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Navigation Finsler metrics on a gradient Ricci soliton

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Abstract. In this paper, we study a class of Finsler metrics defined by a vector field on a gradient Ricci soliton. We obtain a necessary and sufficient condition for these Finsler metrics on a compact gradient Ricci soliton to be of isotropic S-curvature by establishing a new integral inequality. Then we determine the Ricci curvature of navigation Finsler metrics of isotropic S-curvature on a gradient Ricci soliton generalizing result only known in the case when such soliton is of Einstein type. As its application, we obtain the Ricci curvature of *all* navigation Finsler metrics of isotropic S-curvature on Gaussian shrinking soliton.

§1 Introduction

An important approach in studying Finsler geometry is the navigation problem. Let (M^n, Φ) be an n-dimensional Finsler manifold and V be a vector field on M^n with $\Phi(x, V_x) < 1$, $\forall x \in M^n$. Let $F = F(x, y)$ denote the Finsler metric on M^n defined by

$$\Phi(x, \frac{y}{F} + V_x) = 1.$$

We say F is a *navigation Finsler metric* with respect to V on (M^n, Φ).

Recently, the study of navigation Finsler metrics has attracted a lot of attention. Huang-Mo showed that the flag curvature of navigation Finsler metrics is non-increasing by homothetic navigation problem [12]. Furthermore, they gave a geometric description of the geodesics of such navigation Finsler metric [8]. Shen-Xia and Xia determined the flag curvature of the navigation Finsler metrics on a Randers manifold with some special curvature properties by the conformal navigation problem [19, 22]. Later on, Huang-Mo determined the flag curvature of the navigation Finsler metrics on any Finsler manifold in terms of conformal navigation problem and therefore they provided a unifying frame work for results due to Bao-Robles-Shen, Cheng-Shen, Foulon and Mo-Huang [9].

In Finsler geometry, there are several important non-Riemannian quantities, such as the distortion τ, the mean Carton torsion I and the S-curvature, etc. They all vanish for Riemannian metrics, hence they said to be non-Riemannian. These quantities interact with the

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Riemann curvature in a delicate way. The S-curvature (mean covariation in an alternative terminology in [16]) be introduced in [17]. Z. Shen proved that the S-curvature and the Ricci curvature determine the local behavior for the Busemann-Hausdorff measure of small metric balls around a point [18]. An n-dimensional Finsler metric is said to have *isotropic S-curvature* if $\mathbf{S}(x, y) = (n+1)c(x)F(x, y)$, where c is a scalar function on M.

Recently, great progress has been made in discussing Finsler metrics of isotropic S-curvature. In 2009, Cheng-Shen characterized (α, β)-metrics with isotropic S-curvature [5]. Lately, Zhou found an equation that characterizes spherically symmetric Finsler metrics of isotropic S-curvature [23]. Mo-Yang gave an explicit construction of a 3-parameter family of non-locally projectively flat Finsler metrics of non-constant isotropic S-curvature [13].

In this paper, we obtain a necessary and sufficient condition for navigation Finsler metrics on a compact Ricci soliton to be of isotropic S-curvature by establishing a new integral inequality. We have the following:

Theorem 1.1. *Let $(M^n,\, h,\, f,\, \rho)$ be a compact gradient Ricci soliton, and let V be a vector field on M. Let F be the navigation Finsler metric with respect to $(h,\, V)$. Then*

$$\int_M \left[\rho h(x, V)^2 - |\nabla V|^2 - \frac{n-2}{n}(divV)^2 + \langle \nabla f, \nabla_V V + (divV)V\rangle_h\right] dv \leq 0, \tag{1}$$

and the equality holds if and only if F has isotropic S-curvature, $\mathbf{S} = (n+1)cF$, where c is a scalar function on M. In this case, we have the following

$$\int_M [1 + h(x,\, V)]^2 \left[\mathrm{Ric}(x,\, V) - (n-1)\left(\frac{3c_{x^i}V^i}{F(x,\, V)} - c^2 + 2c_{x^i}V^i\right) F(x,\, V)^2\right] dv \geq 0, \tag{2}$$

where $c_{x^i} = \frac{\partial c}{\partial x^i}$ and $V = V^i \frac{\partial}{\partial x^i}$.

Recall that a Riemannian manifold $(M,\, h)$ is called a *gradient Ricci soliton* if the equation $\mathrm{Ric}_f = \rho h^2$ holds for some function f and constant ρ where

$$\mathrm{Ric}_f := {}^{\mathrm{h}}\mathrm{Ric} + \mathrm{Hess}(f) \tag{3}$$

is the Bakry-Émery curvature where ${}^{h}\mathrm{Ric}$ denotes the Ricci curvature of $(M,\, h)$ and $\mathrm{Hess}(f)$ the Hessian of f. The function f is called a *potential function.* Gradient Ricci solitons play an important role in Hamilton's Ricci flow as they correspond to self-similar solitons, and often arise as singularity models.

For the proof of Theorem 1.1, see the proof of Theorem 4.2 below. We mention that there are many compact gradient Ricci solitons. One obtains, besides Einstein metrics, the example of Koiso, Cao and Wang etc. on compact $n(\geq 4)$-dimensional gradient shrinking Ricci solitons that are not Einstein [1, 10, 21].

Theorem 1.1 tells us that for navigation Finsler metrics with respect to (h, V) of isotropic S-curvature on a compact gradient Ricci solution (M^n, h, f, ρ), its Ricci curvature in the direction V has restriction (2). In fact, we determine Ricci curvature in each direction of *all* navigation Finsler metrics of isotropic S-curvature on a gradient Ricci solution.

Theorem 1.2. *Let $(M^n,\, h,\, f,\, \rho)$ be a gradient Ricci soliton, and let V be a vector field on M^n. Let F be the navigation Finsler metric with respect to $(h,\, V)$. Assume that F has isotropic S-curvature, $\mathbf{S} = (n+1)cF$. Then the Ricci curvature of F satisfies*

$$\mathrm{Ric}(x,\, y) = (n-1)\left[\frac{\rho}{n-1} + \frac{3c_{x^i}y^i}{F} - c^2 + 2c_{x^i}V^i\right] F(x,\, y)^2 - \mathrm{Hess}(f)\,(\xi,\, \xi) \tag{4}$$

where $\xi = y + F(x, y)V$.

Gradient Ricci solitons contain both Einstein Riemannian manifolds when the potential functions f are constant functions and Gaussian shrinking soliton, namely, the flat Euclidean space $(\mathbb{R}^n, g_0)$ with the potential function $f = \frac{|x|^2}{4}$. In Theorem 4.2 in [4], Cheng-Shen showed an expression of Ricci curvature for navigation Finsler metrics of isotropic S-curvature on a Einstein Riemannian manifold. It follows that Theorem 1.2 extends Cheng-Shen's expression (4.2) in [4]. As an application of Theorem 1.2, we obtain the Ricci curvature of *all* navigation Finsler metrics of isotropic S-curvature on a Gaussian shrinking soliton (see Example 5.2 below).

§2 Preliminaries

Let (M, h) be a Riemannian manifold. It is known that a navigation Finsler metric F on (M, h) can be expressed in the following form:

$$F = \frac{\sqrt{h(x, y)^2 - [h(x, V_x)^2 h(x, y)^2 - \langle y, V_x\rangle_h^2]}}{1 - h(x, V_x)^2} + \frac{\langle y, V_x\rangle_h}{1 - h(x, V_x)^2}, \tag{5}$$

where V is a vector field on M^n with $h(x, V_x) < 1$ for all $x \in M$ and $\langle\ ,\ \rangle_h$ denotes the inner product defined by h. We have the following

Lemma 2.1. *Let (M, h) be an n-dimensional Riemannian manifold, and let V be a vector field on M. Then V is conformal if and only if the navigation Finsler metric with respect to (h, V) has isotropic S-curvature.*

Recall that a vector field V on an n-dimensional Riemannian manifold (M, h) is *conformal with dilation* σ if the Lie derivative of the metric with respect to V satisfies

$$\mathcal{L}_V h^2 = 2\sigma(x)h^2,$$

where $\sigma : M \to \mathbb{R}$. This is equivalent to the fact the one-parameter group of diffeomorphisms generated by V consists of conformal transformations.

Let (M, h) be a Riemannian manifold, V a smooth vector field and ρ a constant. The system (M, h, V, ρ) is said to be a *Ricci soliton* if

$$2\mathrm{Ric} + \mathcal{L}_V h^2 = 2\rho h^2 \tag{6}$$

where $\mathcal{L}$ denotes the Lie-derivative operator and Ric the Ricci tensor of h [2]. Thus a Ricci soliton is a generalization of an Einstein metric for which $V = 0$ or V is a Killing field. If the vector field X is the gradient of a smooth function f, i.e $V = \nabla f$, then (M, h, f, ρ) is called a *gradient Ricci soliton*, in which case, the Equation (6) becomes

$$\mathrm{Ric} + \mathrm{Hess}(f) = \rho h^2 \tag{7}$$

where Hess denotes the Hessian operator with respect to h. An important result of Perelman tells us that a compact Ricci soliton is gradient [15].

Suppose that $\omega_1, \cdots, \omega_n$ is an orthonormal coframe of (M, h). Hence we have

$$h^2 = \omega_1^2 + \cdots + \omega_n^2,$$

$$d\omega_i = -\sum_j \omega_{ij} \wedge \omega_j, \qquad \omega_{ij} + \omega_{ji} = 0 \tag{8}$$

(the first structure equation) and

$$d\omega_{ij} = -\sum_k \omega_{ik} \wedge \omega_{kj} + \Omega_{ij}, \qquad \Omega_{ij} = \frac{1}{2}\sum_{k,l} R_{ijkl}\omega_k \wedge \omega_l \tag{9}$$

(the second structure equation). The Ricci curvature of (M, h) is defined by

$$\mathrm{Ric} = \sum_{i,j} R_{ikjk}\omega_i \otimes \omega_j. \tag{10}$$

Consider a 1-form $\theta := \sum \theta_i\omega_i$. By (8), we obtain

$$d\theta = \sum_i D\theta_i \wedge \omega_i, \tag{11}$$

where

$$D\theta_i := d\theta_i - \sum_j \theta_j\omega_{ji} =: \sum_j \theta_{ij}\omega_j. \tag{12}$$

Differentiating (12) and using (8) and (9), one deduces

$$\sum_j D\theta_{ij} \wedge \omega_j = \sum_j \theta_j\Omega_{ji}, \tag{13}$$

where

$$D\theta_{ij} := d\theta_{ij} - \sum_k \theta_{ik}\omega_{kj} - \sum_k \theta_{kj}\omega_{ki} =: \sum_k \theta_{ijk}\omega_k. \tag{14}$$

Plugging (14) and (9) into (13), we have

$$\theta_{ijk} = \theta_{ikj} - \sum_l \theta_l R_{likj}. \tag{15}$$

§3 Some lemmas

Recall that a *metric measure space* $(M, h, e^{-f}dv)$ is a Riemannian manifold (M, h) together with a weighted volume form $e^{-f}dv$ on M, where f is a scalar function on M and dv the volume element induced by the metric h. A gradient Ricci soliton is just right a metric measure space satisfying $\mathrm{Ric}_f = \rho h^2$ for some constant ρ. The Bakry-Émery curvature Ric_f associated to smooth metric measure space $(M, h, e^{-f}dv)$ is defined in (3) where Ric denotes the Ricci curvature of (M, h), $\mathrm{Hess}(f)$ the Hessian of f and dv the volume element induced by the metric h. We know that

$$\mathrm{Hess}(f) = \sum_{i,j} f_{ij}\omega_i \otimes \omega_j \tag{16}$$

where $\omega_1, \cdots, \omega_n$ is an orthonormal coframe of (M, g) and

$$\sum_j f_{ij}\omega_j = df_i - \sum_j f_j\omega_{ji}, \quad \sum_i f_i\omega_i = df. \tag{17}$$

We have the following result (see [11]):

Lemma 3.1. *Let (M, h) be an n-dimensional Riemannian manifold, and let V be a vector field on M. Then V is conformal if and only if $V = \sum_{i=1}^n V_ie_i$ satisfies the following:*

$$V_{ij} + V_{ji} = \frac{2}{n}(divV)\delta_{ij} \tag{18}$$

for $i, j = 1, \cdots, n$ where $e_1, \cdots, e_n$ is the dual frame of $\omega_1, \cdots, \omega_n$ and

$$\sum_j V_{ij}\omega_j = dV_i - \sum_j V_j\omega_{ji}. \tag{19}$$

Lemma 3.2. *Let $(M, h, e^{-f}dv)$ be a metric measure space, and let V be a vector field on M. Take*

$$\widetilde{V} := \nabla_V V + (\langle \nabla f, V\rangle_h - divV)V \tag{20}$$

where ∇ denotes the Levi-Civita connection of h and div is the divergence with respect to h. Then the divergence of $\widetilde{V}$ is given by

$$div\widetilde{V} = \mathrm{Ric}_f(V,V) - (divV)^2 + \sum_{i,j} V_{ij}V_{ji} + \langle \nabla f, \nabla_V V + (divV)V\rangle_h \tag{21}$$

where Ric_f denotes the Bakry-Émery curvature.

Proof In fact,

$$\begin{aligned}\nabla V &= \sum_i \nabla(V_i e_i) \\ &= \sum_i dV_i \otimes e_i + \sum_i V_i \nabla e_i \\ &= \sum_i (dV_i - \sum_j V_j \omega_{ji}) \otimes e_i = \sum_{ij} V_{ij}\omega_j \otimes e_i\end{aligned} \tag{22}$$

where we have used the fact

$$\nabla e_i = -\sum_j \omega_{ji} e_j. \tag{23}$$

It follows that

$$\nabla_V V = \sum_{i,j} V_{ij}\omega_j(V)e_i = \sum_{i,j} V_j V_{ij} e_i. \tag{24}$$

Note that

$$divV = \sum_i V_{ii}, \quad \langle \nabla f, V\rangle_h = \sum_i f_i V_i. \tag{25}$$

Plugging (24) and (25) into (20) yields

$$\widetilde{V} = \sum_{i,j} (V_i V_{ji} - V_{ii}V_j + f_i V_i V_j) e_j$$

It follows that

$$div\widetilde{V} = \sum_{i,j} \left[V_i V_{ji} + (f_i V_i - V_{ii})V_j\right]_j = \sum_{i,j} V_{ij}V_{ji} + (I) - (II) - \sum_{i,j} V_{iij}V_j \tag{26}$$

where

$$(II) := \sum_i V_{ii} \sum_j V_{jj} = (divV)^2 \tag{27}$$

$$\begin{aligned}(I) &:= \sum_{i,j} (f_i V_i V_j)_j + \sum_{i,j} V_i V_{jij} \\ &= \sum_{i,j} f_{ij}V_iV_j + \sum_{i,j} f_i V_{ij} V_j + \sum_{i,j} f_i V_i V_{jj} + \sum_{i,j} V_i (V_{jji} - \sum_k V_k R_{kjji}) \\ &= \mathrm{Hess}(f)(V,V) + \langle \nabla f, \nabla_V V\rangle_h + \langle \nabla f, V\rangle_h divV + \sum_{i,j} V_j V_{iij} \\ &\quad + \sum_{i,j,k} V_i V_k R_{kjij}\end{aligned}$$

$$
\begin{aligned}
&= \mathrm{Hess}(f)(V,V) + \langle \nabla f, \nabla_V V + (divV)V\rangle_h + \sum_{i,j} V_{iij}V_j + \mathrm{Ric}(V,V) \\
&= \mathrm{Ric}_f(V,V) + \sum_{i,j} V_{iij}V_j + \langle \nabla f, \nabla_V V + (divV)V\rangle_h
\end{aligned}
\tag{28}
$$

where we have used (26), (16), (25), (3) and the following Ricci identity

$$V_{ijk} = V_{ikj} - \sum V_l R_{likj}$$

where V_{ijk} is defined by

$$\sum_k V_{ijk}\omega_k = dV_{ij} - \sum_k V_{ik}\omega_{kj} - \sum_k V_{kj}\omega_{ki}.$$

Plugging (27) and (28) into (26) yields (22). $\square$

We mention that Lemma 3.2 refines Lemma 2.2 in [11] when the potential functions f is a constant.

§4 Integral inequalities and their applications

In this section, first we are going to show a generalization of Theorem 1.1. Let V be a vector field on a Riemannian manifold $(M,\, h)$. We define the square of the length of ∇V by

$$|\nabla V|^2 = \sum_{i,j} V_{ij}^2. \tag{29}$$

Theorem 4.1. *Let $(M, h, e^{-f}dv)$ be an n-dimensional compact metric measure space, and let V be a vector field on M. Then we have*

$$\int_M \big[\mathrm{Ric}_f(V,V) - |\nabla V|^2 - \frac{n-2}{n}(divV)^2 + \langle \nabla f, \nabla_V V + (divV)V\rangle_h\big]dv \le 0 \tag{30}$$

and the equality holds if and only if the navigation Finsler metric on $(M,\, h)$ with respect to V has isotropic S-curvature.

Proof By using (21) and (29), we have

$$
\begin{aligned}
0 &= \int_M (div\widetilde{V})dv \\
&= \int_M \big[\mathrm{Ric}_f(V,V) - (divV)^2 + \sum_{i,j} V_{ij}V_{ji} + g(\nabla f, \nabla_V V + (divV)V)\big]dv \\
&= \int_M \big[\mathrm{Ric}_f(V,V) - (divV)^2 + \sum_{i,j} V_{ij}V_{ji} + \frac{1}{2}\sum_{i,j} V_{ij}^2 + \frac{1}{2}\sum_{i,j} V_{ji}^2 - \sum_{i,j} V_{ij}^2 \\
&\quad + \langle \nabla f, \nabla_V V + (divV)V\rangle_h\big]dv \\
&= \int_M \big[\mathrm{Ric}_f(V,V) - (divV)^2 + \frac{1}{2}\sum_{i,j}(V_{ij}+V_{ji})^2 - |\nabla V|^2 \\
&\quad + \langle \nabla f, \nabla_V V + (divV)V\rangle_h\big]dv \\
&= \int_M \big[\mathrm{Ric}_f(V,V) - |\nabla V|^2 - \frac{n-2}{n}(divV)^2 + \langle \nabla f, \nabla_V V + (divV)V\rangle_h\big]dv \\
&\quad + \frac{1}{2}\int_M \Big\{\sum_{i,j}\big[V_{ij} + V_{ji} - \frac{2}{n}\delta_{ij}(divV)\big]^2\Big\}dv.
\end{aligned}
$$

It follows that

$$\begin{aligned}&\int_M \big[Ric_f(V,V) - |\nabla V|^2 - \frac{n-2}{n}(divV)^2 + \langle \nabla f, \nabla_V V + (divV)V\rangle_h\big]dv \\ &= -\frac{1}{2}\int_M \Big\{\sum_{i,j}\big[V_{ij} + V_{ji} - \frac{2}{n}\delta_{ij}(divV)\big]^2\Big\}dv \le 0.\end{aligned} \tag{31}$$

It implies (30).

If the equality holds in (30), then the equality in (31) must hold. Thus we must have (18) for $i, j = 1, \cdots, n$; that is, V is conformal. According to Lemma 2.1, the navigation Finsler metric F with respect to V on (M, h) has isotropic S-curvature. □

We mention that when the potential function f is vanishing, Theorem 4.1 had been studied in [11].

For a compact Ricci soliton, we have the following:

Theorem 4.2. *Let (M^n, h, X, ρ) be a compact Ricci soliton, and let V be a vector field on M. Then*

$$\int_M \big[\rho h(x,V)^2 - |\nabla V|^2 - \frac{n-2}{n}(divV)^2 + \langle X, \nabla_V V + (divV)V\rangle_h\big]dv \le 0, \tag{32}$$

where $X = \nabla f$ and the equality holds if and only if the navigation Finsler metric on (M, h) with respect to V has isotropic S-curvature. In this case, we have (2) holds.

Proof By Perelman's result, there is a smooth function f, such that

$$X = \nabla f. \tag{33}$$

It follows that (7) holds. Substituting (7) and (33) into (30) yields (32). By using Theorem 1.1 in [14], we have (2) holds. □

We mention that there are many compact (gradient) Ricci solitons. One obtains, besides Einstein metrics, the examples of nontrivial compact gradient shrinking (Ricci) solitons [1, 10, 21]. Below are two important examples.

(a) For real dimension 4, the first example of a compact shrinking Ricci soliton was constructed in the early 90's by Cao and Koiso independently on compact complex surface $\mathbb{CP}^2\sharp(-\mathbb{CP}^2)$, where $(-\mathbb{CP}^2)$ denotes the complex projective plane with the opposite orientation [1, 10]. It has $U(2)$ symmetry and positive Ricci curvature. More generally, they found $U(n)$-invariant Kähler-Ricci solitons on twisted projective line bundle over $\mathbb{CP}^{n-1}$ for $n \ge 2$ [10].

(b) In [21], Wang etc. found a gradient Ricci soliton on $\mathbb{CP}^2\sharp 2(-\mathbb{CP}^2)$ which has $U(1)\times U(1)$ symmetry. More generally, they proved the existence of gradient Kähler-Ricci solitons on all Fano toric varieties of complex dimension $n \ge 2$ with non-vanishing Futaki invariant.

§5 Ricci curvature of navigation Finsler metrics

In this section, we are going to determine Ricci curvature in each direction of *all* navigation Finsler metrics of isotropic S-curvature on a gradient Ricci solution. As its application, we obtain the Ricci curvature of *all* navigation Finsler metrics of isotropic S-curvature on Gaussian shrinking soliton.

Proof of Theorem 1.2 Note that h is a Riemannian metric. Hence its navigation Finsler metric F is a Randers metric by (5). By using [9] or (3.7) in [14], we have

$$ {}^{h}Ric(\xi,\xi) = Ric(x,y) - (n-1)\left\{\frac{3c_{x^i}y^i}{F} - c^2 + 2c_{x^i}V^i\right\}F^2(x,y), \tag{34} $$

where $\xi := y + F(x,y)V$ satisfies

$$ h(x,\xi) = F(x,y). \tag{35} $$

For gradient Ricci soliton (M,h,f,ρ), we have the following

$$ {}^{h}Ric + Hess(f) = \rho h^2 $$

from (3) where ${}^{h}Ric$ denotes the Ricci tensor of h. Plugging this into (34) and using (35), we get

$$ \begin{aligned} R(x,y) &= {}^{h}Ric(\xi,\xi) + (n-1)\left\{\frac{3c_{x^i}y^i}{F} - c^2 + 2c_{x^i}V^i\right\}F^2(x,y) \\ &= \rho h(x,\xi)^2 - Hess(f)(\xi,\xi) + (n-1)\left\{\frac{3c_{x^i}y^i}{F} - c^2 + 2c_{x^i}V^i\right\}F^2(x,y) \\ &= \rho F(x,y)^2 + (n-1)\left\{\frac{3c_{x^i}y^i}{F} - c^2 + 2c_{x^i}V^i\right\}F^2(x,y) - Hess(f)(\xi,\xi) \\ &= (n-1)\left\{\frac{\rho}{n-1} + \frac{3c_{x^i}y^i}{F} - c^2 + 2c_{x^i}V^i\right\}F^2(x,y) - Hess(f)(\xi,\xi). \end{aligned} $$

Thus we obtain (4). □

Example 5.2 Consider the following Gaussian shrinker, namely, the flat Euclidean space $(\mathbb{R}^n, h_0)$ with the potential function $\frac{|x|^2}{4}$. Then [7]

$$ Hess\left(\frac{|x|^2}{4}\right) = \frac{1}{2}h_0^2, \quad Ric_{\frac{|x|^2}{4}} = \frac{1}{2}h_0^2. \tag{36} $$

We obtain the gradient Ricci soliton $(\mathbb{R}^n, h_0, \frac{|x|^2}{4}, \frac{1}{2})$ [3, 7]. Let F be the navigation Finsler metric with respect to V on $(\mathbb{R}^n,\ h_0)$ where V is a vector field on $\mathbb{R}^n$. Assume $n \geq 3$. Then F has isotropic S-curvature, that is, $\mathbf{S} = (n+1)cF$ for some scalar function c, if and only if,

$$ c = \delta + \langle a, x\rangle \tag{37} $$

where δ is a constant and $a = (a^i) \in \mathbb{R}^n$ is a constant vector, and V is given by

$$ V = -2\left[(\delta + \langle a, x\rangle)x - \frac{|x|^2}{2}a\right] + xQ + b \tag{38} $$

where Q is a fixed anti-symmetric matrix and b is a constant vector [6]. Now we are going to calculate the Ricci curvature of all navigation Finsler metrics of isotropic S-curvature on Gaussian shrinking soliton $(\mathbb{R}^n, h_0, \frac{|x|^2}{4}, \frac{1}{2})$ by using (4). From (35) and the first equation of (36), we have

$$ Hess\left(\frac{|x|^2}{4}\right)(\xi,\xi) = \frac{1}{2}h_0^2(x,\xi) = \frac{1}{2}F^2(x,y). \tag{39} $$

By (37),

$$ c_{x^j} = a^j. \tag{40} $$

It follows that

$$ c_{x^j}y^j = \langle a, y\rangle. \tag{41} $$

Using (38) and (40), we have,

$$c_{x^j}V^j = -2\left[(\delta + \langle a, x\rangle)\langle a, x\rangle - \frac{|x|^2}{2}|a|^2\right] + \langle xQ,\, a\rangle + \langle a, b\rangle. \tag{42}$$

Substituting (41), (37), (39) and (42) into (4), we get

$$\begin{aligned} Ric(x, y) &= (n-1)\Bigg\{\frac{1}{2(n-1)} + \frac{3\langle a, y\rangle}{F} - (\delta + \langle a, x\rangle)^2 + 2\Big[- 2(\delta + \langle a, x\rangle)\langle a, x\rangle \\ &\quad + |a|^2|x|^2 + \langle xQ, a\rangle + \langle a, b\rangle\Big]\Bigg\}F^2(x, y) - \frac{1}{2}F^2(x, y) \\ &= (n-1)\Bigg[\frac{3\langle a, y\rangle}{F} - (\delta + \langle a, x\rangle)^2 - 4(\delta + \langle a, x\rangle)\langle a, x\rangle \\ &\quad - 2(|a|^2|x|^2 + \langle xQ, a\rangle + \langle a, b\rangle)\Bigg]F^2(x, y). \end{aligned}$$

It follows that navigation Finsler metric F is of Einstein type if and only if F has constant S-curvature, $\mathbf{S} = (n+1)\delta F$.

Declarations

Conflict of interest The authors declare no conflict of interest.

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