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# Existence and numerical approximation of a solution to frictional contact problem for electro-elastic materials

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**Abstract**. In this paper, a frictional contact problem between an electro-elastic body and an electrically conductive foundation is studied. The contact is modeled by normal compliance with finite penetration and a version of Coulomb's law of dry friction in which the coefficient of friction depends on the slip. In addition, the effects of the electrical conductivity of the foundation are taken into account. This model leads to a coupled system of the quasi-variational inequality of the elliptic type for the displacement and the nonlinear variational equation for the electric potential. The existence of a weak solution is proved by using an abstract result for elliptic variational inequalities and a fixed point argument. Then, a finite element approximation of the problem is presented. Under some regularity conditions, an optimal order error estimate of the approximate solution is derived. Finally, a successive iteration technique is used to solve the problem numerically and a convergence result is established.

## §1 Introduction

The study of piezoelectric materials remains an active research area and success of adaptive devices has attracted the attention of industry and engineering researchers. Due to the intrinsic coupling between mechanical and electrical energy, these materials can serve as sensors, actuators or transducers. This ability is widely used in various technical devices as ultrasonic medical equipment, fuel injection pistons or smart composites with integrated piezoelectric layers. For this reason, considerable progress has been made with the modelling and analysis of contact problems, and the engineering literature concerning this topic is rather extensive. Different models have been developed to describe the interaction between the electrical and mechanical fields which can be found in [18,16] and the references therein. A static frictional contact problem for electro-elastic materials was considered in [14,17] under the assumption

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that the foundation is insulated, and [4,16,17] under the assumption that the foundation is electrically conductive. Some comprehensive references on analysis and numerical approximation of variational inequalities arising from contact problems with or without friction for piezoelectric materials include [10,11,9] and, more recently, [1,5,20,21].

The present paper is devoted to variational and numerical analysis of a problem of frictional contact under a small deformation hypothesis. The process is static and the friction is described by a slip dependent friction coefficient and a nonlocal regularized contact stress. The material's behavior is described by a linear electro-elastic constitutive law and the contact is modeled with a normal compliance condition of such a type that the penetration is limited with unilateral constraint and a regularized electrical conductivity condition. The resulting variational formulation of the problem is different from that in [4] and represents a new mathematical model, which is in a form of a system coupling a nonlinear variational inequality for the displacement field and a nonlinear variational equation for the electric potential. We show the existence of a unique weak solution for the model. Then, we perform a numerical analysis of the problem and derive error estimates for the numerical approximations based on discrete schemes.

The rest of the paper is structured as follows. In Section 2, we introduce some notation and preliminary and present a model for the process of frictional contact between the electro-elastic body and the conductive foundation. In Section 3, we list assumptions on the data, derive a variational formulation of the model and state our main result, the existence of a unique weak solution of the problem in Theorem 3.1. The proof of the theorem is given in Section 4, where it is carried out in several steps and is based on arguments of elliptic variational inequalities and the Schauder fixed point theorem. In Section 5, we introduce a finite element approximation for the variational inequality problem and we present the results of some error estimates for the numerical approximation. Finally, in Section 6, we propose an iterative solution scheme to solve the problem numerically and we prove its convergence.

## §2 Problem statement

We consider a piezoelectric body occupying, in its reference configuration, a bounded domain $\Omega \subset \mathbb{R}^d$, $d = 2, 3$ with a sufficiently regular boundary $\Gamma$, partitioned into three disjoint measurable parts $\Gamma_1$, $\Gamma_2$ and $\Gamma_3$, such that $meas(\Gamma_1) > 0$. A volume force of density $f_0$ and volume electric charges of density $q_0$ act in $\Omega$. The body is clamped on $\Gamma_1$ and a surface traction of density $f_2$ acts on $\Gamma_2$. To describe the electric constraints of the body, we consider a partition of $\Gamma_1 \cup \Gamma_2$ into two disjoint parts $\Gamma_a$ and $\Gamma_b$, such that $meas(\Gamma_a) > 0$. We assume that the electrical potential vanishes on $\Gamma_a$ and a surface electrical charge of density $q_2$ is prescribed on $\Gamma_b$. In the initial configuration, the body may come in contact over $\Gamma_3$ with a rigid foundation.

To simplify the notation, we do not indicate explicitly the dependence of various functions on the spatial variable $x \in \overline{\Omega}$. The indices $i, j, k, l$ run between 1 and $d$, the summation convention over repeated indices is used and the index that follows a comma indicates the partial derivative with respect to the corresponding component of the independent variable, e.g. $u_{i,j} = \frac{\partial u_i}{\partial x_j}$. We denote by $\mathbb{S}^d$ the linear space of second order symmetric tensors on $\mathbb{R}^d$. The inner products and the corresponding norms on $\mathbb{R}^d$ and $\mathbb{S}^d$ are given by

$$u \cdot v = u_i v_i \,, \; \|v\| = (v \cdot v)^{\frac{1}{2}}, \forall u, v \in \mathbb{R}^d, \quad \sigma \cdot \tau = \sigma_{ij}\tau_{ij} \,, \; \|\tau\| = (\tau \cdot \tau)^{\frac{1}{2}} \,, \forall \sigma, \tau \in \mathbb{S}^d.$$

Throughout the paper, we adopt the following notation $u = (u_i) : \Omega \to \mathbb{R}^d$ for the displacement field, $\sigma = (\sigma_{ij}) : \Omega \to \mathbb{S}^d$ for the stress tensor, $D = (D_i) : \Omega \to \mathbb{R}^d$ for the electric displacement field and $E(\varphi) = (E_i(\varphi)) = -\nabla\varphi$ for the electric vector field, where $\varphi : \Omega \to \mathbb{R}$ is an electric potential. Moreover, $\varepsilon(u) = (\varepsilon_{ij}(u))$ where $\varepsilon_{ij}(u) = \frac{1}{2}(u_{i,j} + u_{j,i})$ denotes the linearized strain tensor, and "Div", "div" are the divergence operators for tensor and vector valued functions. Let $\nu$ be the unit outward normal vector on $\Gamma$, then the normal and tangential components of the displacement field and stress tensor are given by

$$v_\nu = v \cdot \nu \,, \; v_\tau = v - v_\nu \nu \,, \quad \sigma_\nu = \sigma\nu \cdot \nu \,, \; \sigma_\tau = \sigma\nu - \sigma_\nu \nu.$$

The governing equations consist of the equilibrium equations, constitutive relations, strain-displacement and electric field-potential relations. The equilibrium equations are given by

$$\operatorname{Div} \sigma + f_0 = 0, \quad \operatorname{div} D = q_0 \quad \text{in } \Omega. \tag{1}$$

The linear constitutive equations that couple the mechanical and electrical quantities in the piezoelectric materials can be written in the following form

$$\sigma = \mathfrak{F}\varepsilon(u) - \mathcal{E}^* E(\varphi), \;\; D = \mathcal{E}\varepsilon(u) + \beta E(\varphi) \quad \text{in } \Omega, \tag{2}$$

where $\mathfrak{F} = (f_{ijkl}) : \Omega \times \mathbb{S}^d \to \mathbb{S}^d$ is a linear elasticity operator, in which the elasticity coefficients $f(x) = (f_{ijkl}(x))$ may be function of position in a non-homogeneous materials, $\mathcal{E} = (e_{ijk}) : \Omega \times \mathbb{S}^d \to \mathbb{R}^d$ is a linear piezoelectric operator, $\beta = (\beta_{ij}) : \Omega \times \mathbb{R}^d \to \mathbb{R}^d$ is a linear electric permittivity operator. We use $\mathcal{E}^*$ to denote the transpose tensor of $\mathcal{E}$ given by

$$\mathcal{E}\sigma \cdot v = \sigma \cdot \mathcal{E}^* v, \quad \forall\, \sigma \in \mathbb{S}^d, v \in \mathbb{R}^d. \tag{3}$$

The elastic strain-displacement and electric field-potential relations are given by

$$\varepsilon(u) = \frac{1}{2}(\nabla u + (\nabla u)^\perp), \;\; E(\varphi) = -\nabla\varphi \quad \text{in } \Omega. \tag{4}$$

where $(\nabla u)^\perp = (u_{j,i})$ is the transpose of $\nabla u$. Next, we prescribe the boundary conditions by

$$u = 0 \quad \text{on } \Gamma_1, \;\; \sigma\nu = f_2 \quad \text{on } \Gamma_2, \tag{5}$$

$$\varphi = 0 \quad \text{on } \Gamma_a, \;\; D \cdot \nu = q_2 \quad \text{on } \Gamma_b. \tag{6}$$

We model the frictional contact on $\Gamma_3$ with

$$\left.\begin{aligned} & u_\nu \le g, \sigma_\nu + h_\nu(\varphi - \varphi_F) p_\nu(u_\nu) \le 0, \\ & (u_\nu - g)(\sigma_\nu + h_\nu(\varphi - \varphi_\nu) p_\nu(u_\nu)) = 0, \end{aligned}\right\} \quad \text{on } \Gamma_3, \tag{7}$$

$$\left.\begin{aligned} & \|\sigma_\tau\| \le \mu(\|u_\tau\|)|R\sigma_\nu(u,\varphi)| \\ & \|\sigma_\tau\| < \mu(\|u_\tau\|)|R\sigma_\nu(u,\varphi)| \Rightarrow u_\tau = 0 \\ & \|\sigma_\tau\| = \mu(\|u_\tau\|)|R\sigma_\nu(u,\varphi)| \Rightarrow \exists\, \lambda \in \mathbb{R}^+, \; \sigma_\tau = -\lambda u_\tau \end{aligned}\right\} \quad \text{on } \Gamma_3, \tag{8}$$

$$D \cdot \nu = p_e(u_\nu) h_e(\varphi - \varphi_F) \quad \text{on } \Gamma_3. \tag{9}$$

In conditions (7), the function $g$ represents the maximum interpenetration of body's and foundations asperities and $\varphi_F$ denotes the electric potential of the foundation. This condition (7) introduced in [3], represents the normal compliance contact condition with finite penetration in which $p_\nu$ is a prescribed nonnegative function depending on the difference between the potential of the foundation and the body's surface and which vanishes when its argument is negative and $h_\nu$ is a positive function. We note that (7) shows that when there is no contact (i.e. $u_\nu < 0$), the reaction of the foundation vanishes and, therefore, $\sigma_\nu = 0$. When $0 \le u_\nu < g$, then we have $-\sigma_\nu = h_\nu(\varphi - \varphi_\nu)p_\nu(u_\nu)$, which means that the reaction of the foundation depends on the normal displacement and the difference between the potential of the foundation and the body's surface. When $u_\nu = g$, then $-\sigma_\nu \ge h_\nu(\varphi - \varphi_\nu)p_\nu(g)$. We note that if $p_\nu = 0$, the condition (7) becomes the classical Signorini's conditions with a gap $g$, i.e.,

$$u_\nu \le g, \quad \sigma_\nu \le 0, \quad \sigma_\nu(u_\nu - g) = 0.$$

Relations (8) represent the Coulomb's friction law in which $\mu$ is the coefficient of friction and $R$ is a regularization operator. The introduction of the nonlocal smoothing operator $R$ is used for technical reasons, since the trace of the stress tensor on the boundary is too rough. We note that the coefficient of friction $\mu$ is assumed to dependi on the slip $\|u_\tau\|$, which leads to a nonstandard frictional contact problem.

Finally, (9) is a regularized electrical contact condition (see [12]) where $p_e$ represents the electrical conductivity coefficient, which vanishes when its argument is nonnegative, and $h_e$ is a given function depending on the difference of the electric potential of the body's surface and the foundation.

This condition shows that when there is no contact at a point on the surface (i.e., $u_\nu < 0$), then the normal component of the electric displacement field vanishes, and when there is contact (i.e., $u_\nu \ge 0$) then there may be electrical charges which depend on the potential difference between the foundation and the contact surface.

Now, we collect all the above conditions to obtain the following mathematical model,

**Problem** $(P)$**.** *Find a displacement field* $u : \Omega \to \mathbb{R}^d$ *and an electric potential* $\varphi : \Omega \to \mathbb{R}$ *such that* (1)-(9) *hold.*

The variational analysis of the frictional contact Problem $(P)$ will be presented in the next sections, where we give our main existence and uniqueness result of the weak solution of $(P)$.

## §3 Variational formulation and main result

In this section, we state the hypotheses and derive the weak formulation of Problem $(P)$. First, we introduce the following functional spaces

$$H = \{u = (u_i) \mid u_i \in L^2(\Omega)\}, \quad \mathcal{H} = \{\sigma = (\sigma_{ij}) \mid \sigma_{ij} = \sigma_{ji} \in L^2(\Omega)\},$$
$$H_1 = \{u = (u_i) \mid u_i \in H^1(\Omega)\}, \quad \mathcal{H}_1 = \{\sigma \in \mathcal{H} \mid \operatorname{Div} \sigma \in H\}.$$

These are real Hilbert spaces endowed with the inner products

$$(u, v)_H = \int_\Omega u_i v_i \, dx, \quad (\sigma, \tau)_{\mathcal{H}} = \int_\Omega \sigma_{ij} \tau_{ij} \, dx,$$

$$(u,v)_{H_1} = (u,v)_H + (\varepsilon(u),\varepsilon(v))_{\mathcal{H}}, \quad (\sigma,\tau)_{\mathcal{H}_1} = (\sigma,\tau)_{\mathcal{H}} + (\mathrm{Div}\,\sigma, \mathrm{Div}\,\tau)_H,$$

and their associated Euclidean norms $\|\cdot\|_H$, $\|\cdot\|_{\mathcal{H}}$, $\|\cdot\|_{H_1}$ and $\|\cdot\|_{\mathcal{H}_1}$.

Let $H_\Gamma = (H_\Gamma^{\frac{1}{2}})^d$ and let $\gamma : H_1 \to H_\Gamma$ be the trace map. For every element $v \in H_1$, we also use the notation $v$ to denote the trace $\gamma v$ of $v$ on $\Gamma$. Let $H_\Gamma^{'}$ be the dual of $H_\Gamma$ and let $\langle \cdot,\cdot\rangle_\Gamma$ denote the duality pairing between $H_\Gamma^{'}$ and $H_\Gamma$. For every $\sigma \in \mathcal{H}_1$, $\sigma\nu$ can be defined as the element of $H_\Gamma^{'}$ which satisfies the Green formula

$$\langle \sigma\nu, \gamma v\rangle_\Gamma = (\sigma, \varepsilon(v))_{\mathcal{H}} + (\mathrm{Div}\,\sigma, v)_H \text{ for all } v \in H_1.$$

Moreover, if $\sigma$ is continuously differentiable on $\overline{\Omega}$, then

$$\langle \sigma\nu, \gamma v\rangle = \int_\Gamma \sigma\nu \cdot v\, da.$$

Let us also introduce $H_{\Gamma_3}^{\frac{1}{2}} \subset L^2(\Gamma_3)$, the space of normal traces on $\Gamma_3$ given by

$$H_{\Gamma_3}^{\frac{1}{2}} = \{v_\nu \in L^2(\Gamma_3)\ ;\ \ (\exists\ v \in H_1),\ v_\nu = \gamma v \cdot \nu\}. \tag{10}$$

The spaces $H_{\Gamma_3}^{\frac{1}{2}}$ and its dual $H_{\Gamma_3}^{-\frac{1}{2}}$ can be endowed with the following norms

$$\|v_\nu\|_{H_{\Gamma_3}^{\frac{1}{2}}} = \inf_{v \in H_1} \{\|v\|_{H_1}, v_\nu = \gamma v \cdot \nu\} \quad \text{for all} \quad v_\nu \in H_{\Gamma_3}^{\frac{1}{2}}, \tag{11}$$

$$\|\sigma_\nu\|_{H_{\Gamma_3}^{-\frac{1}{2}}} = \sup_{v_\nu \in H_{\Gamma_3}^{\frac{1}{2}}} \frac{\langle \sigma_\nu, v_\nu\rangle_{\Gamma_3}}{\|v_\nu\|_{H_{\Gamma_3}^{\frac{1}{2}}}} \quad \text{for all} \quad v_\nu \in H_{\Gamma_3}^{\frac{1}{2}} \quad \text{and} \quad \sigma_\nu \in H_{\Gamma_3}^{-\frac{1}{2}}, \tag{12}$$

where $\langle\cdot,\cdot\rangle_{\Gamma_3}$ denotes the duality pairing between $H_{\Gamma_3}^{-\frac{1}{2}}$ and $H_{\Gamma_3}^{\frac{1}{2}}$. Recalling the condition (5), we introduce the following subspaces of $H_1$ given by

$$V = \{v \in H_1;\ v = 0 \ \text{ on } \Gamma_1\},$$

and the set $K$ of admissible displacements

$$K = \{v \in V;\ v_\nu - g \le 0 \ \text{ on } \Gamma_3\}.$$

Since $meas(\Gamma_1) > 0$, it follows from the Korn's inequality that there exists a constant $c_k > 0$ depending only on $\Omega$ and $\Gamma_1$ such that

$$\|\varepsilon(v)\|_{\mathcal{H}} \ge c_k \|v\|_{H_1} \quad \forall\, v \in V. \tag{13}$$

Over the space $V$, we consider the following inner product and associated norm

$$(u,v)_V = (\varepsilon(u),\varepsilon(v))_{\mathcal{H}}, \quad \|u\|_V = \|\varepsilon(v)\|_{\mathcal{H}} = (u,u)_V^{\frac{1}{2}}. \tag{14}$$

From (13) it follows that $\|\cdot\|_{H_1}$ and $\|\cdot\|_V$ are equivalent on $V$. Therefore $(V, \|\cdot\|_V)$ is a Hilbert space. Moreover, by the Sobolev trace theorem, (13) and (14), there exists a constant $c_0 > 0$ depending only on $\Omega$, $\Gamma_3$ and $\Gamma_1$ such that

$$\|v\|_{L^2(\Gamma)^d} \le c_0 \|v\|_V \text{ for all } v \in V. \tag{15}$$

For the electric unknowns of the contact problem, we introduce the spaces

$$W = \{\psi \in H^1(\Omega)/\ \psi = 0 \text{ on } \Gamma_a\},$$
$$\mathcal{W} = \{D = (D_i) \in H^1(\Omega)^d/\ (D_i) \in L^2(\Omega)^d,\ \ \mathrm{div}\, D \in L^2(\Omega)\}.$$

These are real Hilbert spaces with the inner products

$$(\varphi,\psi)_W=(\varphi,\psi)_{H^1(\Omega)},\quad (D,E)_{\mathcal{W}}=(D,E)_{L^2(\Omega)^d}+(\operatorname{div}D,\operatorname{div}E)_{L^2(\Omega)}.$$

The associated norms are $\|\cdot\|_W$ and $\|\cdot\|_{\mathcal{W}}$, respectively. Since $meas(\Gamma_a)>0$, the following Friedrichs-Poincaré inequality holds

$$\|\nabla\psi\|_{\mathcal{W}}\geq c_F\|\psi\|_W \text{ for all } \psi\in W. \tag{16}$$

where $c_F>0$ is a constant which depends only on $\Omega$ and $\Gamma_a$. Moreover, by the Sobolev trace theorem, there exists a constant $c_1>0$ depending only on $\Omega$, $\Gamma_a$ and $\Gamma_3$, such that

$$\|\xi\|_{L^2(\Gamma_3)}\leq c_1\|\xi\|_W \text{ for all } \xi\in W. \tag{17}$$

When $D\in\mathcal{W}$ is a sufficiently regular function, the following Green's type formula holds,

$$(D,\nabla\xi)_{L^2(\Omega)^d}+(\operatorname{div}D,\xi)_{L^2(\Omega)}=\int_\Gamma D\cdot\nu\,\xi\,da \text{ for all } \xi\in H^1(\Omega).$$

To study Problem $(P)$, we need the following assumptions on the data's problem.

$(h_1)$ The elasticity operator $\mathfrak{F}=(f_{ijkl}):\Omega\times\mathbb{S}^d\to\mathbb{S}^d$, the electric permittivity tensor $\beta=(\beta_{ij}):\Omega\times\mathbb{R}^d\to\mathbb{R}^d$ satisfy the usual properties of symmetry and ellipticity

$$f_{ijkl}=f_{jikl}=f_{lkij}\in L^\infty(\Omega)\ ,\quad M_{\mathfrak{F}}=\sup_{i,j,k,l}\|f_{ijkl}\|_{L^\infty(\Omega)},$$

$$\beta_{ij}=\beta_{ji}\in L^\infty(\Omega)\ ,\quad M_\beta=\sup_{i,j}\|\beta_{ij}\|_{L^\infty(\Omega)},$$

and there exists $m_{\mathfrak{F}},m_\beta$ such that for all $\xi\in\mathbb{S}^d$, $\zeta\in\mathbb{R}^d$

$$f_{ijkl}(x)\,\xi_{ij}\xi_{kl}\geq m_{\mathfrak{F}}\,\|\xi\|^2\ ,\quad \beta_{ij}(x)\,\zeta_i\zeta_j\geq m_\beta\,\|\zeta\|^2,\ \text{a.e. } x\in\Omega.$$

$(h_2)$ The piezoelectric tensor $\mathcal{E}=(e_{ijk}):\Omega\times\mathbb{S}^d\to\mathbb{R}^d$, satisfies

$$e_{ijk}=e_{ikj}\in L^\infty(\Omega)\ ,\quad M_{\mathcal{E}}=\sup_{i,j,k}\|e_{ijk}\|_{L^\infty(\Omega)}.$$

$(h_3)$ The function $p_r:\Gamma_3\times\mathbb{R}\to\mathbb{R}_+$, $(r=e,\nu)$ satisfies the following hypothesis

(a) there exists $M_{p_r}>0$ such that $|p_r(x,u)|\leq M_{p_r}$ for all $u\in\mathbb{R}$, a.e. $x\in\Gamma_3$,

(b) $x\mapsto p_r(x,u)$ is measurable on $\Gamma_3$ for all $u\in\mathbb{R}$ and is zero for all $u\leq 0$.

$(h_4)$ The function $h_r:\Gamma_3\times\mathbb{R}\to\mathbb{R}$, $(r=e,\nu)$ satisfies the following hypothesis

(a) there exists $M_{h_e}>0$ such that $|h_e(x,\varphi)|\leq M_{h_e}$ for all $\varphi\in\mathbb{R}$, a.e. $x\in\Gamma_3$,

(b) there exists $M_{h_\nu}>0$ such that $0\leq h_\nu(x,\varphi)\leq M_{h_\nu}$ for all $\varphi\in\mathbb{R}$, a.e. $x\in\Gamma_3$,

(c) $x\mapsto h_r(x,\varphi)$ is measurable on $\Gamma_3$ for all $\varphi\in\mathbb{R}$.

$(h_5)$The coefficient of friction $\mu:\Gamma_3\times\mathbb{R}_+\to\mathbb{R}_+$ satisfies

(a) there exists $\mu^*>0$ such that $0\leq\mu(x,u)\leq\mu^*$ for all $u\in\mathbb{R}_+$, a.e. $x\in\Gamma_3$,

(b) the function $x\mapsto\mu(x,u)$ is measurable on $\Gamma_3$ for all $u\in\mathbb{R}$.

$(h_6)$ We assume that $p_r$, $h_r$ and $\mu$ are lipschitz continuous functions in the following sense

(a) $\exists\, L_{p_r} > 0,\ \ |p_r(x,u_1) - p_r(x,u_2)| \le L_{p_r}|u_1 - u_2|$ for all $u_1, u_2 \in \mathbb{R}$, a.e. $x \in \Gamma_3$,

(b) $\exists\, L_{h_r} > 0,\ \ |h_r(x,\varphi_1) - h_r(x,\varphi_2)| \le L_{h_r}|\varphi_1 - \varphi_2|$ for all $\varphi_1, \varphi_2 \in \mathbb{R}$, a.e. $x \in \Gamma_3$,

(c) $\exists\, L_\mu > 0,\ \ |\mu(x,u) - \mu(x,v)| \le L_\mu|u - v|$ for all $u, v \in \mathbb{R}_+$, a.e. $x \in \Gamma_3$.

$(h_7)$ We suppose that $R : H_{\Gamma_3}^{-\frac{1}{2}} \to L^\infty(\Gamma_3)$ is a linear and continuous. We denote $\|R\| = c_R$.

$(h_8)$ The forces, the traction, the volume and surface charge densities satisfy

$$f_0 \in L^2(\Omega)^d,\ f_2 \in L^2(\Gamma_2)^d,\ q_0 \in L^2(\Omega),\ q_2 \in L^2(\Gamma_b).$$

$(h_9)$ The potential of the contact surface and the gap satisfy

$$\varphi_F \in L^2(\Gamma_3), \quad g \in L^2(\Gamma_3).$$

Elements $f \in V$ and $q_e \in W$ are defined as

$$(f, v)_V = \int_\Omega f_0 \cdot v\, dx + \int_{\Gamma_2} f_2 \cdot v\, da \quad \text{for all } v \in V, \tag{18}$$

$$(q_e, \xi)_W = \int_\Omega q_0 \xi\, dx - \int_{\Gamma_b} q_2 \xi\, da \quad \text{for all } \xi \in W. \tag{19}$$

We define the following $\mathbb{R}$-valued mappings $j_1$, $j_2$ and $j$ defined on $V \times W \times V$ by

$$j_1(u, \varphi, v) = \int_{\Gamma_3} \mu(\|u_\tau\|)\, |R\sigma_\nu(u, \varphi)|\ \|v_\tau\|\, da, \tag{20}$$

$$j_2(u, \varphi, v) = \int_{\Gamma_3} h_\nu(\varphi - \varphi_F) p_\nu(u_\nu)\, v_\nu\, da, \tag{21}$$

$$j(u, \varphi, v) = j_1(u, \varphi, v) + j_2(u, \varphi, v), \tag{22}$$

and the mapping $j_3 : V \times W \times W \to \mathbb{R}$ given by

$$j_3(u, \varphi, \xi) = \int_{\Gamma_3} p_e(u_\nu) h_e(\varphi - \varphi_F)\, \xi\, da. \tag{23}$$

Keeping in mind $(h_8)$-$(h_9)$ and $(h_3)$-$(h_5)$, it follows that the integrals in (18)-(23) are well-defined. Under these notations, the Green formula implies that if $(u, \sigma, \phi, D)$ are sufficiently regular functions satisfying (1)-(9), then we obtain the variational formulation of Problem $(P)$.

**Problem** $(PV)$**.** *Find a displacement field $u \in K$ and the electric potential $\varphi \in W$ such that :*

$$\begin{aligned}&(\mathfrak{F}\varepsilon(u), \varepsilon(v) - \varepsilon(u))_{\mathcal{H}} + (\mathcal{E}^*\nabla\varphi, \varepsilon(v) - \varepsilon(u))_{L^2(\Omega)^d} + j(u, \varphi, v) - j(u, \varphi, u)\\ &\ge (f, v - u)_V \ \text{ for all } v \in K,\end{aligned} \tag{24}$$

$$(\beta\nabla\varphi, \nabla\xi)_{L^2(\Omega)^d} - (\mathcal{E}\varepsilon(u), \nabla\xi)_{L^2(\Omega)^d} + j_3(u, \varphi, \xi) = (q_e, \xi)_W \ \text{ for all } \xi \in W. \tag{25}$$

Now, we are able to state our main result that we will prove in the next section.

**Theorem 3.1.** *Assume that $(h_1)$-$(h_5)$ and $(h_7)$-$(h_9)$ hold. Then*

1. *Problem $(PV)$ has at least one solution $(u, \varphi) \in K \times W$.*

2. *Under assumptions $(h_6)$, there exists $L^* > 0$ such that if*

$$M_{h\nu}L_{p\nu} + M_{p\nu}L_{h\nu} + M_{he}L_{pe} + M_{pe}L_{he} + \mu^* + L_\mu < L^*,$$

*then, Problem $(PV)$ has a unique solution.*

## §4 Proof of Theorem 3.1

In this section, we assume that assumptions $(h_1)$-$(h_9)$ hold. The proof is based on fixed point arguments, similar to those used in [4,9], but with a different choice of the operators, and it will be carried out in several steps. First, we consider the product spaces $X = V \times W$ and $Y = L^2(\Gamma_3)^3$ endowed with the following inner products

$$\big(x, y\big)_X = (u, v)_V + (\varphi, \xi)_W, \quad \big(\eta, \alpha\big)_Y = \sum_{i=1}^{i=3} (\eta_i, \alpha_i)_{L^2(\Gamma_3)}, \tag{26}$$

for all $x = (u, \varphi), y = (v, \xi) \in X$, $\eta = (\eta_1, \eta_2, \eta_3), \alpha = (\alpha_1, \alpha_2, \alpha_3) \in Y$ and associated Euclidean norms $\|.\|_X$ and $\|.\|_Y$. Let $U = K \times W$ be nonempty closed convex subset of $X$. We define the operator $A : X \to X$, the function $J : X \times X \to \mathbb{R}$ and the element $F$ of $X$ by

$$(Ax, y)_X = (\mathfrak{F}\varepsilon(u), \varepsilon(v))_{\mathcal{H}} + (\mathcal{E}^*\nabla\varphi, \varepsilon(v))_{L^2(\Omega)^d} + (\beta\nabla\varphi, \nabla\xi))_{L^2(\Omega)^d}$$
$$- (\mathcal{E}\varepsilon(u), \nabla\xi)_{L^2(\Omega)^d}, \quad \forall\, x = (u, \varphi),\ y = (v, \xi) \in X, \tag{27}$$

$$(F, y)_X = (f, v)_V + (q_e, \xi)_W, \quad \forall\, y = (v, \xi) \in X, \tag{28}$$

$$J(x, y) = j_1(u, \varphi, v) + j_2(u, \varphi, v) + j_3(u, \varphi, \xi), \quad \forall\, x = (u, \varphi),\ y = (v, \xi) \in X. \tag{29}$$

Then, we have the following equivalence result,

**Lemma 4.1.** *The couple $x = (u, \varphi)$ is solution of Problem $(PV)$ if and only if*

$$(Ax, y - x)_X + J(x, y) - J(x, x) \geq (F, y - x)_X, \ \forall\, y \in U. \tag{30}$$

*Proof.* Let $x = (u, \varphi) \in X$ a solution of (24)-(25), and $y = (v, \xi) \in U$. We replace $\xi$ in (25) by $\xi - \varphi$, add the corresponding inequality to (24), and use (27)-(29) to obtain (30). Conversely, let $x = (u, \varphi)$ be a solution to the quasivariational inequality (30). For any $v \in K$, we take $y = (v, \varphi)$ in (30), to obtain (24). Then for any $\xi \in W$, we take successively $y = (u, \varphi - \xi)$ and $y = (u, \varphi + \xi)$ in (30) to obtain (25). This completes the proof of Lemma 4.1.

Next, let $\eta = (\eta_1, \eta_2, \eta_3) \in Y$ be given, we define the following closed convex sets of $L^2(\Gamma_3)$ $\square$

$$\mathcal{K}_1 = \{\eta_1 \in L^2(\Gamma_3)\,; \ \ \eta_1 \geq 0 \text{ and } \|\eta_1\|_{L^2(\Gamma_3)} \leq k_1\}, \quad \mathcal{K}_2 = \{\eta_2 \in L^2(\Gamma_3)\ ; \ \ \|\eta_2\|_{L^2(\Gamma_3)} \leq k_2\},$$
$$\mathcal{K}_3 = \{\eta_3 \in L^2(\Gamma_3)\ ; \ \ \|\eta_3\|_{L^2(\Gamma_3)} \leq k_3\},$$

where constants $k_1$, $k_2$ and $k_3$ will be specified later. We also define on $X$ the $\mathbb{R}$-valued functions

$$J_1^\eta(y) = \int_{\Gamma_3} \eta_1\, \|v_\tau\|\, da \text{ for all } y = (v, \xi) \in X, \tag{31}$$

$$J_2^\eta(y) = \int_{\Gamma_3} \eta_2\, v_\nu\, da \text{ for all } y = (v, \xi) \in X, \tag{32}$$

$$J_3^\eta(y) = \int_{\Gamma_3} \eta_3\, \xi\, da \text{ for all } y = (v, \xi) \in X, \tag{33}$$

and the element $F^\eta$ of $X$ given by

$$(F^\eta, y)_X = (F, y)_X - J_2^\eta(y) - J_3^\eta(y) \text{ for all } y = (v, \xi) \in X. \tag{34}$$

Using these notations, we construct the following intermediate problem.

**Problem** $(PV^\eta)$**.** *Let $\eta = (\eta_1, \eta_2, \eta_3) \in \mathcal{K}_1 \times \mathcal{K}_2 \times \mathcal{K}_3$. Find $x_\eta = (u_\eta, \varphi_\eta) \in U$ such that*

$$(Ax_\eta, y - x_\eta)_X + J_1^\eta(y) - J_1^\eta(x_\eta) \geq (F^\eta, y - x_\eta)_X \text{ for all } y \in U. \tag{35}$$

The unique solvability of Problem $(PV^\eta)$ follows from the following lemma.

**Lemma 4.2.** *For any $\eta \in \mathcal{K}_1 \times \mathcal{K}_2 \times \mathcal{K}_3$ assumed to be known, we have*

1. *Under $(h_1)$-$(h_2)$ and $(h_8)$-$(h_9)$, Problem $(PV^\eta)$ has unique solution $x_\eta = (u_\eta, \varphi_\eta) \in U$.*

2. *The solution $x_\eta = (u_\eta, \varphi_\eta)$ of Problem $(PV^\eta)$ depends Lipschitz continuously on $\eta$.*

3. *There exists a constant $c_2 > 0$ such that the solution of Problem $(PV^\eta)$ satisfies*

$$\|x_\eta\|_x \leq c_2(\|F\|_X + \|\eta\|_Y). \tag{36}$$

*Proof.* 1. We consider two elements $x_1 = (u_1, \varphi_1)$ and $x_2 = (u_2, \varphi_2)$ of $X$. Using (27), (3), (17) and $(h_1)$, there exists $m_A = \min(m_{\mathfrak{F}}, m_\beta) > 0$ such that

$$(Ax_1 - Ax_2, x_1 - x_2)_X \geq m_A \|x_1 - x_2\|_X^2. \tag{37}$$

In the same way, using $(h_1)$-$(h_3)$, there exists a constant $M_A = 2\max(M_{\mathfrak{F}}, M_\beta, M_{\mathcal{E}})$ such that

$$\|Ax_1 - Ax_2\|_X \leq M_A \|x_1 - x_2\|_X. \tag{38}$$

Then, by combining (37) and (38), we get that the operator $A : X \to X$ is strongly monotone and Lipschitz continuous. Moreover, it follows from (31) and (26) that the function $J_1^\eta$ is convex and Lipschitz continuous and then $J_1^\eta$ is a fortiori lower semicontinuous. From the definitions (28) and (32) it is easy to see that the function $F^\eta$ defined by (34) is an element of $X$. Recalling that $U$ is a closed convex nonempty subset of $X$, it follows from standard arguments on variational inequalities that there exists a unique solution $x_\eta = (u_\eta, \varphi_\eta)$ of Problem $(PV^\eta)$.
2. Let $\eta^1 = (\eta_1^1, \eta_2^1, \eta_3^1), \eta^2 = (\eta_1^2, \eta_2^2, \eta_3^2) \in Y$. It follows from (35) that

$$(Ax_1, y - x_1)_X + J_1^{\eta_1}(y) - J_1^{\eta_1}(x_1) \geq (F^{\eta_1}, y - x_1)_X \text{ for all } y \in U,$$
$$(Ax_2, y - x_2)_X + J_1^{\eta_2}(y) - J_1^{\eta_2}(x_2) \geq (F^{\eta_2}, y - x_2)_X \text{ for all } y \in U.$$

Taking $y = x_2$ in the first inequality and $y = x_1$ in the second inequality, we obtain

$$\begin{aligned}
&(Ax_1 - Ax_2, x_1 - x_2)_X \\
&\leq J_1^{\eta_1}(x_2) - J_1^{\eta_1}(x_1) + J_1^{\eta_2}(x_1) - J_1^{\eta_2}(x_2) + J_2^{\eta_2 - \eta_1}(x_1 - x_2) + J_3^{\eta_2 - \eta_1}(x_1 - x_2) \\
&\leq \int_{\Gamma_3} \left(\eta_1^1 - \eta_1^2\right)\left(|u_{1\tau}| - |u_{2\tau}|\right) da + \int_{\Gamma_3} \left(\eta_2^1 - \eta_2^2\right)(u_{1\nu} - u_{2\nu}) da \\
&\quad + \int_{\Gamma_3} \left(\eta_3^1 - \eta_3^2\right)\left(\varphi_1 - \varphi_2\right) da \\
&\leq \|\eta_1^1 - \eta_1^2\|_{L^2(\Gamma_3)} \|u_1 - u_2\|_{L^2(\Gamma_3)^d} + \|\eta_2^1 - \eta_2^2\|_{L^2(\Gamma_3)} \|u_1 - u_2\|_{L^2(\Gamma_3)^d} \\
&\quad + \|\eta_3^1 - \eta_3^2\|_{L^2(\Gamma_3)} \|\varphi_1 - \varphi_2\|_{L^2(\Gamma_3)}.
\end{aligned}$$

Using (37), (15), (17) and (26) we find that

$$\|x_1 - x_2\|_X \leq \frac{\sqrt{6}\max(c_0, c_1)}{m_A} \|\eta^1 - \eta^2\|_Y. \tag{39}$$

From the previous inequality the second part of Lemma 4.2 is proved.
3. For all $\eta = (\eta_1, \eta_2, \eta_3) \in \mathcal{K}_1 \times \mathcal{K}_2 \times \mathcal{K}_3$, we denote by $x_\eta = (u_\eta, \varphi_\eta)$ the corresponding solution of Problem $(PV^\eta)$. Then we have

$$(Ax_\eta, y - x_\eta)_X + J_1^\eta(y) - J_1^\eta(x_\eta) \geq (F^\eta, y - x_\eta)_X \quad \forall\, y \in U.$$

Taking $y = 0$ in the above inequality, we obtain

$$(Ax_\eta, x_\eta)_X + J_1^\eta(x_\eta) \le (F^\eta, x_\eta)_X. \tag{40}$$

As $\eta_1 \ge 0$, then $J_1^\eta(x_\eta) \ge 0$. it follows from (40), (37), (26), (16), (17) and (34) that

$$\|x_\eta\|_X \le c_2(\|F\|_X + \|\eta_2\|_{L^2(\Gamma_3)} + \|\eta_3\|_{L^2(\Gamma_3)}), \tag{41}$$

where $c_2 = \frac{\max(1,c_0,c_1)}{m_A}$ and it comes from this inequality that

$$\|x_\eta\|_X \le c_2'(\|F\|_X + \|\eta\|_Y),$$

with $c_2' = \frac{\max(1,2\max(c_0,c_1))}{m_A}$, which concludes the proof of Lemma 4.2. □

In this step, we consider the operator $\Lambda : Y \to Y$ defined by

$$\Lambda\,\eta = (\mu(\|u_{\eta\tau}\|)|R\sigma_\nu(u_\eta, \varphi_\eta)|\,,\ h_\nu(\varphi_\eta - \varphi_F)p_\nu(u_{\eta\nu})\,,\ p_e(u_{\eta\nu})h_e(\varphi_\eta - \varphi_F)), \tag{42}$$

where $(u_\eta, \varphi_\eta)$ is the unique solution of $(PV^\eta)$ corresponding to $\eta$. Using assumptions $(h_3)-(h_5)$ and $(h_7)$, we can easily see that operator $\Lambda$ is well defined. Next, we will prove that the operator $\Lambda$ has fixed point and to this end, we need the following result:

**Lemma 4.3.** *Let $x_\eta$ be a solution of $(PV^\eta)$, the mapping $\eta \mapsto x_\eta$ is weakly continuous on $Y$.*

*Proof.* Let $(\eta_n) = (\eta_{1n}, \eta_{2n}, \eta_{3n})$ be a subsequence of $Y$ converging weakly to $\eta = (\eta_1, \eta_2, \eta_3)$. We denote by $x_{\eta_n} = (u_{\eta_n}, \varphi_{\eta_n}) \in U$ the solution of $(PV^\eta)$ corresponding to $\eta_n$. Then we have

$$(Ax_{\eta_n}, y - x_{\eta_n})_X + J_1^{\eta_n}(y) - J_1^{\eta_n}(x_{\eta_n}) \ge (F^{\eta_n}, y - x_{\eta_n})_X \text{ for all } y \in U. \tag{43}$$

Taking $y = 0$ in inequality (43) and using (31), we deduce

$$(Ax_\eta, x_\eta)_X + J_1^\eta(x_\eta) \le (F^\eta, x_\eta)_X. \tag{44}$$

Using $J_1^\eta(x_\eta) \ge 0$ and the strong monotonicity of $A$, it follows from (31), (32) and (26) that

$$\|x_{\eta_n}\|_X \le c_2(\|F\|_X + \|\eta_n\|_Y).$$

Then, we deduce that the sequence $(x_{\eta_n})$ is bounded in $X$. Hence, there exists $\widetilde{x} = (\widetilde{u}, \widetilde{\varphi}) \in X$ and a subsequence, denote again $(x_{\eta_n})$, such that $(x_{\eta_n})$ converges weakly to $\widetilde{x}$, i.e.,

$$u_{\eta_n} \rightharpoonup \widetilde{u} \text{ in } V \quad \text{and} \quad \varphi_{\eta_n} \rightharpoonup \widetilde{\varphi} \text{ in } W.$$

Since $U$ is closed convex set in a real Hilbert space $X$, it is weakly closed set and then $\widetilde{x} \in U$. We next prove that $\widetilde{x}$ is a solution of $(PV^\eta)$. First, we need to prove that

$$(F^{\eta_n}, y - x_{\eta_n})_X \to (F^\eta, y - \widetilde{x})_X. \tag{45}$$

Indeed, we have

$$\begin{aligned} &|J_2^{\eta_n}(v - \widetilde{u}) - J_2^{\eta_n}(v - u_{\eta_n})| + |J_3^{\eta_n}(\xi - \widetilde{\varphi}) - J_3^{\eta_n}(\xi - \varphi_{\eta_n})| \\ &\le \underbrace{\|\eta_n\|_Y}_{bounded} \big(\|u_{\eta_n} - \widetilde{u}\|_{L^2(\Gamma_3)^d} + \|\varphi_{\eta_n} - \widetilde{\varphi}\|_{L^2(\Gamma_3)}\big). \end{aligned}$$

Since the trace map $\gamma_1 : V \to L^2(\Gamma_3)^d$ is compact operator, the weak convergence $u_{\eta_n} \rightharpoonup \widetilde{u}$ in $V$ leads to the strong convergence $u_{\eta_n} \to \widetilde{u}$ in $L^2(\Gamma_3)^d$. Similarly, since the trace map $\gamma_2 : W \to L^2(\Gamma_3)$ is compact, the weak convergence $\varphi_{\eta_n} \rightharpoonup \widetilde{\varphi}$ in $W$ implies the strong convergence $\varphi_{\eta_n} \to \widetilde{\varphi}$ in $L^2(\Gamma_3)$ and hence, we get (45). Now, it follows from (43) that

$$(Ax_{\eta_n}, y - x_{\eta_n})_X \ge (F^{\eta_n}, y - x_{\eta_n})_X - \big(J_1^{\eta_n}(y) - J_1^{\eta_n}(\widetilde{x})\big) - \big(J_1^{\eta_n}(\widetilde{x}) - J_1^{\eta_n}(x_{\eta_n})\big)$$

for all $y = (v, \xi) \in U$ and by using (31) and (26), we can see that

$$|J_1^{\eta_n}(\widetilde{x}) - J_1^{\eta_n}(x_{\eta_n})| \leq \|\eta_n\|_Y \, \|\widetilde{u} - u_{\eta_n}\|_{L^2(\Gamma_3)^d}$$
$$\leq \underbrace{\|\eta_n\|_Y}_{bounded} \, \|\widetilde{x} - x_{\eta_n}\|_{L^2(\Gamma_3)^d \times L^2(\Gamma_3)}.$$

We recall that $(\eta_{1n}, \eta_{2n}, \eta_{3n})$ is said to be weakly convergent to $(\eta_1, \eta_2, \eta_3)$ in $Y$ if

$(\eta_{1n}, a_1)_{L^2(\Gamma_3)} \to (\eta_1, a_1)_{L^2(\Gamma_3)}, (\eta_{2n}, a_2)_{L^2(\Gamma_3)} \to (\eta_2, a_2)_{L^2(\Gamma_3)}, (\eta_{3n}, a_3)_{L^2(\Gamma_3)} \to (\eta_3, a_3)_{L^2(\Gamma_3)},$
for all $(a_1, a_2, a_3) \in Y$.
Let $y = (v, \xi) \in X$ and Taking $a_1 = \|v_\tau\|$, then,

$$(\eta_{1n}, \|v_\tau\|)_{L^2(\Gamma_3)} \to (\eta_1, \|v_\tau\|)_{L^2(\Gamma_3)}.$$

Using (31), we find

$$J_1^{\eta_n}(y) = \int_{\Gamma_3} \eta_{1n} \, \|v_\tau\| da = (\eta_{1n}, \|v_\tau\|)_{L^2(\Gamma_3)} \to J_1^{\eta}(y) = \int_{\Gamma_3} \eta_1 \, \|v_\tau\| da = (\eta_1, \|v_\tau\|)_{L^2(\Gamma_3)},$$

which prove that the convergence "$J_1^{\eta_n}(y) \to J_1^{\eta}(y)$" holds.
Then, we conclude

$$\limsup_{n\to+\infty} (Ax_{\eta_n}, x_{\eta_n} - y)_X \leq (F^\eta, \tilde{x} - y)_X + \big(J_1^\eta(y)\big) - J_1^\eta(\tilde{x})\big), \quad \forall\, y \in U. \tag{46}$$

Moreover, we obtain from (46) that

$$\begin{aligned}
&\limsup_{n\to+\infty} (Ax_{\eta_n}, x_{\eta_n} - \tilde{x})_X \\
&\leq \limsup_{n\to+\infty} \{(Ax_{\eta_n}, x_{\eta_n} - y)_X + \|Ax_{\eta_n}\|_X \|y - \tilde{x}\|_X\} \\
&\leq (F^\eta, \tilde{x} - y)_X + (J_1^\eta(y)) - J_1^\eta(\tilde{x})) + \limsup_{n\to+\infty} \|Ax_{\eta_n}\|_X \|y - \tilde{x}\|_X,
\end{aligned}$$

for all $y \in U$. Since $\|Ax_{\eta_n}\|_X$ is bounded, we take $y = \tilde{x}$ in the previous inequality to get $\limsup_{n\to+\infty} (Ax_{\eta_n}, x_{\eta_n} - \tilde{x})_X \leq 0$. Using the pseudomonocity of the operator $A$, we deduce

$$(A\tilde{x}, \tilde{x} - y)_X \leq \liminf_{n\to+\infty} (Ax_{\eta_n}, x_{\eta_n} - y)_X \text{ for all } y \in U. \tag{47}$$

Combining now (43), (45) and (47), we deduce

$$\begin{cases} \widetilde{x} \in U \\ (A\widetilde{x}, y - \widetilde{x})_X + J_1^\eta(y) - J_1^\eta(\widetilde{x}) \geq (F^\eta, y - \widetilde{x})_X \,, \ \forall\, y = (v, \xi) \in U. \end{cases} \tag{48}$$

From (48), we find that $\widetilde{x}$ is a solution of Problem $(PV^\eta)$ and from the uniqueness of the solution of the variational inequality (48), we obtain $\widetilde{x} = x_\eta$. Since $x_\eta$ is the unique weak limit of any subsequence of $(x_{\eta_n})$, we get that the whole sequence $(x_{\eta_n})$ is weakly convergent to $x_\eta$ in $X$ and that ensures the weak continuous of the mapping $\eta \mapsto x_\eta$ from $Y$ to $X$. $\square$

**Lemma 4.4.** *$\Lambda$ is an operator of $\mathcal{K} = \mathcal{K}_1 \times \mathcal{K}_2 \times \mathcal{K}_3$ into itself and has at least one fixed point.*

*Proof.* Let be $\eta = (\eta_1, \eta_2, \eta_3) \in \mathcal{K} = \mathcal{K}_1 \times \mathcal{K}_2 \times \mathcal{K}_3$. Then, we have

$$\|\eta_1\|_{L^2(\Gamma_3)} \leq k_1 \,, \ \|\eta_2\|_{L^2(\Gamma_3)} \leq k_2 \ \text{ and } \ \|\eta_3\|_{L^2(\Gamma_3)} \leq k_3,$$

which implies that $\|\eta\|_Y \leq k_1 + k_2 + k_3$. From (42), it follows that

$$\|\Lambda(\eta)\|_Y \leq \ \|\mu(\|u_{\eta\tau}\|)|R\sigma_\nu(u_\eta, \varphi_\eta)|\|_{L^2(\Gamma_3)} + \|h_\nu(\varphi_\eta - \varphi_F)p_\nu(u_{\eta\nu})\|_{L^2(\Gamma_3)}$$

$$+\|p_e(u_{\eta\nu})h_e(\varphi_\eta-\varphi_F)\|_{L^2(\Gamma_3)}.$$

Moreover, from assumptions $(h_3)$ and $(h_4)$, we obtain

$$\|h_\nu(\varphi_\eta-\varphi_F)p_\nu(u_{\eta\nu})\|_{L^2(\Gamma_3)}\leq M_{h\nu}M_{p\nu}meas(\Gamma_3)^{\frac{1}{2}}=k_2, \tag{49}$$

$$\|p_e(u_{\eta\nu})h_e(\varphi_\eta-\varphi_F)\|_{L^2(\Gamma_3)}\leq M_{he}M_{pe}meas(\Gamma_3)^{\frac{1}{2}}=k_3. \tag{50}$$

For all $u_\eta\in K$, $\varphi_\eta\in W$ satisfying (1), (4), (5) and $(h_8)$, the element $\sigma_\nu(u_\eta,\varphi_\eta)$ of $H_{\Gamma_3}^{-\frac{1}{2}}$ is defined, for all $v\in V$ with $v_\tau=0$ on $\Gamma_3$, as follows (see [2,7]),

$$\langle\sigma_\nu(u_\eta,\varphi_\eta),v_\nu\rangle_{\Gamma_3}=(\mathfrak{F}\varepsilon(u_\eta)+\mathcal{E}^*\nabla\varphi_\eta,\varepsilon(v))_{\mathcal{H}}-(f,v)_V. \tag{51}$$

Using (12), (51) and the trace theorem (see [11]), there exists $c_F>0$ such that

$$\|\sigma_\nu(u_\eta,\varphi_\eta)\|_{H_{\Gamma_3}^{-\frac{1}{2}}}=\sup_{v\in H_{\Gamma_3}^{\frac{1}{2}}}\frac{\langle\sigma_\nu(u_\eta,\varphi_\eta),v_\nu\rangle_{\Gamma_3}}{\|v_\nu\|_{H_{\Gamma_3}^{\frac{1}{2}}}}\leq c_F\big(\|u_\eta\|_V+\|\varphi_\eta\|_W+\|f\|_V\big). \tag{52}$$

Moreover, it follows from $(h_5)$, $(h_7)$, (36), (52) and (41) that

$$\begin{aligned}\|\mu(\|u_\tau\|)|R\sigma_\nu(u_\eta,\varphi_\eta)|\|_{L^2(\Gamma_3)}&\leq\mu^*meas(\Gamma_3)^{\frac{1}{2}}c_Rc_F\big(\sqrt{2}\|x_\eta\|_X+\|f\|_V\big)\\&\leq\mu^*meas(\Gamma_3)^{\frac{1}{2}}c_Rc_F\big[c_2\sqrt{2}\big(\|\eta_2\|_{L^2(\Gamma_3)}+\|\eta_3\|_{L^2(\Gamma_3)}+\|F\|_X\big)+\|f\|_V\big]\\&\leq\mu^*meas(\Gamma_3)^{\frac{1}{2}}c_Rc_F\big[c_2\sqrt{2}\big(k_2+k_3+\|F\|_X\big)+\|f\|_V\big]=k_1.\end{aligned} \tag{53}$$

Combining (53), (49) and (50), we get

$$\|\Lambda(\eta)\|_Y\leq k_1+k_2+k_3.$$

Hence, $\Lambda$ is an operator from $\mathcal{K}=\mathcal{K}_1\times\mathcal{K}_2\times\mathcal{K}_3$ into itself. Note that $\mathcal{K}$ is a nonempty convex and closed subset of the reflexive space $Y$. Then, $\mathcal{K}$ is weakly compact. Using the proprieties of $p_e$, $h_e$, $\mu$, $h_\nu$, $p_\nu$ and $R$, we can deduce that $\Lambda$ is weakly continuous and then, by the Schauder fixed point theorem, the operator $\Lambda$ has at least one fixed point. □

In the last step, we have all the ingredients to provide the proof of Theorem (3.1). For the existence part, let $\eta^*$ be the fixed point of $\Lambda$. We denote by $x^*=(u^*,\varphi^*)$, the solution of Problem $(PV^\eta)$ for $\eta=\eta^*$. The definition of $\Lambda$ and $(PV^\eta)$ imply that $x^*$ is a solution of $(PV)$ and that leads to the existence part of Theorem (3.1). For the uniqueness part, let $x_1=(u^1,\varphi^1)$ and $x_2=(u^2,\varphi^2)$ denote two solutions of Problem $(PV)$. It comes from (30) that

$$(Ax_1,y-x_1)_X+J(x_1,y)-J(x_1,x_1)\geq(F,y-x_1)_X \text{ for all } y\in U,$$

$$(Ax_2,y-x_2)_X+J(x_2,y)-J(x_2,x_2)\geq(F,y-x_2)_X \text{ for all } y\in U.$$

Taking $y=x_2$ in the first inequality, $y=x_1$ in the second, we add obtained inequalities to get

$$(Ax_1-Ax_2,x_2-x_1)_X\leq J(x_1,x_2)-J(x_1,x_1)+J(x_2,x_1)-J(x_2,x_2). \tag{54}$$

By using (29), we obtain

$$(Ax_1-Ax_2,x_2-x_1)_X\leq G_1+G_2+G_3, \tag{55}$$

where

$$G_1=j_1(u_1,\varphi_1,u_2)-j_1(u_1,\varphi_1,u_1)+j_1(u_2,\varphi_2,u_1)-j_1(u_2,\varphi_2,u_2),$$

$$G_2=j_2(u_1,\varphi_1,u_2)-j_2(u_1,\varphi_1,u_1)+j_2(u_2,\varphi_2,u_1)-j_2(u_2,\varphi_2,u_2),$$

$$G_3=j_3(u_1,\varphi_1,\varphi_2)-j_3(u_1,\varphi_1,\varphi_1)+j_3(u_2,\varphi_2,\varphi_1)-j_3(u_2,\varphi_2,\varphi_2).$$

From (20), we deduce

$$G_1 = \int_{\Gamma_3} \mu(\|u_{1\tau}\|)\{|R\sigma_\nu(u_1,\varphi_1)| - |R\sigma_\nu(u_2,\varphi_2)|\}\,(\|u_{1\tau}\| - \|u_{2\tau}\|)\,da$$
$$+ \int_{\Gamma_3} |R\sigma_\nu(u_2,\varphi_2)|\,\{\mu(\|u_{1\tau}\|) - \mu(\|u_{2\tau}\|)\}\,(\|u_{1\tau}\| - \|u_{2\tau}\|)\,da. \tag{56}$$

Using $(h_5)$, $(h_6)(c)$, (15) and (26), we obtain, after some algebraic manipulations that

$$G_1 \leq \mu^* c_R c_F\,(\|\varphi_1 - \varphi_2\|_W + \|u_1 - u_2\|_V)\,\|u_1 - u_2\|_{L^2(\Gamma_3)^d}$$
$$+ \|R\sigma_\nu(u_2,\varphi_2)\|_{L^\infty(\Gamma_3)} L_\mu \|u_1 - u_2\|_{L^2(\Gamma_3)^d}\|u_1 - u_2\|_{L^2(\Gamma_3)^d}$$
$$\leq \{2\mu^* c_R c_F c_0 + L_\mu c_0^2 \|R\sigma_\nu(u_2,\varphi_2)\|_{L^\infty(\Gamma_3)}\}\,\|x_1 - x_2\|_X^2. \tag{57}$$

Next, it follows from (21) that

$$G_2 = \int_{\Gamma_3} h_\nu(\varphi_1 - \varphi_F)\{p_\nu(u_{1\nu}) - p_\nu(u_{2\nu})\}\,(u_{1\nu} - u_{2\nu})\,da$$
$$+ \int_{\Gamma_3} p_\nu(u_{2\nu})\{h_\nu(\varphi_1 - \varphi_F) - h_\nu(\varphi_2 - \varphi_F)\}\,(u_{1\nu} - u_{2\nu})\,da. \tag{58}$$

Keeping in mind $(h_3)$-$(h_4)$ and $(h_6)(b)$, it follows from (15), (17) and (26) that

$$G_2 \leq M_{h\nu} L_{p\nu} \|u_1 - u_2\|_{L^2(\Gamma_3)^d}\|u_1 - u_2\|_{L^2(\Gamma_3)^d}$$
$$+ M_{p\nu} L_{h\nu} \|\varphi_1 - \varphi_2\|_{L^2(\Gamma_3)}\|u_1 - u_2\|_{L^2(\Gamma_3)^d}$$
$$\leq \{M_{h\nu} L_{p\nu} c_0^2 + M_{p\nu} L_{h\nu} c_0 c_1\}\,\|x_1 - x_2\|_X^2. \tag{59}$$

Moreover, it follows from (23) that

$$G_3 = \int_{\Gamma_3} p_e(u_{1\nu})\{h_e(\varphi_1 - \varphi_F) - h_e(\varphi_2 - \varphi_F)\}\,(\varphi_1 - \varphi_2)\,da$$
$$+ \int_{\Gamma_3} h_e(\varphi_2 - \varphi_F)\{p_e(u_{1\nu}) - p_e(u_{2\nu})\}\,(\varphi_1 - \varphi_2)\,da. \tag{60}$$

Recalling $(h_3)$-$(h_4)$ and $(h_6)(a)$, it comes from (15), (17) and (26) that

$$G_3 \leq M_{he} L_{pe} \|u_1 - u_2\|_{L^2(\Gamma_3)^d}\|\varphi_1 - \varphi_2\|_{L^2(\Gamma_3)}$$
$$+ M_{pe} L_{he} \|\varphi_1 - \varphi_2\|_{L^2(\Gamma_3)}\|\varphi_1 - \varphi_2\|_{L^2(\Gamma_3)}$$
$$\leq \{M_{he} L_{pe} c_1 c_0 + M_{pe} L_{he} c_1^2\}\,\|x_1 - x_2\|_X^2. \tag{61}$$

Combining (54)-(61), (53) and using (37), then there exists a constant $M_J > 0$, such that

$$\|x_1 - x_2\|_X^2 \leq \frac{M_J}{m_A}\{\mu^* + L_\mu + M_{h\nu}L_{p\nu} + M_{p\nu}L_{h\nu} + M_{he}L_{pe} + M_{pe}L_{he}\}\|x_1 - x_2\|_X^2.$$

Let $L^* = \frac{m_A}{M_J}$, then if we have

$$\mu^* + L_\mu + M_{h\nu}L_{p\nu} + M_{p\nu}L_{h\nu} + M_{he}L_{pe} + M_{pe}L_{he} < L^*,$$

we get $x_1 = x_2$.

## §5 Numerical approximation

Now, we introduce a finite element approximation of $(PV)$ and we derive, under some regularity assumptions, an optimal error estimate. Let $h > 0$ be a discretization parameter, we consider the following finite-dimensional spaces $V^h$ and $W^h$ approximating $V$ and $W$, respectively, given

$$V^h = \{v^h \in C(\overline{\Omega})^d,\ v^h_{|Tr} \in \mathbb{P}_1(Tr),\ Tr \in \tau^h,\ v^h = 0 \text{ on } \Gamma_1\} \subset V,$$
$$W^h = \{\varphi^h \in C(\overline{\Omega}),\ \varphi^h_{|Tr} \in \mathbb{P}_1(Tr),\ Tr \in \tau^h,\ \varphi^h = 0 \text{ on } \Gamma_a\} \subset W,$$

where $\Omega$ is assumed to be a polygonal domain, $\tau^h$ denotes a regular family of triangular finite element partitions of $\overline{\Omega}$, and $\mathbb{P}_1(Tr)$ represents the space of polynomials of global degree less or equal to one in an element $Tr$ of the triangulation. We consider the nonempty finite-dimensional closed convex sets of admissible displacements with $V^h$, defined by

$$K^h = K \cap V^h = \{v^h \in V^h,\ v^h_\nu \leq 0 \ on\ \overline{\Gamma}_3\}.$$

Then, we introduce the following finite element approximation of Problem $(PV)$.

**Problem** $(PV^h)$**.** *Find a displacement $u^h \in K^h$ and the electric potential $\varphi^h \in W^h$ such that*

$$(\mathfrak{F}\varepsilon(u^h), \varepsilon(v^h) - \varepsilon(u^h))_{\mathcal{H}} + (\mathcal{E}^*\nabla\varphi^h, \varepsilon(v^h) - \varepsilon(u^h))_{L^2(\Omega)^d} + j(u^h, \varphi^h, v^h)$$
$$- j(u^h, \varphi^h, u^h) \geq (f, v^h - u^h)_V, \quad \forall\, v^h \in K^h, \tag{62}$$
$$(\beta\nabla\varphi^h, \nabla\xi^h)_{L^2(\Omega)^d} - (\mathcal{E}\varepsilon(u^h), \nabla\xi^h)_{L^2(\Omega)^d} + j_3(u^h, \varphi^h, \xi^h) = (q_e, \xi^h)_W,\ \forall\, \xi^h \in W^h. \tag{63}$$

The unique solvability of the Problem $(PV^h)$ follows from arguments similar to those used in the proof of Theorem 3.1. Our main purpose here is to estimate the numerical errors $u - u^h$ and $\varphi - \varphi^h$. Therefore, for the sake of simplicity, everywhere in the sequel $C$ will denote a positive constant which may depend on the data but independent of $h$ and whose value may change from place to place. Moreover, we derive the following error estimate result.

**Theorem 5.1.** *Assume the conditions of Theorem 3.1 hold. Let $(u, \varphi)$ and $(u^h, \varphi^h)$ denote the solutions of $(PV)$ and $(PV^h)$, respectively. We have the following error estimate :*

$$\|u - u^h\|_V^2 + \|\varphi - \varphi^h\|_W^2 \leq C\,\{\|u - v^h\|_V^2 + \|u - v^h\|_{L^2(\Gamma)^d} + \|\varphi - \xi^h\|_W^2 + \|\varphi - \xi^h\|_{L^2(\Gamma)}\}, \tag{64}$$

*for all $(v^h, \xi^h) \in K^h \times W^h$, where $C$ is linearly depending on $\|u\|_V$ and $\|\varphi\|_W$.*

*Proof.* Let $U^h = K^h \times W^h \subset U$. Using Lemma 4.1, it is easy to see that $x^h = (u^h, \varphi^h)$ is a solution of Problem $(PV^h)$ if and only if

$$(Ax^h, y^h - x^h)_X + J(x^h, y^h) - J(x^h, x^h) \geq (F, y^h - x^h)_X \text{ for all } y^h \in U^h. \tag{65}$$

We take $y = x^h$ in (30) and we combine the obtained inequality with (65) to get

$$(Ax - Ax^h, x - x^h)_X \leq (Ax^h, y^h - x)_X + (F, x - y^h)_X + G \text{ for all } y^h \in U^h, \tag{66}$$

where

$$\begin{aligned} G &= J(x^h, y^h) - J(x^h, x^h) + J(x, x^h) - J(x, x) \\ &= \{J(x, y^h) - J(x, x)\} + \{J(x^h, y^h) - J(x^h, x) + J(x, x) - J(x, y^h)\} \\ &\quad + \{J(x^h, x) - J(x^h, x^h) + J(x, x^h) - J(x, x)\}. \end{aligned}$$

Using (37), (29) and the above inequality, we obtain

$$m_A\|x - x^h\|_X^2 \leq \mathcal{S}_1 + \mathcal{S}_2 + \mathcal{S}_3 + \mathcal{S}_4, \tag{67}$$

where

$$\mathcal{S}_1 = (Ax^h, y^h - x)_X + (F, x - y^h)_X,$$

$$\begin{aligned}\mathcal{S}_2 &= j_1(u,\varphi,v^h) - j_1(u,\varphi,u) + j_2(u,\varphi,v^h) - j_2(u,\varphi,u)\\ &\quad + j_3(u,\varphi,\xi^h) - j_3(u,\varphi,\varphi),\end{aligned}$$

$$\begin{aligned}\mathcal{S}_3 &= j_1(u^h,\varphi^h,v^h) - j_1(u^h,\varphi^h,u) + j_1(u,\varphi,u) - j_1(u,\varphi,v^h)\\ &\quad + j_2(u^h,\varphi^h,v^h) - j_2(u^h,\varphi^h,u) + j_2(u,\varphi,u) - j_2(u,\varphi,v^h)\\ &\quad + j_3(u^h,\varphi^h,\xi^h) - j_3(u^h,\varphi^h,\varphi) + j_3(u,\varphi,\varphi) - j_3(u,\varphi,\varphi^h),\end{aligned}$$

$$\begin{aligned}\mathcal{S}_4 &= j_1(u^h,\varphi^h,u) - j_1(u^h,\varphi^h,u^h) + j_1(u,\varphi,u^h) - j_1(u,\varphi,u)\\ &\quad + j_2(u^h,\varphi^h,u) - j_2(u^h,\varphi^h,{}^h) + j_2(u,\varphi,u^h) - j_2(u,\varphi,u)\\ &\quad + j_3(u^h,\varphi^h,\varphi) - j_3(u^h,\varphi^h,\varphi^h) + j_3(u,\varphi,\varphi^h) - j_3(u,\varphi,\varphi).\end{aligned}$$

Now, by using property (38) of the operator $A$, we have

$$\begin{aligned}\mathcal{S}_1 &= (Ax^h - Ax, y^h - x)_X + (Ax, y^h - x)_X + (F, x - y^h)_X\\ &\le M_A\|x^h - x\|_X\|y^h - x\|_X + M_A\|x\|_X\|y^h - x\|_X + \|F\|_X\|x - y^h\|_X. \end{aligned}\tag{68}$$

From (20)-(23) and $(h_3)$-$(h_7)$ we obtain

$$\begin{aligned}\mathcal{S}_2 &= \int_{\Gamma_3}\mu(\|u_\tau\|)|R\sigma_\nu(u,\varphi)|\left(|v_\tau^h| - |u_\tau|\right)da + \int_{\Gamma_3}h_\nu(\varphi-\varphi_F)p_\nu(u_\nu)\left(v_\nu^h - u_\nu\right)\\ &\quad + \int_{\Gamma_3}p_e(u_\nu)h_e(\varphi-\varphi_F)\left(\xi^h - \varphi\right)da \qquad (69)\\ &\le \mu^*\|R\sigma_\nu(u,\varphi)\|_{L^\infty(\Gamma_3)}\mathrm{meas}(\Gamma_3)^{\frac12}\|v^h - u\|_{L^2(\Gamma_3)^d}\\ &\quad + M_{h\nu}M_{p\nu}\mathrm{meas}(\Gamma_3)^{\frac12}\|v^h - u\|_{L^2(\Gamma_3)^d} + M_{pe}M_{he}\mathrm{meas}(\Gamma_3)^{\frac12}\|\xi^h - \varphi\|_{L^2(\Gamma_3)}.\end{aligned}$$

We proceed now estimate $\mathcal{S}_3$. First, we use (20) to see that

$$\begin{aligned}&j_1(u^h,\varphi^h,v^h) - j_1(u^h,\varphi^h,u) + j_1(u,\varphi,u) - j_1(u,\varphi,v^h)\\ &= \int_{\Gamma_3}|R\sigma_\nu(u,\varphi)|\{\mu(\|u_\tau^h\|) - \mu(\|u_\tau\|)\}\{\|v_\tau^h\| - \|u_\tau\|\}da\\ &\quad + \int_{\Gamma_3}\mu(\|u_\tau^h\|)\{|R\sigma_\nu(u^h,\varphi^h)| - |R\sigma_\nu(u,\varphi^h)|\}\{\|v_\tau^h\| - \|u_\tau\|\}da.\end{aligned}$$

Recalling $(h_5)$, $(h_6)(c)$ and $(h_7)$, it comes from (15), (16) and (26) that

$$\begin{aligned}&j_1(u^h,\varphi^h,v^h) - j_1(u^h,\varphi^h,u) + j_1(u,\varphi,u) - j_1(u,\varphi,v^h)\\ &\qquad \le C\left\{\mu^* + L_\mu\|R\sigma_\nu(u,\varphi)\|_{L^\infty(\Gamma_3)}\right\}\|x - x^h\|_X\|x - y^h\|_X.\end{aligned}\tag{70}$$

After easy algebraic manipulations, using $(h_3)$,$(h_4)$,$(h_6)(a)(b)$, (15),(16), and (26) we get

$$\begin{aligned}&j_2(u^h,\varphi^h,v^h) - j_2(u^h,\varphi^h,u) + j_2(u,\varphi,u) - j_2(u,\varphi,v^h)\\ &\qquad \le C\{M_{h_\nu}L_{p_\nu} + M_{p_\nu}L_{h_\nu}\}\|x - x^h\|_X\|x - y^h\|_X.\end{aligned}\tag{71}$$

Similarly, from (23) and using the assumptions on $p_e$ and $h_e$, we have

$$\begin{aligned}&j_3(u^h,\varphi^h,\xi^h) - j_3(u^h,\varphi^h,\varphi) + j_3(u,\varphi,\varphi) - j_3(u,\varphi,\varphi^h)\\ &\qquad \le C\{M_{p_e}L_{h_e} + M_{h_e}L_{p_e}\}\|x - x^h\|_X\|x - y^h\|_X.\end{aligned}\tag{72}$$

Then, we combine (70), (71) and (72) to obtain

$$\mathcal{S}_3 \le C\{\mu^* + L_\mu\|R\sigma_\nu(u,\varphi)\|_{L^\infty(\Gamma_3)} + M_{h_\nu}L_{p_\nu} + M_{p_\nu}L_{h_\nu} + M_{p_e}L_{h_e} + M_{h_e}L_{p_e}\}\|x - x^h\|_X\|x - y^h\|_X. \tag{73}$$

Proceeding in a similar way, we can obtain the following inequality

$$\mathcal{S}_4 \le C\{\mu^* + L_\mu\|R\sigma_\nu(u,\varphi)\|_{L^\infty(\Gamma_3)} + M_{h_\nu}L_{p_\nu} + M_{p_\nu}L_{h_\nu} + M_{p_e}L_{h_e} + M_{h_e}L_{p_e}\}\|x - x^h\|_X^2. \tag{74}$$

Finally, by introducing (68)-(79) into (67) and applying Young's inequality $ab \le \eta a^2 + \frac{1}{4\eta}b^2$ for $a = \|x - x^h\|_X$, $b = \|x - y^h\|_X$ and $\eta > 0$, we get after some simplifications that

$$\|x - x^h\|_X^2 \le C\{\|x - y^h\|_X^2 + \|v^h - u\|_{L^2(\Gamma)^d} + \|\xi^h - \varphi\|_{L^2(\Gamma)}\}, \quad \forall\, y^h \in U^h.$$

Then, the previous inequality combined with (26) leads to (64). □

We notice that the above error estimate is the basis of the convergence order analysis. Moreover, we assume the following additional regularity conditions

$$u \in H^2(\Omega)^d,\ u_{|\Gamma_3} \in H^2(\Gamma_3)^d,\ \varphi \in H^2(\Omega),\ \varphi_{|\Gamma_3} \in H^2(\Gamma_3). \tag{75}$$

Let $\Pi^h u \in V^h$ and $\Pi^h\varphi \in W^h$ be the piecewise linear interpolant of $u$ and $\varphi$, respectively. The standard finite element interpolation theory yields (see [6,p. 133] and [19, p. 54] for details)

$$\|u - \Pi^h u\|_V \le c\,h\,|u|_{H^2(\Omega)^d}, \qquad \|\varphi - \Pi^h\varphi\|_W \le c\,h\,|\varphi|_{H^2(\Omega)},$$

$$\|u - \Pi^h u\|_{L^2(\Gamma_3)^d} \le c\ h^2\ |u|_{H^2(\Gamma_3)^d}, \qquad \|\varphi - \Pi^h\varphi\|_{L^2(\Gamma_3)} \le c\ h^2\ |\varphi|_{H^2(\Gamma_3)}.$$

We state the following result which is a direct consequence of (64).

**Theorem 5.2.** *Assume the conditions of Theorem 3.1 hold. Let $(u,\varphi)$ and $(u^h,\varphi^h)$ denote the solutions of Problem $PV$ and $PV^h$, respectively. Under the regularity assumptions* (75)*, we have the optimal order error estimate*

$$\|u - u^h\|_V + \|\varphi - \varphi^h\|_W \le C\,h.$$

The previous theorem states that, under the regularity conditions (75), the convergence order for the numerical solution is optimal. Furthermore, if the regularity conditions are different, the error estimate need to be changed accordingly, but it follows easily from (64).

## §6 Iteration Method

In this section, we propose an iterative solution scheme for the finite element system (62)-(63) which is based on the method of successive approximations by a fixed-point iteration method. This follows from a discrete analog of the proof of Theorem 3.1. Given an initial guess $(u_0^h,\varphi_0^h)$, we define the sequence $(u_n^h,\varphi_n^h) \in K^h \times W^h$ as follows:

$$(\mathfrak{F}\varepsilon(u_{n+1}^h), \varepsilon(v^h) - \varepsilon(u_{n+1}^h))_{\mathcal{H}} + (\mathcal{E}^*\nabla\varphi_{n+1}^h, \varepsilon(v^h) - \varepsilon(u_{n+1}^h))_{L^2(\Omega)^d} + j(u_n^h,\varphi_n^h,v^h) - j(u_n^h,\varphi_n^h,u_{n+1}^h) \ge (f, v^h - u_{n+1}^h)_V \text{ for all } v^h \in K^h, \tag{76}$$

$$(\beta\nabla\varphi_{n+1}^h, \nabla\xi^h)_{L^2(\Omega)^d} - (\mathcal{E}\varepsilon(u_{n+1}^h), \nabla\xi^h)_{L^2(\Omega)^d} + j_3(u_n^h,\varphi_n^h,\xi^h) = (q_e,\xi^h)_W \text{ for all } \xi^h \in W^h. \tag{77}$$

The following theorem gives the convergence analysis of the iterative method given by (76)-(77).

**Theorem 6.1.** *Under the assumptions of Theorem 3.1 and the same value of $L^*$, the iteration method defined by* (76)-(77) *converges, i.e.,*

$$\|u_n^h - u^h\|_V \to 0\,,\ \|\varphi_n^h - \varphi^h\|_W \to 0 \quad as \quad n \to +\infty.$$

*Proof.* Using Lemma 4.1, we see that $x_n^h = (u_n^h, \varphi_n^h)$ is the solution of the problem (76)-(77) if

$$(Ax_{n+1}^h, y^h - x_{n+1}^h)_X + J(x_n^h, y^h) - J(x_n^h, x_{n+1}^h) \geq (F, y^h - x_{n+1}^h)_X \text{ for all } y^h \in U^h. \tag{78}$$

Taking $y^h = x_{n+1}^h$ in (65), $y^h = x^h$ in (78) and adding the two obtained inequalities to get

$$(Ax^h - Ax_{n+1}^h, x^h - x_{n+1}^h)_X \leq J(x^h, x_{n+1}^h) - J(x^h, x^h) + J(x_n^h, x^h) - J(x_n^h, x_{n+1}^h).$$

Then, as done in the proof of Theorem 3.1, we obtain, after some manipulations, that

$$\begin{aligned} &J(x^h, x_{n+1}^h) - J(x^h, x^h) + J(x_n^h, x^h) - J(x_n^h, x_{n+1}^h) \\ &\leq M_J\big\{\mu^* + L_\mu + M_{h\nu}L_{p\nu} + M_{p\nu}L_{h\nu} \\ &\qquad + M_{he}L_{pe} + M_{pe}L_{he}\big\}\|x^h - x_n^h\|_X\|x^h - x_{n+1}^h\|_X. \end{aligned} \tag{79}$$

Thus, we have

$$\|x^h - x_{n+1}^h\|_X \leq \frac{\mu^* + L_\mu + M_{h\nu}L_{p\nu} + M_{p\nu}L_{h\nu} + M_{he}L_{pe} + M_{pe}L_{he}}{L^*}\|x^h - x_n^h\|_X.$$

Under the stated assumptions, we have $k = \frac{\mu^* + L_\mu + M_{h\nu}L_{p\nu} + M_{p\nu}L_{h\nu} + M_{he}L_{pe} + M_{pe}L_{he}}{L^*} < 1$. Then

$$\|x_n^h - x^h\|_V \leq c\,k^n. \tag{80}$$

Finally, the following convergence result hold,

$$\|x_n^h - x^h\|_V \to 0 \text{ as } n \to +\infty.$$

□

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**Conflict of interest** The authors declare no conflict of interest.

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