sqrt{6}\mu}}{\exp^{\sqrt{6}\mu} + \exp^{\frac{\sqrt{6}}{2}\mu}} + \frac{9}{2}\sqrt{\frac{3}{4}}\frac{\exp^{\sqrt{6}\mu}\exp^{\frac{\sqrt{6}}{2}\mu}}{(\exp^{\sqrt{6}\mu} + \exp^{\frac{\sqrt{6}}{2}\mu})^2}\frac{\tau^{\beta}}{\Gamma(\beta+1)} \\ &+ \frac{81}{4}\sqrt{\frac{3}{4}}\frac{\exp^{\sqrt{6}\mu}\exp^{\frac{\sqrt{6}}{2}\mu}(-\exp^{\sqrt{6}\mu} + \exp^{\frac{\sqrt{6}}{2}\mu})}{(\exp^{\sqrt{6}\mu} + \exp^{\frac{\sqrt{6}}{2}\mu})^3}\frac{\tau^{2\beta}}{\Gamma(2\beta+1)} + \cdots .\end{aligned} \tag{54}$$

Particularly, putting $\beta = 1$, we get the exact solution

$$\psi(\mu,\tau) = \sqrt{\frac{3}{4}}\frac{\exp^{\sqrt{6}\mu}}{\exp^{\sqrt{6}\mu} + \exp^{(\frac{\sqrt{6}}{2}\mu - \frac{9}{2}\tau)}}. \tag{55}$$

§5 Conclusion

The analytical view of fractional Newell-Whitehead-Segel is done via the Homotopy perturbation transform method. The closed contact between the exact and obtained solution is shown by its graphical representation. The plot of fractional order problem solutions has confirmed the convergence of fractional solutions towards a solution at integer order of the targeted problems. The higher accuracy is achieved with a small number of calculations. The effective and straightforward implementation attracts the researchers to analyze the fractional view of other problems in different areas of applied science.

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Declarations

Conflict of interest The authors declare no conflict of interest.

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