Appl. Math. J. Chinese Univ.
2024, 39(2): 253-265

Multivariate form of Hermite sampling series

Rashad M. Asharabi

Abstract. In this paper, we establish a new multivariate Hermite sampling series involving samples from the function itself and its mixed and non-mixed partial derivatives of arbitrary order. This multivariate form of Hermite sampling will be valid for some classes of multivariate entire functions, satisfying certain growth conditions. We will show that many known results included in Commun Korean Math Soc, 2002, 17: 731-740, Turk J Math, 2017, 41: 387-403 and Filomat, 2020, 34: 3339-3347 are special cases of our results. Moreover, we estimate the truncation error of this sampling based on localized sampling without decay assumption. Illustrative examples are also presented.

§1 Introduction

Let $\mathcal{E}_\sigma^n$ be the class of all multivariate entire functions which satisfy one of the following growth conditions

$$|f(z)| \leq \frac{C\exp\left(\sum_{j=1}^{n}\sigma_j|\Im z_i|\right)}{\prod_{i=1}^{n}(1+|\Re z_i|)}, \qquad |f(z)| \leq \frac{C\exp\left(\sum_{j=1}^{n}\sigma_j|\Im z_i|\right)}{\prod_{i=1}^{n}(1+|\Im z_i|)}, \tag{1.1}$$

where z is the complex vector $z := (z_1, \ldots, z_n) \in \mathbb{C}^n$ and C is a positive number. Throughout this paper, the multivariate entire function f is an entire function of each z_j when the other variables are kept fixed. $\mathcal{E}_\sigma^n$-functions are of exponential type $\sigma := (\sigma_1, \sigma_2, \ldots, \sigma_n)$ which do not necessarily belong to $L^p(\mathbb{R}^n)$, $1 \leq p < \infty$, when restricted to $\mathbb{R}^n$. The n-variate entire function f is said to be of exponential type σ if there exists positive constant C such that

$$|f(z)| \leq C\exp\left(\sum_{j=1}^{n}\sigma_j|\Im z_i|\right), \quad z := (z_1, \ldots, z_n) \in \mathbb{C}^n. \tag{1.2}$$

Shin (2002) establishes the class $\mathcal{E}_\sigma^1$, cf. [16], while the class $\mathcal{E}_\sigma^2$ is introduced in [6]. He proves that if $f \in \mathcal{E}_\sigma^1$, then we can expand it via the following generalized Hermite sampling series

$$f(z_1) = \sum_{n=-\infty}^{\infty}\sum_{i+j+\ell=r} f^{(i)}(nh)\,\frac{\sin^{r+1}\left(\pi h^{-1}z_1\right)}{i!\,\ell!\,(z_1-nh)^{j+1}}\left[\frac{d^\ell}{dz_1^\ell}\left(\frac{z_1-nh}{\sin\left(\pi h^{-1}z_1\right)}\right)^{r+1}\right]_{z_1=nh}, \tag{1.3}$$

Received: 2020-01-22. Revised: 2021-12-25.
MR Subject Classification: 94A20, 30E10, 30D10, 41A63, 40B05.
Keywords: multidimensional sampling series, sampling with partial derivatives, contour integral, truncation error.
Digital Object Identifier(DOI): https://doi.org/10.1007/s11766-024-4024-9.

which converges uniformly on any compact subset of $\mathbb{C}$ where $z_1 \in \mathbb{C}$, $h := (r+1)\pi/\sigma$ and $r \in \mathbb{N}_\circ$, cf. [16]. Asharabi and Al-Haddad (2016) study a bivariate form of generalized Hermite sampling series and they prove that if $f \in \mathcal{E}_\sigma^2$, then we can reconstruct it via the following sampling series, [6],

$$f(z_1, z_2) = \sum_{k\in\mathbb{Z}^2} \sum_{i+s+\ell=r} \sum_{j+\tau+l=r} f^{(i,j)}(k_1h_1, k_2h_2) \frac{\delta_\ell^r(k_1)\delta_l^r(k_2)\sin^{r+1}\left(\pi h_1^{-1}z_1\right)\sin^{r+1}\left(\pi h_2^{-1}z_2\right)}{i!\ell!j!l!(z_1-k_1h_1)^{s+1}(z_2-k_2h_2)^{\tau+1}}, \tag{1.4}$$

where $(z_1, z_2) \in \mathbb{C}^2$, $k := (k_1, k_2) \in \mathbb{Z}^2$, $h_j := (r+1)\pi/\sigma_j$, $j = 1, 2$ and δ_l^r is defined by

$$\delta_l^r(\nu) := \left[\frac{d^l}{d\zeta^l}\left(\frac{\zeta - \nu h}{\sin\left(\pi h^{-1}\zeta\right)}\right)^{r+1}\right]_{\zeta=\nu h}. \tag{1.5}$$

Series (1.4) converges uniformly on any compact subset of $\mathbb{C}^2$, [6]. The series (1.4) involves samples from the function and its mixed and non-mixed partial derivatives.

Fang and Li (2006) introduced a multivariate form of Hermite sampling series involving only samples from all the first partial derivatives for functions from Bernstein space. The Bernstein space, $B_{\sigma,p}^n$, is the class of all entire functions of n-variables which satisfy the growth condition (1.2) and belong to $L^p(\mathbb{R}^n)$ when restricted to $\mathbb{R}^n$. They prove that if $f \in B_{2\sigma,p}^n$, then we have the multivariate sampling series, [8,9],

$$f(x) = \sum_{k\in\mathbb{Z}^n}\left\{f\left(\frac{k\pi}{\sigma}\right) + \sum_{j=1}^n f'_{x_j}\left(\frac{k\pi}{\sigma}\right)\left(x_j - \frac{k_j\pi}{\sigma_j}\right)\right\}\prod_{j=1}^n \operatorname{sinc}^2\left(\sigma_j x_j - k_j\pi\right), \tag{1.6}$$

where $k := (k_1, \ldots, k_n) \in \mathbb{Z}^n$, $k/\sigma := (k_1/\sigma_1, \ldots, k_n/\sigma_n)$ $x := (x_1, \ldots, x_n) \in \mathbb{R}^n$ and $f'_{x_j} = \partial f/\partial x_j$ for all $j = 1, \ldots, n$. Series (1.6) converges absolutely and uniformly on $\mathbb{R}^n$. In fact, we can verify that formula in (1.6) is also justified on $\mathbb{C}^n$ and converges uniformly on any compact subset of $\mathbb{C}^n$. The sinc function is defined via

$$\operatorname{sinc}(t) := \begin{cases} \dfrac{\sin(\pi t)}{\pi t}, & t \neq 0, \\ 1, & t = 0. \end{cases} \tag{1.7}$$

We would like to mention here that the series in (1.3) is valid for all functions in $B_{\sigma,p}^1$, [16,17], and series (1.4) validates for $f \in B_{\sigma,p}^2$, [6].

The classical multivariate sampling, which only involves samples from the function, has been extensively investigated in many studies, see e.g. [2,10,14,15,19,20] and references therein, while the studies of the multivariate sampling involving samples from the function itself and its partial derivatives are very few. Motivated by Fang-Li's sampling formula (1.6) and the bivariate Hermite formula (1.4), we derive a multivariate form of Hermite sampling series for functions from the classes $\mathcal{E}_\sigma^n$ and $B_{\sigma,p}^n$. The results of Shin (2002), Fang and Li (2006), Asharabi and Al-Haddad (2017) and Norvidas (2020) will be special cases of our multivariate Hermite sampling series. To establish this multivariate form of Hermite sampling, we will use techniques slightly different from those in the bivariate case. The layout of the paper is as follows: The next section is devoted to establishing a multivariate Hermite sampling formula for all functions which belong to the classes $\mathcal{E}_\sigma^n$ and $B_{\sigma,p}^n$. The truncation error estimate will be introduced in Section 3. Section 4 deals with illustrative examples.

§2 Multivariate Hermite sampling

In this section, we establish a multivariate Hermite sampling for functions from the classes $\mathcal{E}_\sigma^n$ and $B_{\sigma,p}^n$, which are defined above. We show that some known results will be special cases for our sampling formula. Before introducing the main results of this section, we introduce some notions and auxiliary results which benefit us in the proof of these results.

Let M_j be positive numbers and define $\mathcal{S}_j$, $1 \le j \le n$ to be

$$\mathcal{S}_j := \{z_j \in \mathbb{C} : |z_j| \le M_j \quad \text{and} \quad z_j \ne k_j h_j, \quad k_j \in \mathbb{Z}\}, \tag{2.8}$$

where $h_j = (r+1)\pi/\sigma_j$ and $r \in \mathbb{N}_\circ$. Denote $\mathcal{R}_{j,N}$ as the rectangular path whose vertices are $\pm\tau_{j,N} \pm \mathrm{i}\tau_{j,N}$, $\tau_{j,N} = (N+1/2)h_j$. Assume that $f(\ldots,\zeta_j,\ldots) \in \mathcal{E}_\sigma$ as a function of ζ_j and $\{\zeta_k\}_{j\ne k=1}^n$ are arbitrary fixed complex parameters. Then we have, as it appears in Lemma 2.1 of [6], for all $1 \le j \le n$

$$\int_{\mathcal{R}_{j,N}} \frac{f(\zeta)\,\mathrm{d}\zeta_j}{(\zeta_j - z_j)\sin^{r+1}(\pi h_j^{-1}\zeta_j)} \quad \text{converges to zero uniformly on } S_j \text{ as } N \to \infty, \tag{2.9}$$

wherever $z_j \in S_j$. Define the entire function $P_{z,r}$ to be

$$P_{z,r}(\zeta) := -\prod_{j=1}^n \left(\sin^{r+1}(\pi h_j^{-1}\zeta_j) - \sin^{r+1}(\pi h_j^{-1}z_j)\right) + \prod_{j=1}^n \sin^{r+1}(\pi h_j^{-1}\zeta_j), \tag{2.10}$$

where $z := (z_1,\ldots,z_n), \zeta := (\zeta_1,\ldots,\zeta_n) \in \mathbb{C}^n$ and $r \in \mathbb{N}_\circ$. Consider the kernel function

$$\mathcal{K}_{z,r}(\zeta) := \frac{f(\zeta)P_{z,r}(\zeta)}{\prod_{j=1}^n (\zeta_j - z_j)\sin^{r+1}(\pi h_j^{-1}\zeta_j)}, \tag{2.11}$$

where f is a multivariate entire function. The kernel $\mathcal{K}_{z,r}$ has a singularity of order one at all the points of the set $\{\zeta \in \mathbb{C}^n : \zeta_j = z_j \text{ for some } j\}$ and a singularity of order $r+1$ at all the points of the set $\{\zeta \in \mathbb{C}^n : \zeta_j = k_j h_j \text{ for some } j\}$. Using Leibniz formula n-times, as the authors of [6, Lemma 2.2] are done, we obtain for any multivariate entire function f

$$\left[\frac{\partial^{s_1+\ldots+s_n}}{\partial\zeta_1^{s_1}\ldots\partial\zeta_n^{s_n}} f(\zeta)P_{z,r}(\zeta)\right]_{\zeta=kh} = (-)^{n-1}\prod_{j=1}^n \sin^{r+1}(\pi h_j^{-1}z_j) f^{(s_1,\ldots,s_n)}(kh), \tag{2.12}$$

where $z,\zeta \in \mathbb{C}^n$, $kh := (k_1h_1,\ldots,k_nh_n)$, $z := (z_1,\ldots,z_n)$ and $0 \le s_j \le r$. The following axillary result will be use in the proof of Theorem 2.2.

Lemma 2.1. *Let $f \in \mathcal{E}_\sigma^n$ and $r \in \mathbb{N}_\circ$, then we have*

$$\int_{\mathcal{R}_{n,N}} \cdots \int_{\mathcal{R}_{1,N}} \mathcal{K}_{z,r}(\zeta)\mathrm{d}\zeta_1\ldots\mathrm{d}\zeta_n \text{ converges to zero uniformly on } \mathcal{S}_1\times\ldots\times\mathcal{S}_n \text{ as } N \to \infty. \tag{2.13}$$

Proof. Let A_k be a subset of $I := \{1,2,\ldots,n\}$ such that it contains any k elements and $B_k := I\backslash A_k$. Denote $p(I)$ to the set of all subsets of I. Using (2.10), it is not hard to see that

$$P_{z,r}(\zeta) \;=\; -\sum_{k=1}^n (-1)^{n-k} \sum_{A_k\in p(I)} \prod_{j\in A_k} \sin^{r+1}(\pi h_j^{-1}\zeta_j) \prod_{l\in B_k} \sin^{r+1}(\pi h_l^{-1}z_l). \tag{2.14}$$

Substituting (2.14) into (2.11), we obtain

$$\int_{\mathcal{R}_{n,N}} \cdots \int_{\mathcal{R}_{1,N}} \mathcal{K}_{z,r}(\zeta)\,\mathrm{d}\zeta_1\ldots\mathrm{d}\zeta_n = -\sum_{k=1}^n (-1)^{n-k} \sum_{A_k\in p(I)} \prod_{l\in B_k} \sin^{r+1}(\pi h_l^{-1}z_l)$$

$$\int_{\mathcal{R}_{n,N}} \cdots \int_{\mathcal{R}_{1,N}} \frac{f(\zeta)\prod_{j\in A_k}\sin^{r+1}(\pi h_j^{-1}\zeta_j)}{\prod_{j\in A_k}(\zeta_j - z_j)\prod_{l\in B_k}(\zeta_l - z_l)\sin^{r+1}(\pi h_l^{-1}\zeta_l)}\,\mathrm{d}\zeta_1 \ldots \mathrm{d}\zeta_n. \tag{2.15}$$

Let $\prod_{j\in A_k}\mathcal{R}_{j,N}$ be the Cartesian product of all rectangular paths in ζ_j-plane, $j \in A_k$. From the Cauchy integral formula in higher dimensions, cf. e.g. [18, p.8] and [11, p.26], we have

$$\left\{\{f(\zeta)\}_{\zeta_j = z_j}\right\}_{j\in A_k}\prod_{j\in A_k}\sin^{r+1}(\pi h_j^{-1}z_j) = \int_{\prod\limits_{j\in A_k}\mathcal{R}_{j,N}} \frac{f(\zeta)\prod_{j\in A_k}\sin^{r+1}(\pi h_j^{-1}\zeta_j)}{\prod_{j\in A_k}(\zeta_j - z_j)}\prod_{j\in A_k}\mathrm{d}\zeta_j, \tag{2.16}$$

where we have considered $\{\zeta_j\}_{j\in B_k}$ to be arbitrary fixed parameters. We combine (2.16) and (2.15) to get

$$\begin{aligned}\int_{\mathcal{R}_{n,N}} \cdots \int_{\mathcal{R}_{1,N}} \mathcal{K}_{z,r}(\zeta)\,\mathrm{d}\zeta_1 \ldots \mathrm{d}\zeta_n = -\prod_{l=1}^{n}\sin^{r+1}(\pi h_l^{-1}z_l)\sum_{k=1}^{n}(-1)^{n-k}\sum_{A_k\in p(I)} \\ \int_{\prod\limits_{j\in B_k}\mathcal{R}_{j,N}} \frac{\left\{\{f(\zeta)\}_{\zeta_j=z_j}\right\}_{j\in A_k}}{\prod_{l\in B_k}(\zeta_l - z_l)\sin^{r+1}(\pi h_l^{-1}\zeta_l)}\prod_{j\in B_k}\mathrm{d}\zeta_j.\end{aligned} \tag{2.17}$$

According to (2.9), all integral in the right-hand side of (2.17) converge to zero uniformly on $\prod_{j\in A_k} S_j$ for all $A_k \in p(I)$ as $N \to \infty$. Therefore, the integral in the left-hand side of (2.17) converges to uniformly on $\mathcal{S}_1 \times \ldots \times \mathcal{S}_n$ as $N \to \infty$. $\square$

Now we are ready to establish the first fundamental result of this section.

Theorem 2.2. *Assume that* $k := (k_1, \ldots, k_n) \in \mathbb{Z}^n$, $kh := (k_1h_1, \ldots, k_nh_n)$, $z := (z_1, \ldots, z_n) \in \mathbb{C}^n$ *and* $s_j, m_j, \ell_j \in \mathbb{N}_\circ$, $1 \le j \le n$. *If* $f \in \mathcal{E}_\sigma^n$, *then it can be expanded as the following multivariate Hermite sampling expansion*

$$f(z) = \sum_{k\in\mathbb{Z}^n}\sum_{s_n+m_n+\ell_n=r}\cdots\sum_{s_1+m_1+\ell_1=r} f^{(s_1,\ldots,s_n)}(kh)\prod_{j=1}^{n}\frac{\delta_{\ell_j}^{r}(k_j)\sin^{r+1}\left(\pi h_j^{-1}z_j\right)}{s_j!\ell_j!(z_j - k_jh_j)^{m_j+1}}, \tag{2.18}$$

which converges uniformly on any compact subsets of $\mathbb{C}^n$.

Proof. Let $z_j \in \mathcal{S}_j$ be arbitrary fixed complex parameters. We consider the kernel $\mathcal{K}_{z,r}$ as a function of ζ_1 and $\{\zeta_j\}_{j=2}^n$ are the arbitrary fixed complex parameters. Applying the classical Cauchy integral formula on ζ_1-plane, see e.g. [1, p.141], we obtain

$$\frac{1}{2\pi i}\oint_{\mathcal{R}_{1,N}}\mathcal{K}_{z,r}(\zeta)\,\mathrm{d}\zeta_1 = \mathrm{Res}\left(\mathcal{K}_{z,r};(z_1,\zeta_2,\ldots,\zeta_n)\right) + \sum_{|k_1|\le N}\mathrm{Res}\left(\mathcal{K}_{z,r};(k_1h_1,\zeta_2,\ldots,\zeta_n)\right), \tag{2.19}$$

where $\mathcal{R}_{1,N}$ is the rectangular path which is defined above.
The residues, $\mathrm{Res}\left(\mathcal{K}_{z,r};(\cdot,\zeta_2,\ldots,\zeta_n)\right)$, on ζ_1-plane are given as follows

$$\mathrm{Res}\left(\mathcal{K}_{z,r};(z_1,\zeta_2,\ldots,\zeta_n)\right) := \frac{f(z_1,\zeta_2,\ldots,\zeta_n)\,P_{z,r}(z_1,\zeta_2,\ldots,\zeta_n)}{\sin^{r+1}(\pi h_1^{-1}z_1)\,\mathfrak{P}_k(\zeta)},$$

and for $-N \le k_1 \le N$

$$\mathrm{Res}\left(\mathcal{K}_{z,r};(k_1h_1,\zeta_2,\ldots,\zeta_n)\right) := \frac{\mathfrak{P}_2(\zeta)}{r!}\left\{\frac{\partial^r}{\partial\zeta_1^r}\left(\frac{f(\zeta)P_{z,r}(\zeta)}{(\zeta_1 - z_1)}\left(\frac{\zeta_1 - k_1h_1}{\sin(\pi h_1^{-1}\zeta_1)}\right)^{r+1}\right)\right\}_{\zeta_1=k_1h_1},$$

where

$$\mathfrak{P}_k(\zeta) := \prod_{j=k}^{n} (\zeta_j - z_j) \sin^{r+1}(\pi h_j^{-1} \zeta_j). \tag{2.20}$$

Now we consider the right-hand side of (2.19) as a function of ζ_2 and $\{\zeta_j\}_{j=3}^n$ are the arbitrary fixed complex parameters. Again applying the classical Cauchy integral formula on ζ_2-plane, we get

$$\begin{aligned} \frac{1}{(2\pi i)^2} \oint_{\mathcal{R}_{2,N}} \oint_{\mathcal{R}_{1,N}} \mathcal{K}_{z,r}(\zeta)\, \mathrm{d}\zeta_1 \mathrm{d}\zeta_2 &= \operatorname{Res}\left(\mathcal{K}_{z,r}; (z_1, z_2, \zeta_3, \ldots, \zeta_n)\right) \\ &+ \sum_{|k_1| \le N} \sum_{|k_2| \le N} \operatorname{Res}\left(\mathcal{K}_{z,r}; (k_1 h_1, k_2 h_2, z_n, \ldots, \zeta_n)\right), \end{aligned} \tag{2.21}$$

where the residues, $\operatorname{Res}(\mathcal{K}_{z,r}; (z_1, \cdot, \zeta_3, \ldots, \zeta_n))$ and $\operatorname{Res}(\mathcal{K}_{z,r}; (k_1 h_1, \cdot, \zeta_3, \ldots, \zeta_n))$, on ζ_2-plane are given as follows

$$\operatorname{Res}\left(\mathcal{K}_{z,r}; (z_1, z_2, \zeta_3, \ldots, \zeta_n)\right) := \frac{f\left(z_1, z_2, \zeta_3, \ldots, \zeta_n\right) P_{z,r}\left(z_1, z_2, \zeta_3, \ldots, \zeta_n\right)}{\prod_{k=1}^{2} \sin^{r+1}(\pi h_k^{-1} z_k)\, \mathfrak{P}_3(\zeta)},$$

and for $-N \le k_1, k_2 \le N$

$$\begin{aligned} &\operatorname{Res}\left(\mathcal{K}_{z,r}; (k_1 h_1, k_2 h_2, z_n, \ldots, \zeta_n)\right) := \\ &\frac{\mathfrak{P}_3(\zeta)}{(r!)^2} \left\{ \frac{\partial^{2r}}{\partial \zeta_2^r \partial \zeta_1^r} \left(\frac{f(\zeta) P_{z,r}(\zeta)}{\prod_{j=1}^{2} (\zeta_j - z_j)} \left(\prod_{k=1}^{2} \frac{\zeta_k - k_k h_k}{\sin(\pi h_k^{-1} \zeta_k)} \right)^{r+1} \right) \right\}_{\substack{\zeta_1 = k_1 h_1 \\ \zeta_2 = k_2 h_2}}. \end{aligned}$$

Repeatedly applying the classical Cauchy integral formula on ζ_j-plane, $3 \le j \le n$ and calculating the residues for all times, we obtain

$$\frac{1}{(2\pi i)^n} \oint_{\mathcal{R}_{n,N}} \cdots \oint_{\mathcal{R}_{1,N}} \mathcal{K}_{z,r}(\zeta)\, \mathrm{d}\zeta_1 \ldots \mathrm{d}\zeta_n = \operatorname{Res}\left(\mathcal{K}_{z,r}; z\right) + \sum_{|k_n| \le N} \cdots \sum_{|k_1| \le N} \operatorname{Res}\left(\mathcal{K}_{z,r}; kh\right), \tag{2.22}$$

where $\operatorname{Res}(\mathcal{K}_{z,r}; \cdot)$ is given as follows

$$\operatorname{Res}\left(\mathcal{K}_{z,r}; z\right) := f(z),$$

and for all $-N \le k_1, \ldots, k_n \le N$

$$\begin{aligned} \operatorname{Res}\left(\mathcal{K}_{z,r}; kh\right) &:= \frac{1}{(r!)^n} \left\{ \frac{\partial^{nr}}{\partial \zeta_n^r \ldots \partial \zeta_1^r} \left(\frac{f(\zeta) P_{z,r}(\zeta)}{\prod_{j=1}^{n} (\zeta_j - z_j)} \left(\prod_{k=1}^{n} \frac{\zeta_k - k_k h_k}{\sin(\pi h_k^{-1} \zeta_k)} \right)^{r+1} \right) \right\}_{\zeta = kh} \\ &= \frac{(-1)^n}{(r!)^n} \sum_{s_n + m_n + \ell_n = r} \cdots \sum_{s_1 + m_1 + \ell_1 = r} \begin{pmatrix} r \\ s_k, m_k, \ell_k \end{pmatrix} \\ &\quad \times \left[\frac{\partial^{s_1 + \ldots + s_n}}{\partial \zeta_1^{s_1} \ldots \partial \zeta_n^{s_n}} f(\zeta) P_{z,r}(\zeta) \right]_{\zeta = kh} \prod_{j=1}^{n} \frac{m_j\, \delta_{\ell_j}^r(k_j)}{s_j! \ell_j! (z_j - k_j h_j)^{m_j + 1}}, \end{aligned} \tag{2.23}$$

where we have used the generalized Leibniz formula r-times in the last step of (2.23). Combining

(2.23) with (2.12) implies

$$\operatorname{Res}\left(\mathcal{K}_{z,r};kh\right)=-\sum_{s_n+m_n+\ell_n=r}\cdots\sum_{s_1+m_1+\ell_1=r}f^{(s_1,\ldots,s_n)}(kh)\prod_{j=1}^{n}\frac{\delta^r_{\ell_j}(k_j)\sin^{r+1}\left(\pi h_j^{-1}z_j\right)}{s_j!\ell_j!(z_j-k_jh_j)^{m_j+1}}. \tag{2.24}$$

Therefore

$$\begin{aligned} f(z) \quad &- \sum_{\substack{|k_j|\le N,\\ 1\le j\le n}}\sum_{s_n+m_n+\ell_n=r}\cdots\sum_{s_1+m_1+\ell_1=r}f^{(s_1,\ldots,s_n)}(kh)\prod_{j=1}^{n}\frac{\delta^r_{\ell_j}(k_j)\sin^{r+1}\left(\pi h_j^{-1}z_j\right)}{s_j!\ell_j!(z_j-k_jh_j)^{m_j+1}}, \\ &= \frac{1}{(2\pi i)^n}\oint_{\mathcal{R}_{n,N}}\cdots\oint_{\mathcal{R}_{1,N}}\mathcal{K}_{z,r}(\zeta)\,\mathrm{d}\zeta_1\ldots\mathrm{d}\zeta_n. \end{aligned} \tag{2.25}$$

Since $f\in\mathcal{E}^n_\sigma$, then the integral in the right-hand side of (2.25) converges uniformly to zero on $S_1\times\ldots\times S_n$ as $N\to\infty$, see Lemma 2.1, and the sampling series (2.18) converges uniformly on $S_1\times\ldots\times S_n$. It is not hard to check that the equality (2.18) holds for any point of the set $\{\zeta\in\mathbb{C}^n:\zeta_j=k_jh_j \text{ for some } j\}$. Hence the sampling expansion (2.18) holds for any $\zeta\in\mathbb{C}^n$ such that $|\zeta_j|\le M$ for all $1\le j\le n$. Since $M>0$ is arbitrary, then the series (2.18) is convergent uniformly on any compact subset of $\mathbb{C}^n$. □

Remark 2.3. *Setting $n=1$ in the expansion* (2.18)*, we get Theorem* 3.1 *and Remark* 3.2 *of Shin, [16]. Also letting $n=2$ in* (2.18) *implies Theorem* 3.1 *and* 3.2 *of Asharabi and Al-Haddad, [6].*

Let $A\subseteq I:=\{1,2,\ldots,n\}$. Define $E^n_\sigma(A)$ to be the class of all entire functions of n-variables satisfying the condition

$$|f(z)|\le\frac{C\,\exp\left(\sum_{i=1}^n\sigma_j|\Im z_j|\right)}{\prod_{j\in I\setminus A}\left(1+|\Re z_i|\right)\prod_{j\in A}\left(1+|\Im z_i|\right)}, \tag{2.26}$$

where $z:=(z_1,\ldots,z_n)\in\mathbb{C}^n$ and C is a positive number. Note that conditions (1.1) are special cases of condition (2.26). Taking A to be the empty set $\emptyset$, we get the first condition in (1.1) and when $A=I$ condition (2.26) becomes the second condition in (1.1). Therefore, we have $\mathcal{E}^n_\sigma=E^n_\sigma(I)\cup E^n_\sigma(\emptyset)$. In the following theorem, we extend our expansion for functions from the class $E^n_\sigma(A)$.

Theorem 2.4. *Let A be any subset of I. For $f\in E^n_\sigma(A)$, the multivariate expansion* (2.18) *holds and it converges uniformly on any compact subset of $\mathbb{C}^n$.*

Proof. The proof is basically similar to Theorem 2.1. □

Now, we show that the expansion (2.18) holds for any function from Bernstein space. In the proof of this theorem, we will use a different path to the proof of last one benefiting from the properties of the Bernstein space, $B^n_{\sigma,p}$. In the sequel of the paper, we use $f^{(r)}_{z_j}:=\frac{\partial^r f}{\partial z_j^r}$.

Theorem 2.5. *Let $f\in B^n_{\sigma,p}$, $1\le p<\infty$. Then, the multivariate Hermite sampling series* (2.18) *is valid and it converges uniformly on any compact subset of $\mathbb{C}^n$.*

Proof. We consider z_j, $2\le j\le n$ to be arbitrary fixed complex parameters, and we regard the function $f(z_1,\ldots,z_n)$ as a function of z_1. Since $f\in B^n_{\sigma,p}$, then f, as a function of variable z_1, belongs to $B^1_{\sigma,p}$. Since the expansion (1.3) holds for Bernstein's functions $B^1_{\sigma,p}$, as we mention in Section 1, then we apply the expansion (1.3) on z_1-plane to obtain

$$f(z) = \sum_{k_1\in\mathbb{Z}} \sum_{s_1+m_1+\ell_1=r} f^{(s_1)}_{z_1}(k_1h_1, z_2, \ldots, z_n) \frac{\delta^r_{\ell_1}(k_1)\sin^{r+1}\left(\pi h_1^{-1} z_1\right)}{s_1!\ell_1!(z_1-k_1h_1)^{m_1+1}}, \tag{2.27}$$

which converges uniformly on any compact subsets of z_1-plane. Since $f \in B^n_{\sigma,p}$, then $f^{(s_1)}_{z_1} \in B^n_{\sigma,p}$, see [12, pp. 123–124]. Now, we consider $f^{(s_1)}_{z_1}$ as a function of variable z_2 and z_j $3 \le j \le n$ are the arbitrary fixed complex parameters. Therefore, we have $f^{(s_1)}_{z_1} \in B^1_{\sigma,p}$, as a function of variable z_2. Again applying the expansion (1.3) on z_2-plane, we obtain

$$f^{(s_1)}_{z_1}(k_1h_1, z_2, \ldots, z_n) = \sum_{k_2\in\mathbb{Z}} \sum_{s_2+m_2+\ell_2=r} f^{(s_1,s_2)}_{z_1,z_2}(k_1h_1, k_2h_2, z_3, \ldots, z_n) \frac{\delta^r_{\ell_2}(k_2)\sin^{r+1}\left(\pi h_2^{-1} z_2\right)}{s_2!\ell_2!(z_2-k_2h_2)^{m_2+1}}, \tag{2.28}$$

which converges uniformly on any compact subsets of z_2-plane. Substituting the right-hand side of (2.28) for $f^{(s_1)}_{z_1}$ in (2.27) implies

$$f(z) = \sum_{(k_1,k_2)\in\mathbb{Z}^2} \sum_{s_1+m_1+\ell_1=r} \sum_{s_2+m_2+\ell_2=r} f^{(s_1,s_2)}_{z_1,z_2}(k_1h_1, k_2h_2, z_3, \ldots, z_n) \prod_{j=1}^{2} \frac{\delta^r_{\ell_j}(k_j)\sin^{r+1}\left(\pi h_j^{-1} z_j\right)}{s_j!\ell_j!(z_j-k_jh_j)^{m_j+1}}. \tag{2.29}$$

Series (2.28) also converges uniformly on any compact subsets of (z_1-plane)×(z_2-plane), see [6, Theorem 3.3]. Repeatedly applying the expansion (1.3) and using Lemma 2.1 as above, we obtain the desired expansion (2.18). □

Remark 2.6. *Setting $n = 1$ and $n = 2$ in Theorem 2.4, we get the series of [16, Theorem 3.3] and [6, Theorem 3.3] respectively.*

The following multivariate sampling series which goes back to Parzen [14], Peterson-Middleton [15] and Gosselin [10] is a special case of our series (2.18).

Corollary 2.7. *If $f \in B^n_{\sigma,p}$, then we have*

$$f(z) = \sum_{k\in\mathbb{Z}^n} f\left(\frac{k_1\pi}{\sigma_1}, \ldots, \frac{k_n\pi}{\sigma_n}\right) \prod_{j=1}^{n} \operatorname{sinc}\left(\sigma_j z_j - k_j\pi\right), \tag{2.30}$$

where $k := (k_1, \ldots, k_n) \in \mathbb{Z}^n$ and $z := (z_1, \ldots, z_n) \in \mathbb{C}^n$. Series (2.30) converges uniformly on any compact subsets of $\mathbb{C}^n$.

Proof. Letting $r = 0$ in (2.18) implies $s_j = m_j = \ell_j = 0$ for all $1 \le j \le n$ and $h_j = \pi/\sigma_j$. From (1.5), we have $\delta^0_0(k_j) = (-1)^{k_j}/\sigma_j$ for all $1 \le j \le n$ and consequently $\delta^0_0(k_j)\sin(\pi h_j^{-1} z_j) = \sin(\sigma z_j - k_j\pi)/\sigma_j$. Therefore (2.30) is proved. □

The Fang and Li's multivariate Hermite sampling series (1.6) only uses samples from the function itself and its first partial derivatives while our multivariate Hermite sampling series (2.18) uses samples from the function itself and its mixed and non-mixed derivatives up to order n. The following multivariate Hermite sampling series, which is a special case of our formula (2.18), was given in [13, Theorem 1.1].

Corollary 2.8. *Let $f \in B^n_{2\sigma,p}$. Then it can be expanded as*

$$f(z) = \sum_{k\in\mathbb{Z}^n} \sum_{s_n+m_n=1} \cdots \sum_{s_1+m_1=1} \prod_{i=1}^{n} \left(z_i - \frac{k_i\pi}{\sigma_i}\right)^{1-m_i} f^{(s_1,\ldots,s_n)}\left(\frac{k\pi}{\sigma}\right) \prod_{j=1}^{n} \operatorname{sinc}^2\left(\sigma_j z_j - k_j\pi\right), \tag{2.31}$$

where $k := (k_1, \ldots, k_n) \in \mathbb{Z}^n$, $k/\sigma := (k_1/\sigma_1, \ldots, k_n/\sigma_n)$ and $z := (z_1, \ldots, z_n) \in \mathbb{C}^n$. Series (2.31) converges uniformly on any compact subsets of $\mathbb{C}^n$.

Proof. Letting $r = 1$ in (1.5), we have for all $1 \le j \le n$

$$\delta^1_{\ell_j}(k_j) = \begin{cases} 0, & \ell_j = 1, \\ \\ 1/\sigma_j^2, & \ell_j = 0. \end{cases} \tag{2.32}$$

Therefore any term in (2.18) contains $\delta^1_1(k_j)$ will be equal zero. Since $f \in B^n_{\sigma,p}$, then the bandwidth of the functions is equal to 2σ and hence $h = \pi/\sigma$. Combining (2.32) and (2.18) the expansion (2.18) will be

$$f(z) = \sum_{k\in\mathbb{Z}^n} \sum_{s_n+m_n=1} \cdots \sum_{s_1+m_1=1} f^{(s_1,\ldots,s_n)}\left(\frac{k\pi}{\sigma}\right) \prod_{j=1}^{n} \frac{\sin^2(\sigma_j z_j)}{\sigma_j^2 \left(z_j - k_j\pi/\sigma_j\right)^{m_j+1}}, \tag{2.33}$$

which is equivalent form of expansion (2.31). $\square$

As special cases of expansion (2.31), we refer to the three know series

- If $f \in B^1_{2\sigma,p}$, $1 \le p < \infty$, then we have the classical Hermite sampling

$$f(z) = \sum_{n=-\infty}^{\infty} \left\{ f\left(\frac{n\pi}{\sigma}\right) + \left(z - \frac{n\pi}{\sigma}\right) f'\left(\frac{n\pi}{\sigma}\right) \right\} \operatorname{sinc}^2\left(\sigma z - n\pi\right), \quad z \in \mathbb{C}, \tag{2.34}$$

  which converges uniformly on any compact subsets of $\mathbb{C}$, see e.g. [4]. Also Fang-Li's expansion (1.6) turns into series (2.34) when $n = 1$.

- If $f \in B^2_{2\sigma,p}$, $1 \le p < \infty$, then we have the following expansion for $z := (z_1, z_2)$

$$\begin{aligned} f(z) \;=\; & \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} \left\{ f\left(\frac{n\pi}{\sigma_1}, \frac{m\pi}{\sigma_2}\right) + \left(z_1 - \frac{n\pi}{\sigma_1}\right) f'_{z_1}\left(\frac{n\pi}{\sigma_1}, \frac{m\pi}{\sigma_2}\right) \right. \\ & + \left. \left(z_2 - \frac{m\pi}{\sigma_2}\right) f'_{z_2}\left(\frac{n\pi}{\sigma_1}, \frac{m\pi}{\sigma_2}\right) + \left(z_1 - \frac{n\pi}{\sigma_1}\right)\left(z_2 - \frac{m\pi}{\sigma_2}\right) f''_{z_1 z_2}\left(\frac{n\pi}{\sigma_1}, \frac{m\pi}{\sigma_2}\right) \right\} \\ & \times \; \operatorname{sinc}^2\left(\sigma_1 z_1 - n\pi\right) \operatorname{sinc}^2\left(\sigma_2 z_2 - m\pi\right), \end{aligned} \tag{2.35}$$

  which is introduced in [6, Theorem 3.3] with $\sigma_1 = \sigma_2$. Series (2.35) converges uniformly on any compact subsets of $\mathbb{C}^2$.

- If $f \in B^3_{2\sigma,p}$, $1 \le p < \infty$, then we have the following expansion for $z := (z_1, z_2, z_3)$

$$\begin{aligned} f(z) \;=\; & \sum_{k\in\mathbb{Z}^3} \left\{ f\left(\frac{k\pi}{\sigma}\right) + \sum_{j=1}^{3} \left(z_j - \frac{k_j\pi}{\sigma_j}\right) f'_{z_j}\left(\frac{k\pi}{\sigma}\right) + \prod_{j=1}^{2} \left(z_j - \frac{k_j\pi}{\sigma_j}\right) f''_{z_1 z_2}\left(\frac{k\pi}{\sigma}\right) \right. \\ & + \prod_{j=1, j\ne 2}^{3} \left(z_j - \frac{k_j\pi}{\sigma_j}\right) f''_{z_1 z_3}\left(\frac{k\pi}{\sigma}\right) + \prod_{j=2}^{3} \left(z_j - \frac{k_j\pi}{\sigma_j}\right) f''_{z_2 z_3}\left(\frac{k\pi}{\sigma}\right) \end{aligned}$$

$$+\prod_{j=1}^{3}\left(z_j-\frac{k_j\pi}{\sigma_j}\right)f'''_{z_1z_2z_3}\left(\frac{k\pi}{\sigma}\right)\Bigg\}\prod_{j=1}^{3}\operatorname{sinc}^2\left(\sigma_i z_j-k_j\pi\right), \tag{2.36}$$

where $k:=(k_1,k_2,k_3)\in\mathbb{Z}^3$ and $k/\sigma:=(k_1/\sigma_1,k_2/\sigma_2,k_3/\sigma_3)$. Series (2.31) converges uniformly on any compact subsets of $\mathbb{C}^3$. This series was given in [13].

§3 Truncation error

Expansion (2.18) requires us to know the exact samples of a function f and its derivatives $f^{(s_1,\dots,s_n)}$ at infinitely many points. In practice, only finitely many samples are available and hence the truncation error appears. The bounds of the truncation error of the sampling series (1.3) have been extensively studied in [5]. Ye (2011) studies the truncation error of the multivariate series (2.30), which is a special case of our series (2.18) as we show in Corollary 2.7, based on localized sampling without decay assumption, cf. [20]. Also, the authors of [6,7] estimate the truncation error for the bivariate case of (2.18), (when $n=2$). Here we extend the Ye's technique to establish a bound for the truncation error of our multivariate form (2.18). For any $N:=(N_1,\dots,N_n)\in\mathbb{Z}^n$, we truncate the series (2.18) as follows

$$f_{r,h,N}(z)=\sum_{k\in\mathbb{Z}^n_N(z)}\sum_{s_n+m_n+\ell_n=r}\cdots\sum_{s_1+m_1+\ell_1=r}f^{(s_1,\dots,s_n)}(kh)\prod_{j=1}^{n}\frac{\delta^r_{\ell_j}(k_j)\sin^{r+1}\left(\pi h_j^{-1}z_j\right)}{s_j!\ell_j!(z_j-k_jh_j)^{m_j+1}}, \tag{3.37}$$

where $k:=(k_1,\dots,k_n)\in\mathbb{Z}^n$, $kh:=(k_1h_1,\dots,k_nh_n)$, $z:=(z_1,\dots,z_n)\in\mathbb{C}^n$ and

$$\mathbb{Z}^n_N(z):=\Big\{k\in\mathbb{Z}^n:\ |\lfloor h_j^{-1}\Re z_j\rfloor-k_j|\le N_j,\ 1\le j\le n\Big\}. \tag{3.38}$$

Here $\lfloor x\rfloor$ is the integer part of x and $N:=(N_1,\dots,N_n)$. That is, if we want to estimate $f(z)$ we only sum over values of f on a part of $h\mathbb{Z}^n$ near $\Re z$. In this way we can study a bound on the truncation error

$$\mathcal{T}_{r,h,N}(z)=f(z)-f_{r,h,N}(z),\quad z\in\mathbb{C}^n. \tag{3.39}$$

Theorem 3.9. *Let $f\in B^n_{\sigma,p}$, $1\le p<\infty$ then for any $z\in\mathbb{C}^n$, we have the following bound*

$$|\mathcal{T}_{r,h,N}(x)|\ \le\ \eta\|f\|_p\sum_{s_n+m_n+\ell_n=r}\cdots\sum_{s_1+m_1+\ell_1=r}\beta_{s,m,\ell}\left(\sum_{j=1}^{n}\frac{1}{N_j^{m_j+1/p}}\right)\exp\left(\sum_{j=1}^{n}(r+1)\pi h_j^{-1}\Im z_j\right), \tag{3.40}$$

where $r\in\mathbb{N}_\circ$, $N:=(N_1,\dots,N_n)\in\mathbb{Z}^n$ and

$$\eta:=p\,A\left(\prod_{j=1}^{n}h_j^{-1}\right)^{1/p},\quad \beta_{s,m,\ell}:=\prod_{j=1}^{n}\frac{\sigma_j^{s_j}|\pi h_j^{-1}|^{m_j+1}|\delta^r_{\ell_j}(k_j)|}{s_j!\ \ell_j!}.$$

Proof. Since $f\in B^n_{\sigma,p}$, then the expansion (2.18) holds. Using (3.39) and the general triangle inequality implies

$$|\mathcal{T}_{r,h,N}(z)|\ \le\ \sum_{k\notin\mathbb{Z}^n_N(z)}\sum_{s_n+m_n+\ell_n=r}\cdots\sum_{s_1+m_1+\ell_1=r}\left|f^{(s_1,\dots,s_n)}(kh)\prod_{j=1}^{n}\frac{\delta^r_{\ell_j}(k_j)\sin^{r+1}\left(\pi h_j^{-1}z_j\right)}{s_j!\ell_j!(z_j-k_jh_j)^{m_j+1}}\right|$$

$$= \sum_{s_n+m_n+\ell_n=r} \cdots \sum_{s_1+m_1+\ell_1=r} \sum_{k\notin\mathbb{Z}_N^n(z)} \left| f^{(s_1,\ldots,s_n)}(kh) \prod_{j=1}^{n} \frac{\delta_{\ell_j}^r(k_j)\sin^{r+1}\left(\pi h_j^{-1}z_j\right)}{s_j!\ell_j!(z_j-k_jh_j)^{m_j+1}} \right|, \tag{3.41}$$

where the interchange of the sums in the last step of (3.41) is justified by the absolute convergence of (2.18). Applying the general Hölder's inequality yields

$$\begin{aligned} &\sum_{k\notin\mathbb{Z}_N^n(z)} \left| f^{(s_1,\ldots,s_n)}(kh) \prod_{j=1}^{n} \frac{\delta_{\ell_j}^r(k_j)\sin^{r+1}\left(\pi h_j^{-1}z_j\right)}{s_j!\ell_j!(z_j-k_jh_j)^{m_j+1}} \right| \\ &\le \left(\sum_{k\notin\mathbb{Z}_N^n(z)} \left| f^{(s_1,\ldots,s_n)}(kh) \right|^p \right)^{1/p} \left(\sum_{k\notin\mathbb{Z}_N^n(z)} \left| \prod_{j=1}^{n} \frac{\delta_{\ell_j}^r(k_j)\sin^{r+1}\left(\pi h_j^{-1}z_j\right)}{s_j!\ell_j!(z_j-k_jh_j)^{m_j+1}} \right|^q \right)^{1/q}. \end{aligned} \tag{3.42}$$

Since $f\in B_{\sigma,p}^n$, then $f^{(s_1,\ldots,s_n)}\in B_{\sigma,p}^n$ for all $0\le s_j\le r$ and we have (see [12, pp. 123–124])

$$\begin{aligned} \left(\sum_{k\notin\mathbb{Z}_N^n(z)} \left| f^{(s_1,\ldots,s_n)}(kh) \right|^p \right)^{1/p} &\le \left(\sum_{k\in\mathbb{Z}^n} \left| f^{(s_1,\ldots,s_n)}(kh) \right|^p \right)^{1/p} \\ &\le A \left(\prod_{j=1}^{n} h_j^{-1} \right)^{1/p} \|f^{(s_1,\ldots,s_n)}\|_p \\ &\le A \left(\prod_{j=1}^{n} h_j^{-1} \right)^{1/p} \|f\|_p \prod_{j=1}^{n} \sigma_j^{s_j}, \end{aligned} \tag{3.43}$$

where A is a positive constant. In the last step of (3.43), we have used the Bernstein inequality, see [12, p. 116]. For all $j=1,\ldots,n$, it is easy to see that

$$\frac{\sin^{r+1}\left(\pi h_j^{-1}z_j\right)}{(z_j-k_jh_j)^{m_j+1}} = (-1)^{(m_j+1)k_j}\left(\pi h_j^{-1}\right)^{m_j+1}\sin^{r-m_j}\left(\pi h_j^{-1}z_j\right)\operatorname{sinc}^{m_j+1}\left(\pi h_j^{-1}z_j-k_j\pi\right). \tag{3.44}$$

Using (3.44) and the well known inequality $|\sin(\zeta)|\le\exp(|\Im\zeta|)$, we obtain

$$\begin{aligned} &\left(\sum_{k\notin\mathbb{Z}_N^n(z)} \left| \prod_{j=1}^{n} \frac{\delta_{\ell_j}^r(k_j)\sin^{r+1}\left(\pi h_j^{-1}z_j\right)}{s_j!\ell_j!(z_j-k_jh_j)^{m_j+1}} \right|^q \right)^{1/q} \\ &\le p(z) \left(\sum_{k\notin\mathbb{Z}_N^n(z)} \left| \prod_{j=1}^{n} \operatorname{sinc}^{(m_j+1)q}\left(\pi h_j^{-1}z_j-k_j\pi\right) \right| \right)^{1/q}, \end{aligned} \tag{3.45}$$

where p is defined by

$$p(z) = \left(\prod_{j=1}^{n} \frac{|\pi h_j^{-1}|^{m_j+1}|\delta_{\ell_j}^r(k_j)|}{s_j!\ \ell_j!} \right) \exp\left(\sum_{j=1}^{n} (r-m_j)\pi h_j^{-1}\Im z_j \right).$$

Note that the sequence $|\delta_{\ell_j}^r(k_j)|$ is independent of k_j for all $1\le j\le n$, see [6, Lemma 2.4]. To complete the proof, we will estimate the sum in the right-hand side of (3.45). From the definition of $\mathbb{Z}_N^n(z)$ and by applying the general Minkowski inequality, it is not hard to see that

$$\begin{aligned}
&\left(\sum_{k\notin\mathbb{Z}_N^n(z)}\left|\prod_{j=1}^{n}\operatorname{sinc}^{(m_j+1)q}\left(\pi h_j^{-1}z_j-k_j\pi\right)\right|\right)^{1/q}\\
&\le\sum_{j=1}^{n}\left(\sum_{k_j\notin\mathbb{Z}_{N_j}(z_j)}\left|\operatorname{sinc}^{(m_j+1)q}\left(\pi h_j^{-1}z_j-k_j\pi\right)\right|\right)^{1/q}\\
&\times\prod_{j\neq i=1}^{n}\left(\sum_{k_i\in\mathbb{Z}}\left|\operatorname{sinc}^{(m_i+1)q}\left(\pi h_i^{-1}z_i-k_i\pi\right)\right|\right)^{1/q},
\end{aligned}\tag{3.46}$$

where $\mathbb{Z}_N(\zeta):=\left\{k\in\mathbb{Z}:\ |\lfloor h^{-1}\Re\zeta\rfloor-k|\le N\right\}$. From [6, Lemma 4.1] and [3, Lemma 2.5], we have the following inequality which hold for all $\zeta\in\mathbb{C}$ and $m\in\mathbb{Z}^+$

$$\left(\sum_{k\notin\mathbb{Z}_N(\zeta)}\left|\operatorname{sinc}^{(m+1)q}\left(\pi h^{-1}\zeta-k\pi\right)\right|\right)^{1/q}\le \mathrm{e}^{(m+1)\pi h^{-1}|\Im\zeta|}\,N^{-m-1/p},\tag{3.47}$$

$$\left(\sum_{k=-\infty}^{\infty}\left|\operatorname{sinc}\left(\pi h^{-1}\zeta-k\pi\right)\right|^{q}\right)^{1/q}\le p\,\mathrm{e}^{\pi h^{-1}|\Im\zeta|},\tag{3.48}$$

respectively. Using the well known inequality $|\operatorname{sinc}(\zeta)|\le\exp(|\Im\zeta|)$ and combining (3.41)–(3.43) and (3.45)–(3.48), we obtain (3.40) and the proof is completed. $\square$

In the following corollary, we show that Ye's bound, [20, Theorem 1], is special case of (3.40).

Corollary 3.10. *Let $f\in B^n_{\sigma,p}$, $1\le p<\infty$ then for any $x\in\mathbb{R}^n$, we have the uniform bound*

$$|\mathcal{T}_{0,h,N}(x)|\le\frac{A}{\pi^{n/p}}\left(\sum_{j=1}^{n}\frac{1}{N_j^{1/p}}\right)\left(\prod_{j=1}^{n}\sigma_j\right)^{1/p}\|f\|_p.\tag{3.49}$$

Proof. Letting $r=0$ implies $s_j=m_j=\ell_j=0$, $h_j=\pi/\sigma_j$ and $|\delta_0^0(k_j)|=1/\sigma_j$ for all $1\le j\le n$. In the case $r=0$ and $\Im z_j=0$ for all $1\le j\le n$, the estimation (3.40) gives us (3.49). $\square$

§4 Illustrative examples

In this section, we discuss two examples which are devoted to a numerically comparison between our formula (2.31) and Fang-Li's formula (1.6). We restrict ourselves to functions from the space $B^3_{\sigma,p}$. The case when $f\in B^2_{\sigma,p}$ is given in [6] and formula (2.31) turns into (1.6) in the case $f\in B^1_{\sigma,p}$. Our formula (2.31) gives us a high accuracy approximations comparing with the results of (1.6). We truncate the series (1.6) for functions from the space $B^3_{\sigma,p}$ as follows

$$f_N(x)=\sum_{k\notin\mathbb{Z}_N^n(x)}\left\{f\left(\frac{2k\pi}{\sigma}\right)+\sum_{j=1}^{n}f'_{x_j}\left(\frac{2k\pi}{\sigma}\right)\left(x_j-\frac{2k_j\pi}{\sigma_j}\right)\right\}\prod_{j=1}^{n}\operatorname{sinc}^2\left((\sigma_j/2)x_j-k_j\pi\right),\tag{4.50}$$

and then the truncation error associated with this series will be $T_N[f](x):=f(x)-f_N(x)$.

Example 4.11. *Consider the function* $f(x) = \mathrm{sinc}\left(\sqrt{1+x_1^2+x_2^2+x_3^2}\right)$, $x = (x_1, x_2, x_3) \in \mathbb{R}^3$, *which belongs to the space* $B^3_{\sigma,2}$, $\sigma = (1,1,1)$. *In Table 1, we approximate* f *at some points using sampling formula (2.31) and Fang-Li's formula (1.6) with* $N = (12, 12, 12)$.

Table 1. Absolute error for two formulas.

$x \in \mathbb{R}^3$	Fang-Li's formula (1.6)	Sampling formula (2.31)
	$\lvert f(x) - f_N(x) \rvert$	$\lvert f(x) - f_{1,h,N}(x) \rvert$
$(\frac{1}{2}, \frac{1}{2}, \frac{1}{2})$	0.0220122	8.06393×10^{-5}
$(\frac{1}{2}, 1, 1,)$	0.0423094	9.44323×10^{-5}
$(1, 1, \frac{3}{2})$	0.0933885	3.37872×10^{-5}
$(\frac{3}{2}, \frac{3}{2}, \frac{3}{2})$	0.0684493	1.11348×10^{-5}

Example 4.12. *In this example, we approximate the function* $f(z) = \prod_{j=1}^{3} \mathrm{sinc}^{(1)}(z_j)$, $z = (z_1, z_2, z_3) \in \mathbb{C}^3$, *which belongs to* $B^3_{\sigma,p}$, $\sigma = (1,1,1)$, *using formulas (2.31) and (1.6) with* $N = (15, 15, 15)$. *As we mentioned in Section 1, Fang-Li's formula (1.6) is also justified on* $\mathbb{C}^3$. *We summarize the numerical results on Table 2.*

Table 2. Absolute error for two formulas.

$z \in \mathbb{R}^3$	Fang-Li's formula (1.6)	Sampling formula (2.31)
	$\lvert f(z) - f_N(z) \rvert$	$\lvert f(z) - f_{1,h,N}(z) \rvert$
$(1+\mathrm{i}, 1+\mathrm{i}, 1+\mathrm{i})$	0.106645	8.76907×10^{-7}
$(2+\mathrm{i}, 1+\mathrm{i}, 1+\mathrm{i})$	0.126314	1.52696×10^{-6}
$(2+\mathrm{i}, 2+\mathrm{i}, 1+\mathrm{i})$	0.149698	2.48755×10^{-6}
$(2+\mathrm{i}, 2+\mathrm{i}, 2+\mathrm{i})$	0.177531	3.79328×10^{-6}

Declarations

Conflict of interest The authors declare no conflict of interest.

References

[1] L Ahlfors. *Complex Analysis*, McGraw-Hill, New York, 1979.

[2] M H Annaby. *Multivariate sampling theorems associated with multiparameter differential operators*, Proc Edin Math Soc, 2005, 48: 257-277.

[3] M H Annaby, R M Asharabi. *On sinc-based method in Computing eigenvalues of boundary-value problems*, SIAM J Numer Anal, 2008, 46: 671-690.

[4] M H Annaby, R M Asharabi. *Error analysis associated with uniform Hermite interpolation of bandlimited functions*, J Korean Math Soc, 2010, 47: 1299-1316.

[5] R M Asharabi, H S Al-Abbas. *Truncation error estimates for generalized Hermite sampling*, Numer Algor, 2017, 74: 481-497.

[6] R M Asharabi, H A Al-Haddad. *A bivariate sampling series involving mixed partial derivatives*, Turk J Math, 2017, 41: 387-403.

[7] R M Asharabi, J Prestin. *On two-dimensional classical and Hermite sampling*, IMA J Numer Anal, 2016, 36: 851-871.

[8] G Fang, Y Li. *Multidimensional sampling theorem of Hermite type and estimates for aliasing error on Sobolev classes*, J Chinese Ann Math Ser A, 2006, 27: 217-230.(in Chinese).

[9] H A Li, G S Fang. *Sampling theorem of Hermite type and aliasing error on the Sobolev class of functions*, Front Math China, 2006, 2: 252–271.

[10] R P Gosselin. *On the L^p theory of cardinal series*, Ann of Math, 1963, 78: 567-581.

[11] L Hörmander. *An Introduction to Complex Analysis in Several Variables*, North-Holland, Amsterdam, 1990.

[12] S N Nikol'skii. *Approximation of Function of Several Variables and Imbedding Theorems*, Springer-Verlag, New York, 1975.

[13] S Norvidas. *On a sampling expansion with partial derivatives for functions of several variables*, Filomat, 2020, 34: 3339-3347.

[14] E Parzen. *A simple proof and some extensions of sampling theorems*, Technical Report 7, Stanford University, California, 1956, https://doi.org/10.21236/ad0117999.

[15] D P Peterson, D Middleton. *Sampling and reconstruction of wave number-limited function in N-dimensional Euclidean space*, Inform and Control, 1962, 5: 279-323.

[16] C E Shin. *Generalized Hermite interpolation and sampling theorem involving derivatives*, Commun Korean Math Soc, 2002, 17: 731-740.

[17] J Voss. *Irregular Sampling: Error Analysis, Applications and Extensions*, Mitt Math Sem Giessen, 1999, 238: 1-86.

[18] R M Range. *Holomorphic Functions and Integral Representations in Several Complex Variables*, Springer-Verlag, New York, 1986.

[19] J Wang, G Fang. *Multidimensional sampling theorem and estimate of aliasing error*, ACTA Mathematicae Applicatae Sinica, 1996, 19: 481-488.(in Chinese).

[20] P Ye. *Error bounds for multidimensional Whittaker-Shannon sampling expansion*, International Conference on Multimedia and Signal Processing (CMSP), 2011, 2: 33-36.

Department of Mathematics, College of Arts and Sciences, Najran University, Najran, Saudi Arabia.
Email: rashad1974@hotmail.com