On traceable iterated line graph and hamiltonian path index

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Abstract. Xiong and Liu [21] gave a characterization of the graphs G for which the n-iterated line graph $L^n(G)$ is hamiltonian, for $n \geq 2$. In this paper, we study the existence of a hamiltonian path in $L^n(G)$, and give a characterization of G for which $L^n(G)$ has a hamiltonian path. As applications, we use this characterization to give several upper bounds on the hamiltonian path index of a graph.

§1   Introduction

Graphs considered in this paper are finite, undirected and loopless. Undefined notations and terminologies will follow [2].

Let G be a graph, then $V(G)$ and $E(G)$ denote the sets of vertices and edges of G, respectively. Define $V_i(G) = \{v \in V(G) : d_G(v) = i\}$ and $W(G) = V(G) \backslash V_2(G)$. A *branch* in G is a nontrivial path with ends in $W(G)$ and with internal vertices, if any, of degree 2. We denote by $B(G)$ the set of branches of G. Define $B_1(G) = \{b \in B(G) : V(b) \cap V_1(G) \neq \emptyset\}$. The distance between two subgraphs H_1 and H_2 of G, denoted by $d_G(H_1, H_2)$, is $\min\{d_G(v_1, v_2) : v_1 \in V(H_1)$ and $v_2 \in V(H_2)\}$. For a subgraph $H \subseteq G$, we denote by $G - V(H)$, or briefly, $G - H$, the graph obtained from G by deleting all the vertices of H together with all the edges incident with the vertices of H.

The *line graph* $L(G)$ of a graph G has $E(G)$ as its vertex set and two vertices are adjacent in $L(G)$ if and only if they are adjacent as edges in G. A trail T of G is *dominated* if each edge of G is incident with at least one vertex of T. Harary and Nash-Williams characterized those graphs G (especially, non-hamiltonian graphs) for which $L(G)$ is hamiltonian.

Theorem 1. *(Harary and Nash-Williams, [10]) Let G be a connected graph with at least three edges. Then $L(G)$ is hamiltonian if and only if G has a dominating closed trail.*

The *n-iterated line graph* of a graph G is $L^n(G) = L(L^{n-1}(G))$, where $L^1(G)$ denotes the line graph $L(G)$ of G, and $L^{n-1}(G)$ is assumed to have a nonempty edge set. Chartrand [3] considered the hamiltonianity of $L^n(G)$, and introduced the *hamiltonian index* of a graph, denoted by $h(G)$, i.e., the minimum number n such that $L^n(G)$ is hamiltonian. Since then, many results of $h(G)$ for special graphs have been proved, such as Chartrand and Wall [4] for trees (other than paths) and connected graphs G with $\delta(G) \geq 3$, Han et al. [8] for 2-connected graph with $\kappa(G) \geq \alpha(G) - t$ (where t is a nonnegative integer). Ryjáček et al. [18] showed that the problem to decide whether the hamiltonian index of a given graph is less than or equal to a given constant is NP-complete. The following theorem characterized those graphs G for which $L^n(G)$ is hamiltonian.

Theorem 2. *(Xiong and Liu, [21]) Let G be a connected graph with at least three edges and $n \geq 2$. Then $L^n(G)$ is hamiltonian if and only if $EU_n(G) \neq \emptyset$, where $EU_n(G)$ denotes the set of those subgraphs H of G that satisfy the following conditions:*

(I) $d_H(x) \equiv 0 \,(\mathrm{mod}\ 2)$ for every $x \in V(H)$;

(II) $V_0(H) \subseteq \bigcup_{i=3}^{\Delta(G)} V_i(G) \subseteq V(H)$;

(III) $d_G(H_1, H - H_1) \leq n - 1$ for every subgraph H_1 of H;

(IV) $|E(b)| \leq n + 1$ for every branch $b \in B(G)$ with $E(b) \cap E(H) = \emptyset$;

(V) $|E(b)| \leq n$ for every branch $b \in B_1(G)$.

With the help of Theorem 2, one can deduce $h(G) \leq n$ if it is convenient to check that $EU_n(G)$ is nonempty. For more results of $h(G)$, see [12-14,19].

As a weakening of hamiltonianity, the existence of hamiltonian paths of a graph G or a line graph $L(G)$ has also got a lot of attention. A graph G is *traceable* if it has a hamiltonian path. Let $\sigma_2(G) \min\{d_G(u) + d_G(v) \,|\, uv \notin E(G)\}$. Momège [17] showed that a connected graph G of order n with $\sigma_2(G) \geq 2n/3$ and $K_{1,4}$-free is traceable. He and Yang [11] proved that there exist at least $\max\{1, \lfloor \frac{1}{8}\delta(G) \rfloor - 1\}$ edge-disjoint hamiltonian paths between any two vertices in a hamiltonian-connected line graph $L(G)$. For more results on the traceability of graphs or line graphs, see [6,15,22]. The following, which characterized the traceability of line graphs, is a similar result as Theorem 1.

Theorem 3. *(Lai, Shao and Zhan, [15], Xiong and Zong, [22]) Let G be a connected graph with at least three edges. Then the line graph $L(G)$ is traceable if and only if G has a dominating trail.*

In 1983, Clark and Wormald [7] extended the idea of the hamiltonian index and introduced the hamiltonian-connected index, which is the least integer n such that the iterated line graph

$L^n(G)$ is hamilton-connected. Since then, several hamiltonian-like indices were proposed. For more results on the hamiltonian-like indices, see [5,7] for the hamilton-connected index, [23] for the s-hamiltonian index, [16] for the panconnected index, [24] for the s-vertex pancyclic index, and [20] for the s-fully cycle extendable index.

The current research is motivated by the results above. In this paper, we study the existence of a hamiltonian path in the n-iterated line graph $L^n(G)$, i.e., the traceability of $L^n(G)$.

We present a few needed notations. For a graph G, we use $O(G)$ to denote the set of odd degree vertices of G. In the definition of $EU_n(G)$, (I) guarantees that each vertex in $V(H)$ has even degree, i.e., $O(H) = \emptyset$. When concerning about the traceability of $L^n(G)$, we can allow that H has at most two odd degree vertices. Moreover, by the fact that H may contain a branch $b \in B_1(G)$ or a part of b, we just need (V) holds for every branch $b \in B_1(G)$ with $E(b) \cap E(H) = \emptyset$.

Let $EUP_n(G)$ denote the set of subgraphs H of a graph G that satisfy the following conditions (I)$'$ and (V)$'$, and (II)-(IV) in the definition of $EU_n(G)$.

(I)$'$ $|O(H)| \leq 2$;

(V)$'$ $|E(b)| \leq n$ for every branch $b \in B_1(G)$ with $E(b) \cap E(H) = \emptyset$.

Obviously, $EU_n(G) \subseteq EUP_n(G)$. Now we are ready to present the main result.

Theorem 4. *Let G be a connected graph with at least three edges and $n \geq 2$. Then $L^n(G)$ is traceable if and only if $EUP_n(G) \neq \emptyset$.*

Moreover, as applications of Theorem 4, we also examine the *hamiltonian path index* (which is proposed as a new hamiltonian-like index) of a graph G, denoted by $h_p(G)$, i.e., the minimum number n such that $L^n(G)$ is traceable. Regard $h_p(G) = 0$ if G is traceable.

In Section 2, we will present some auxiliary results, which will be used to prove Theorem 4 in Section 3. As applications, we will give some upper bounds of the hamiltonian path index $h_p(G)$ in the last section.

§2   Preliminaries and auxiliary results

In this section, we present several auxiliary results, which will be used in the proof of Theorem 4.

The multi-graph of order 2 with two edges will be called 2-*cycle*. Let G be a graph and H a subgraph of G, then $\bar{E}(H)$ denotes the set of all edges of G that are incident with vertices of H. If $u \in V(H)$, then $E_H(u)$ denotes the set of all edges of H that are incident with u, and $d_H(u) = |E_H(u)|$. A graph is called a *circuit* (or equivalently, *eulerian graph*) if it is connected and every vertex has an even degree. Regard K_1 as a circuit.

For any subgraph C of $L(G)$, by $S(G,C)$ we denote the collection of circuits H of G, such that $L(G[\bar{E}(H)])$ contains C, and C contains all edges of $E(H)$. Here and throughout, $G[S]$ denotes the subgraph of G induced by S, where $S \subseteq V(G)$ or $S \subseteq E(G)$.

Lemma 5. *(Xiong and Liu, [21]) Each of the following holds.*

(1) *If C is a cycle of $L(G)$ with $|E(C)| \geq 3$, then $S(G,C)$ is nonempty.*

(2) *If G has a circuit H such that $\bar{E}(H)$ has at least three edges, then $L(G)$ has a cycle C with $V(C) = \bar{E}(H)$.*

Beineke [1] characterized line graphs in terms of nine forbidden induced subgraphs, one of which is $K_{1,3}$.

Lemma 6. *(Beineke, [1]) $K_{1,3}$ is not an induced subgraph of the line graph of any graph.*

The following lemma indicates the relationship between a branch of G and the corresponding branch of $L(G)$.

Lemma 7. *(Xiong and Liu, [21]) Let $b = u_1 u_2 \cdots u_s \ (s \geq 3)$ be a path of G and $e_i = u_i u_{i+1}$. Then $b \in B(G)$ if and only if $b' = e_1 e_2 \cdots e_{s-1} \in B(L(G))$.*

Lemma 8. *Let H be a subgraph of G in $EUP_n(G)$ with a minimum number of components. Then there exists no multiple edges in $\bar{E}(H_1) \cap \bar{E}(H_2)$ for any two components H_1 and H_2 of H.*

A similar result as Lemma 8 was proved for $H \in EU_n(G)$ in [21]. Then arguing similarly, one can obtain Lemma 8. Hence, we omit the details here. An *eulerian subgraph* of G is a circuit which contains at least one cycle of length at least 3.

Lemma 9. *(Xiong and Liu, [21]) Let G be a connected graph and C be an eulerian subgraph of the line graph $L(G)$. Then there exists a subgraph H of G with*

(1) *$d_H(x) \equiv 0 \,(\mathrm{mod}\ 2)$ for every $x \in V(H)$;*

(2) *$d_G(x) \geq 3$ for every $x \in V(H)$ with $d_H(x) = 0$;*

(3) *for any two components H^0, H^{00} of H, there exists a sequence of components $H^0 = H_1, H_2, \ldots, H_s = H^{00}$ of H such that $d_G(H_i, H_{i+1}) \leq 1$ for $i \in \{1, 2, \ldots, s-1\}$;*

(4) *$L(G[\bar{E}(H)])$ contains C, and C contains all edges of $E(H)$.*

Lemma 10. *Each of the following holds.*

(1) *If P is a non-trivial path of $L(G)$, then G has a trail T', such that $L(G[\bar{E}(T')])$ contains P, and P contains all edges of $E(T')$.*

(2) *If G has a connected subgraph H such that $|O(H)| = 2$, then $L(G)$ has a path P with $V(P) = \bar{E}(H)$.*

Proof. (1) The proof just needs a slight modification of the proof of Theorem 1 in [10]. So we omit the details here.

(2) Suppose $O(H) = \{u, v\}$. If $|\bar{E}(H)| = 1$, then $G \cong H \cong K_2$, (2) holds trivially. So we may assume that $\bar{E}(H)$ has at least two edges. Let $e^* = uv$ be a new edge, which doesn't

belong to $E(G)$. Note that if $uv \in E(G)$, then e^* and $uv\,(\in E(G))$ are multiple edges in $G + e^*$. Hence, $H + e^*$ is a circuit of $G + e^*$ such that $\bar{E}(H + e^*)$ has at least three edges. By Lemma 5 (2), $L(G + e^*)$ has a cycle C with $V(C) = \bar{E}(H + e^*)$. Let $P = C - v_{e^*}$, where v_{e^*} is the vertex in $L(G + e^*)$ corresponding to the edge e^* in G. Note that $\bar{E}(H) \cup \{e^*\} = \bar{E}(H + e^*)$. P is a path of $L(G)$ with $V(P) = \bar{E}(H)$.

So Lemma 10 holds.   □

Theorem 2 was derived from the following result, which indicates a close relationship between $EU_n(L(G))$ and $EU_{n+1}(G)$.

Theorem 11. *(Xiong and Liu, [21]) Let G be a connected graph and $k \geq 1$ be an integer. Then $EU_n(L(G)) \neq \emptyset$ if and only if $EU_{n+1}(G) \neq \emptyset$.*

§3   Proof of Theorem 4

In this section, we will prove Theorem 4, which is a direct consequence of the following two theorems. The *symmetric difference* of two non-empty sets A and B, denoted by $A \Delta B$, is the set $(A \cup B) \backslash (A \cap B)$.

Theorem 12. *Let G be a connected graph and $k \geq 1$ be an integer. Then $EUP_k(L(G)) \neq \emptyset$ if and only if $EUP_{k+1}(G) \neq \emptyset$.*

Proof. *Sufficiency.* Suppose that $EUP_{k+1}(G) \neq \emptyset$. Note that if $EU_{k+1}(G) \neq \emptyset$, then by Theorem 11, $EU_k(L(G)) \neq \emptyset$, and hence, $EUP_k(L(G)) \neq \emptyset$ by the fact that $EU_k(G) \subseteq EUP_k(G)$. So we may assume that $EU_{k+1}(G) = \emptyset$, which implies that each subgraph of G in $EUP_{k+1}(G)$ contains exactly two odd vertices.

Now let $H \in EUP_{k+1}(G)$ with a minimum number of components denoted by $C_1, C_2, \ldots, C_t$. If H is connected, then $H \cong C_1$. Without loss of generality, we let $|O(C_1)| = 2$, and then C_i is a circuit for $2 \leq i \leq t$.

Since $|O(C_1)| = 2$ and G is connected, we have $|\bar{E}(C_1)| \geq 2$: for otherwise, $|V(G)| = |V(C_1)| = 2$, Theorem 12 holds obviously. Hence, by Lemma 10 (2), $L(G)$ has a nontrivial path P with $V(P) = \bar{E}(C_1)$. Now we claim that $|\bar{E}(C_i)| \geq 3$ for $2 \leq i \leq t$: if C_i is nontrivial, then we are done; if C_i is an isolated vertex, then by the definition of $EUP_n(G)$, $d_G(C_i) \geq 3$, our claim holds. By Lemma 5 (2), we can find a cycle C_i' in $L(G)$ with $V(C_i') = \bar{E}(C_i)\,(2 \leq i \leq t)$. Let

$$H' = P \cup \Big(\bigcup_{i=2}^{t} C_i' \Big).$$

We will prove that $H' \in EUP_k(L(G))$.

By Lemma 8 and the minimality of t, $E(P) \cap E(C_i') = \emptyset$ and $E(C_i') \cap E(C_j') = \emptyset$ for $2 \leq i, j \leq t$ with $i \neq j$, which implies that $|O(H')| = 2$. (I)$'$ holds.

Since P is nontrivial, and $V(C_i') = \bar{E}(C_i) \geq 3\,(2 \leq i \leq t)$, H' contains no isolated vertex. Note that $\bigcup_{i=3}^{\Delta(G)} V_i(G) \subseteq V(H)$ and $V(H') = \bigcup_{i=1}^{t} \bar{E}(C_i)$. We have

$$\bigcup_{i=3}^{\Delta(L(G))} V_i(L(G)) \subseteq V(H').$$

Hence, H' satisfies (II).

The details of H' satisfying (III), (IV) and (V)$'$ are almost the same as the proof of Theorem 11 in [21], so we omit them here.

It follows that $H' \in EUP_k(L(G))$.

Necessity. Suppose that $EUP_k(L(G)) \neq \emptyset$. Note that if $EU_k(L(G)) \neq \emptyset$, then by Theorem 11, $EU_{k+1}(G) \neq \emptyset$, and hence, $EUP_{k+1}(G) \neq \emptyset$. So we may assume that $EU_k(L(G)) = \emptyset$, which implies that each subgraph of $L(G)$ in $EUP_k(L(G))$ contains exactly two odd vertices.

Let H be a subgraph of $L(G)$ in $EUP_k(L(G))$ with a minimum number of isolated vertices. Then H contains no isolated vertices. For otherwise, suppose $C_1 = \{e_0\}$ is an isolated vertex of H, then by (II), $d_{L(G)}(e_0) \geq 3$. By Lemma 6, there exist $e_1, e_2 \in N_{L(G)}(e_0)$ such that $e_1 e_2 \in E(L(G))$. Now we construct a subgraph H_0 of $L(G)$ as follows.

$$H_0 = \begin{cases} H + \{e_0 e_1, e_1 e_2, e_2 e_0\} & \text{if } e_1 e_2 \notin E(H), \\ H + \{e_0 e_1, e_2 e_0\} - \{e_1 e_2\} & \text{if } e_1 e_2 \in E(H). \end{cases}$$

Obviously $H_0 \in EUP_k(L(G))$ has fewer isolated vertices than H has, a contradiction.

Let $H_1, H_2, \ldots, H_m$ be the components of H, and without loss of generality, let $|O(H_1)| = 2$. Since H has no isolated vertices, H_i is an eulerian subgraph of $L(G)$ for $2 \leq i \leq m$. Then H_1 can be decomposed into a nontrivial path P and several eulerian subgraphs. Let $P, H_1^1, H_1^2, \ldots, H_1^q$ be such a decomposition with q minimized. Then $V(H_1^i) \cap V(H_1^j) = \emptyset$ for $\{i, j\} \subseteq \{1, 2, \ldots, q\}$ with $i \neq j$.

For the path P, by Lemma 10 (1), G has a trail T', such that $L(G[\bar{E}(T')])$ contains P, and P contains all edges of $E(T')$. For any eulerian subgraph H_1^j ($1 \leq j \leq q$) or H_i ($i \in \{2, 3, \ldots, m\}$), by Lemma 9, there exists a subgraph C_1^j or C_i of G, respectively, satisfying (1) to (4) of Lemma 9. Let

$$C' = \big(T \Delta \big(\bigcup_{j=1}^{q} C_1^j \big)\big) \cup \big(\bigcup_{i=2}^{m} C_i \big),$$

where $T \Delta (\bigcup_{j=1}^{q} C_1^j)$ is the subgraph of G with vertex set $V(T \cup (\bigcup_{j=1}^{q} C_1^j))$ and edge set $E(T) \Delta E(\bigcup_{j=1}^{q} C_1^j)$. We construct a subgraph C of G from C' as follows:

$$V(C) = \big(\bigcup_{i=3}^{\Delta(G)} V_i(G) \big) \cup V(C'), \text{ and } E(C) = E(C').$$

We will prove that $C \in EUP_{k+1}(G)$.

Since $V(H_i) \cap V(H_j) = \emptyset$ for $\{i, j\} \subseteq \{1, 2, \ldots, m\}$ with $i \neq j$, $V(H_1^j) \subseteq V(H_1)$ ($1 \leq j \leq q$), and $V(H_1^i) \cap V(H_1^j) = \emptyset$ for $\{i, j\} \subseteq \{1, 2, \ldots, q\}$ with $i \neq j$, we have $E(C_i) \cap E(C_j) = \emptyset$, $E(C_1^i) \cap E(C_1^j) = \emptyset$, and $E(C_i) \cap E(C_1^j) = \emptyset$. It follows that $d_C(x) \equiv 0 \pmod 2$ for every $x \in V(C)$ excepting the end-vertices of the path T, which implies that C satisfies (I)$'$. Since C_i and C_1^j satisfy Lemma 9 (2), $d_G(x) \geq 3$ for every $x \in V(C)$ with $d_C(x) = 0$. Thus, (II) holds.

Arguing similarly as the proof of Theorem 11, we can prove that C satisfies (III), (IV) and (V)$'$.

It follows that $C \in EUP_{k+1}(G)$.

This completes the proof of Theorem 12.　　　　　　　　　　　　　　　　□

Theorem 13. *Let G be a connected graph with at least three edges. Then $L^2(G)$ is traceable if and only if $EUP_2(G) \neq \emptyset$.*

Proof. *Sufficiency.* Suppose that $EUP_2(G) \neq \emptyset$. Note that if $EU_2(G) \neq \emptyset$, then by Theorem 11, $L^2(G)$ is hamiltonian, we are done. So we may assume that $EU_2(G) = \emptyset$, which implies that each subgraph of G in $EUP_2(G)$ contains exactly two odd vertices.

We choose an $H \in EUP_2(G)$ with a minimum number of components that are denoted by $H_1, H_2, \ldots, H_t$, and assume that $|O(H_1)| = 2$. Since $H \in EUP_2(G)$ and $|E(G)| \geq 3$, we have $|\bar{E}(H_1)| \geq 2$, and $|\bar{E}(H_i)| \geq 3$ for $i \in \{2, 3, \ldots, t\}$. Then by Lemma 10 (2), we can find a nontrivial path P of $L(G)$ such that $V(P) = \bar{E}(H_1)$. By Lemma 5 (2), we can find a cycle C_i of $L(G)$ such that $V(C_i) = \bar{E}(H_i)$, $i \in \{2, 3, \ldots, t\}$. Let

$$T = P \cup \Big(\bigcup_{i=2}^{t} C_i \Big).$$

By Lemma 8 and the minimality of t, $P, C_2, C_3, \ldots, C_t$ are edge-disjoint. Hence, T is a subgraph of $L(G)$ with exactly 2 odd vertices. Since $d_G(H', H - H') \leq 1$ for every subgraph H' of H, T is connected.

Note that H satisfies (II), $V(P) = \bar{E}(H_1)$ and $V(C_i) = \bar{E}(H_i)$ for $i \in \{2, 3, \ldots, t\}$. By the fact that any edge in G, which corresponds to a vertex of degree at least 3 in $L(G)$, must be incident to a vertex of degree at least 3 in G,

$$\bigcup_{i=3}^{\Delta(L(G))} V_i(L(G)) \subseteq V(T).$$

Since $H \in EUP_2(G)$, any branch $b \in B(L(G))$ with $E(b) \cap E(C) = \emptyset$ has length at most 2, and any branch in $B_1(L(G))$ has length at most 1. Then by Lemma 7, $\bar{E}(T) = E(L(G))$, which implies that T is a dominating trail of $L(G)$. Hence, $L^2(G)$ is traceable by Theorem 3.

Necessity. Suppose that $L^2(G)$ is traceable. By Theorem 3, $L(G)$ has a dominating trail. Select a dominating trail T of $L(G)$ with a maximum number of vertices of degree at least 3.

Claim 1. $\bigcup_{i=3}^{\Delta(L(G))} V_i(L(G)) \subseteq V(T)$.

The proof of Claim 1 is the same as the proof that H has no isolated vertices in Theorem 12, so we omit it here.

Then T can be decomposed into a nontrivial path P and several eulerian subgraphs. Let $P, H_1, H_2, \ldots, H_q$ be such a decomposition with q minimized. Note that if T is closed, then $T = \cup_{i=1}^{q} H_i$.

For the path P, by Lemma 10 (1), G has a trail T', such that $L(G[\bar{E}(T')])$ contains P, and P contains all edges of $E(T')$. For any eulerian subgraph H_i $(1 \leq i \leq q)$, by Lemma 9, there exists a subgraph C_i of G, satisfying (1) to (4).

Set

$$H' = T' \Delta \Big(\bigcup_{i=1}^{q} C_i \Big)$$

be the subgraph of G with vertex set $V(T' \cup (\bigcup_{i=1}^{q} C_i))$ and edge set $E(T')\Delta E((\bigcup_{i=1}^{q} C_i))$. We construct a subgraph H of G from H' as follows:

$$V(H) = \left(\bigcup_{i=3}^{\Delta(G)} V_i(G) \right) \cup V(H'), \text{ and } E(H) = E(H').$$

We will prove that $H \in EUP_2(G)$. Before this, we present the following claim.

Claim 2. $d_G(x, H) \leq 1$ for any $x \in \bigcup_{i=3}^{\Delta(G)} V_i(G)$.

Proof of Claim 2. If G is either a star or a cycle, then the conclusion holds. For otherwise, then $E_G(x) \cap (\cup_{i=3}^{\Delta(L(G))} V_i(L(G))) \neq \emptyset$ for every vertex x in $\cup_{i=3}^{\Delta(G)} V_i(G)$. Hence, by Claim 1, there exists an edge e_x, which is incident to x in G, has an endvertex in H. Claim 2 holds.

Now we prove $H \in EUP_2(G)$. Obviously, H satisfies (I)$'$. Since $C_i\,(1 \leq i \leq q)$ satisfies (2) of Lemma 9, and by the definition of H, (II) holds. By Claim 2 and (3) of Lemma 9, $d_G(H', H - H') \leq 1$ for every subgraph H' of H, thus H satisfies (III). It follows from Lemma 7 and $E(L(G)) = \bar{E}(T)$ that $|E(b)| \leq 3$ for $b \in B(G)$ with $E(b) \cap E(H) = \emptyset$, and $|E(b)| \leq 2$ for $b \in B_1(G)$ with $E(b) \cap E(H) = \emptyset$. H satisfies (IV) and (V)$'$. Hence, $H \in EUP_2(G)$.

This completes the proof of Theorem 13. □

Now we prove Theorem 4.

Proof of Theorem 4. We proceed by induction on n. Theorem 13 shows that Theorem 4 holds for $n = 2$.

Assume, as an inductive hypothesis, that the theorem is true for $n = k > 2$, i.e., $L^k(G)$ is traceable if and only if $EUP_k(G) \neq \emptyset$. Now let $n = k + 1$. Then $L^{k+1}(G) = L^k(L(G))$ is traceable if and only if $EUP_k(L(G)) \neq \emptyset$. Hence, by Theorem 12, $EUP_k(L(G)) \neq \emptyset$ if and only if $EUP_{k+1}(G) \neq \emptyset$. Theorem 4 holds for $n = k + 1$. Thus, the induction succeeds. □

Theorem 4 doesn't hold for $n = 1$. For example, Fig. 1 shows a graph G with $EUP_1(G) = \emptyset$ while $L(G)$ is traceable. By the definition of $EUP_1(G)$, any subgraph H in $EUP_1(G)$ should be connected, $V_3(G) = \{v_3, v_6, v_9, v_{12}\} \subseteq V(H)$, and the 4 branches $v_1v_2v_3$, $v_3v_4v_5v_6$, $v_6v_7v_8v_9$ and $v_9v_{10}v_{11}$ belong to H. Then $|O(H)| \geq 4$, a contradiction to (I)$'$. Thus, $EUP_1(G) = \emptyset$, but $L(G)$ is traceable by the fact that $v_1v_2 \cdots v_{11}$ is a dominating trail of G and by Theorem 3.
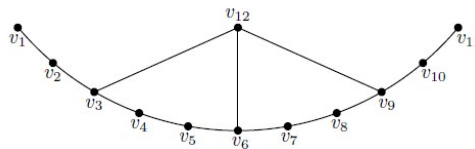
Figure 1. A graph G with $EUP_1(G) = \emptyset$ while $L(G)$ is traceable.

Note that Theorem 4 doesn't hold for $n = 1$. Hence, when we prove it by induction, the basis step is $n = 2$ (Theorem 13).

§4   Applications of Theorem 4

In this section, inspired by the massive upper bounds of $h(G)$, as applications of Theorem 4, we will present some upper bounds on the hamiltonian path index $h_p(G)$ of a graph G. The main idea is to show that $EUP_k(G) \neq \emptyset$, and then by Theorem 4, $h_p(G) \leq k$.

Note that the hamiltonian index $h(G)$ exists for any connected graph G other than a path and $h_p(G) \leq h(G)$. The hamiltonian path index $h_p(G)$ exists for any connected graph.

Comparing the definition of hamiltonian with traceable, we know that being hamiltonian is stronger than being traceable. Then one may believe that the former needs more iterated steps, and hence, $h_p(G) < h(G)$. Unfortunately, this is not true by the fact that $h_p(K_{1,n-1}) = h(K_{1,n-1}) = 1$ ($n \geq 3$). Moreover, Fig. 2 shows a graph G with $h_p(G) = h(G) = k$: one can check that the unique cycle of G is an element in $EU_k(G)$, but $EUP_{k-1}(G) = \emptyset$ by the fact that any element in $EUP_{k-1}(G)$ can't contain all the three pendent paths with length k. Hence, our trivial bound $h_p(G) \leq h(G)$ is the best possible.
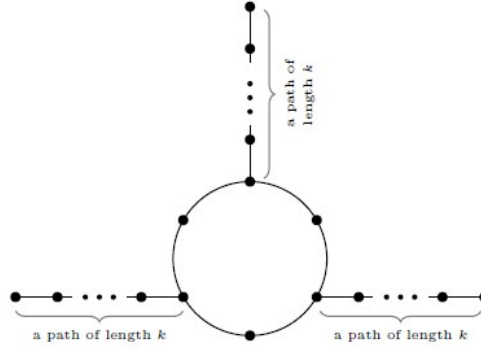

Figure 2. A graph G with $h_p(G) = h(G) = k$.

For a graph G, let $MT^*(G)$ be a trail of G with the most number of vertices, and in this sense, with the least number of vertices in $\bigcup_{i=3}^{\Delta(G)} V_i(G)$. Denote $mt^*(G) = |V(MT^*(G))|$ and $d_{\geq 3}^*(G) = |\bigcup_{i=3}^{\Delta(G)} V_i(G) \backslash V(MT^*(G))|$.

Theorem 14. *Let G be a connected graph of order n. Then $h_p(G) \leq n - mt^*(G) - d_{\geq 3}^*(G) + 2$.*

Proof. Since G is connected, Theorem 14 holds for $|E(G)| < 3$ trivially. So we may assume that $|E(G)| \geq 3$.

Let $MT^*(G)$ be a trail of G satisfying the hypotheses above. Denote $k = n - mt^*(G) - d_{\geq 3}^*(G) + 2$. Note that $k \geq 2$. By Theorem 4, it suffices to prove that $EUP_k(G) \neq \emptyset$.

Let H be the subgraph of G with vertex set $V(MT^*(G)) \cup \left(\bigcup_{i=3}^{\Delta(G)} V_i(G) \right)$ and edge set $E(MT^*(G))$. We will prove that $H \in EUP_k(G)$.

By the definition of H, H satisfies (I)$'$ and (II). Note that $|V(G)| - |V(H)| = n - (mt^*(G) + d_{\geq 3}^*(G)) = k - 2$. Then $d_G(H_1, H - H_1) \leq k - 1$ for every subgraph H_1 of H, $|E(b)| \leq k - 1 (< k + 1)$ for every branch $b \in B(G)$ with $E(b) \cap E(H) = \emptyset$, and $|E(b)| \leq k - 2 (< k)$ for every

branch $b \in B_1(G)$ with $E(b) \cap E(H) = \emptyset$. Hence, H satisfies (III), (IV) and (V)′. Theorem 14 holds. □

The bound of $h_p(G)$ in Theorem 14 is sharp. Fig. 3 shows a graph G with $h_p(G) = n - mt^*(G) - d^*_{\geq 3}(G) + 2$, where s, t are positive integers and $t \geq s + 5$ in the figure. Since $t \geq s+5$, $MT^*(G)$ is the path $x_1 x_2 \cdots x_t w_1 y_t \cdots y_2 y_1$, and then $mt^*(G) = 2t+1$ and $d^*_{\geq 3}(G) = 4$. Note that $n = |V(G)| = 2t + s + 5$. On the one hand, by Theorem 14, we have $h_p(G) \leq n - mt^*(G) - d^*_{\geq 3}(G) + 2 = s + 2$. On the other hand, we will explain $h_p(G) \geq s + 2$ in the following. For $k \leq s + 1$, the k-th iterated line graph $L^k(G)$ is also illustrated in Fig. 3, where the gray ellipse and triangle are the nontrivial hamiltonian subgraph S_1 and S_2 of $L^k(G)$, respectively. Note that $k \leq s + 1$. We have $s - k + 2 \geq 1$, which means that either $|V(S_1) \cap V(S_2)| = 1$ (when $s - k + 2 = 1$), or $d_{L^k(G)}(S_1, S_2) \geq 1$ (when $s - k + 2 \geq 2$). In both cases, $L^k(G)$ is not traceable. Hence, $h_p(G) \geq s + 2$.
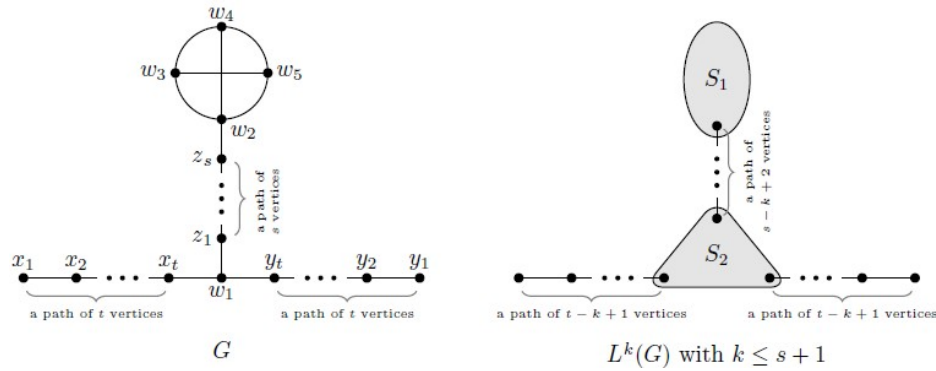

Figure 3. A graph G with $h_p(G) = n - mt^*(G) - d^*_{\geq 3}(G) + 2$ and its iterated line graph $L^k(G)$ with $k \leq s + 1$.

By Theorem 14, we can obtain the following corollary.

Corollary 15. *Let G be a connected graph of order n. Then $h_p(G) \leq \max\{1, n - mt^*(G)\}$.*

Proof. Let $MT^*(G)$ be a maximum trail of G, and in this sense, with the least number of vertices in $\bigcup_{i=3}^{\Delta(G)} V_i(G)$, and let $k = n - mt^*(G)$. If $k \leq 1$, then $MT^*(G)$ is a dominating trail of G. By Theorem 3, $L(G)$ is traceable. Hence, $h_p(G) \leq 1$. So we may assume that $k \geq 2$. Now the proof is divided into three cases.

Case 1. $d^*_{\geq 3}(G) = 0$.

Let $H = MT^*(G)$. We will prove that $H \in EUP_k(G)$, and then $h_p(G) \leq k$. Obviously, H satisfies (I)′, (II) and (III). Note that $d^*_{\geq 3}(G) = 0$. $G - H$ has exactly k vertices. Then $|E(b)| \leq k + 1$ for every branch $b \in B(G)$ with $E(b) \cap E(H) = \emptyset$, and $|E(b)| \leq k$ for every branch $b \in B_1(G)$ with $E(b) \cap E(H) = \emptyset$. Hence, H satisfies (IV) and (V)′.

Case 2. $d^*_{\geq 3}(G) = 1$.

Let v be the vertex of degree at least 3 in $\bigcup_{i=3}^{\Delta(G)} V_i(G) \backslash V(MT^*(G))$. If $|N_G(v)| = 1$, then by $d_{\geq 3}^*(G) = 1$, the neighbour of v belongs to $MT^*(G)$. Hence, the union of $MT^*(G) + v$ and two multiple edges incident to v is a longer trail than $MT^*(G)$, contrary to the maximality of $MT^*(G)$. Then we may assume that $|N_G(v)| \geq 2$.

Let $H = MT^*(G) + v$. We will prove that $H \in EUP_k(G)$. Obviously, H satisfies (I)$'$ and (II). Arguing similarly as the proof of Case 1, H satisfies (IV) and (V)$'$. It remains to prove that $d_G(MT^*(G), v) \leq k - 1$. This holds by the fact that $G - H$ has exactly $k - 1$ vertices, $|N_G(v)| \geq 2$, and the shortest path between v and $MT^*(G)$ contains only one neighbour of v.

Case 3. $d_{\geq 3}^*(G) \geq 2$.

By Theorem 14, $h_p(G) \leq n - mt^*(G) - d_{\geq 3}^*(G) + 2$. Then by $d_{\geq 3}^*(G) \geq 2$, we have $h_p(G) \leq n - mt^*(G)$.

This completes the proof of Corollary 15. $\qquad\qquad\square$

By the sharpness of $h_p(G) \leq n - mt^*(G) - d_{\geq 3}^*(G) + 2$ and the proof of Corollary 15, we know that the bound $h_p(G) \leq \max\{1, n - mt^*(G)\}$ is sharp when $d_{\geq 3}^*(G) \leq 2$.

The *diameter* of a graph G, denoted by $\operatorname{diam}(G)$, is the greatest distance between two vertices of G. Note that $\operatorname{diam}(G) + 1 \leq mt^*(G)$. The following corollary is obvious.

Corollary 16. *Let G be a connected graph of order n. Then $h_p(G) \leq \max\{1, n - \operatorname{diam}(G) - 1\}$.*

For a graph G, let $d_G'(v) = |N_G(v)|$ and $\Delta'(G) = \max\{d_G'(v) : v \in V(G)\}$. Note that if G is simple, then $d_G'(v) = d_G(v)$ and $\Delta'(G) = \Delta(G)$. Let $d_{\geq 3}^{**}(G) = \max\{|(\bigcup_{i=3}^{\Delta(G)} V_i(G)) \backslash N_G(v)| : v \in V(G)$ and $|N_G(v)| = \Delta'(G)\}$. A cycle C of G is called *pendent* if $|V(C) \cap (\bigcup_{i=3}^{\Delta(G)} V_i(G))| = 1$. See Fig. 4 (a) for illustrations of pendent cycles. Let $PC(G)$ be the set of pendent cycles of G.

Theorem 17. *Let G be a connected graph of order n. Then*
$$h_p(G) \leq \left\lfloor \frac{n - \Delta'(G) - d_{\geq 3}^{**}(G)}{3} \right\rfloor + 3.$$

Proof. Let $k = \lfloor (n - \Delta'(G) - d_{\geq 3}^{**}(G))/3 \rfloor + 3$, and v a vertex of G with $|N_G(v)| = \Delta'(G)$ and $|(\bigcup_{i=3}^{\Delta(G)} V_i(G)) \backslash N_G(v)|$ maximized. Note that $k \geq 3$. By Theorem 4, it suffices to prove that $EUP_k(G) \neq \emptyset$.

If $\Delta'(G) \leq 2$, then G is traceable. Hence, $h_p(G) = 0$, and the bound holds trivially. Now we may assume that $\Delta'(G) \geq 3$. Then $B(G)$, the set of branches of G, has at least two elements.

Let $b_1 \in B(G)$ be a branch with $|V(b_1) \cap (V_1(G) \cup V_2(G))|$ maximized, and $b_2 \in B(G) \backslash \{b_1\}$ a branch with $|V(b_2) \cap (V_1(G) \cup V_2(G))|$ maximized. Since G is connected, we can find a trail T_1 (may be trivial) which connects b_1 and b_2. By the fact that the internal vertices (if any) of b_1 and b_2 have degree 2 in G, $T = T_1 \cup b_1 \cup b_2$ is a trail of G.

Let Let H be the subgraph of G with vertex set $\left(\bigcup_{i=3}^{\Delta(G)} V_i(G)\right) \cup V(T) \cup V(PC(G))$ and edge set $E(T) \cup E(PC(G))$. We will prove that $H \in EUP_k(G)$.

Obviously, H satisfies (I)$'$ and (II). By the choice of b_1 and b_2, each of the other branches of G has at most $\lfloor (n - \Delta'(G) - d_{\geq 3}^{**}(G))/3 \rfloor + 1$ vertices in $V_1(G) \cup V_2(G)$, where the +1 is

necessary since each branch may have at most one neighbour of v. Then $|E(b)| \leq \lfloor (n - \Delta'(G) - d_{\geq 3}^{**}(G))/3 \rfloor + 2 = k - 1$ for every branch $b \in B(G)\backslash\{b_1, b_2\}$, and hence, (III), (IV) and (V)$'$ hold.

This completes the proof of Theorem 17.   □

The bound of $h_p(G)$ in Theorem 17 is sharp. Fig. 4 (b) shows a graph G with $n = 3s + 16$, $\Delta'(G) = 6$ and $d_{\geq 3}^{**}(G) = 13$, where s is a positive integer and the three gray cycles are induced K_4. Then by Theorem 17, $h_p(G) \leq \lfloor (n - \Delta'(G) - d_{\geq 3}^{**}(G))/3 \rfloor + 3 = s + 2$. Note that G has 3 branches of length $s + 1$. $L^k(G)$ is not traceable when $k < s + 2$. Then $h_p(G) \geq s + 2$, and hence, $h_p(G) = s + 2$, which implies the sharpness of the upper bound in Theorem 17.

If G is a connected simple graph of order n, then Theorem 17 implies that $h_p(G) \leq \lfloor (n - \Delta(G) - d_{\geq 3}^{**}(G))/3 \rfloor + 3$.
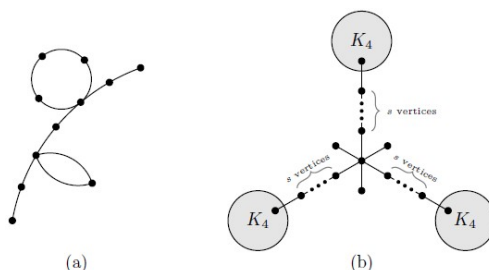

Figure 4. (a) A graph with 2 pendent cycles; and (b) A graph G with $h_p(G) = \lfloor (n - \Delta'(G) - d_{\geq 3}^{**}(G))/3 \rfloor + 3$.

In [19], Saražin proved the following upper bound of $h(G)$.

Theorem 18. *(Saražin, [19]) Let G be a connected simple graph of order n. If $\Delta(G) \geq 3$, then $h(G) \leq n - \Delta(G)$.*

The following corollary, obtained by Theorem 17 immediately, implies that when considering $h_p(G)$, the upper bound in Theorem 18 can be improved evidently.

Corollary 19. *Let G be a connected simple graph of order n. Then*
$$h_p(G) \leq \left\lfloor \frac{n - \Delta(G)}{3} \right\rfloor + 3.$$

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Conflict of interest The authors declare no conflict of interest.

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