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Oscillation properties of eigenfunctions for Sturm-Liouville problems with interface conditions via Prüfer transformation

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Abstract. A class of Sturm-Liouville problems with discontinuity is studied in this paper. The oscillation properties of eigenfunctions for Sturm-Liouville problems with interface conditions are obtained. The main method used in this paper is based on Prüfer transformation, which is different from the classical ones. Moreover, we give two examples to verify our main results.

§1 Introduction

In this paper, we consider the Sturm-Liouville equation

$$Ly := -(py')' + q(x)y = \lambda y, \quad x \in [a, c) \cup (c, b], \tag{1}$$

with separated boundary conditions

$$y(a)\cos\alpha - (py')(a)\sin\alpha = 0, \tag{2}$$

$$y(b)\cos\beta - (py')(b)\sin\beta = 0, \tag{3}$$

and additional interface conditions

$$y(c+0) = y(c-0) + K(py')(c-0), \tag{4}$$

$$y'(c+0) = -Ky(c-0) + (py')(c-0), \tag{5}$$

where the positive real-valued function $p^{-1}(x)$ is locally integrable on $[a, c) \cup (c, b]$, the real-valued function $q(x)$ is continuous on $[a, c) \cup (c, b]$ with finite limits, $\lambda \in \mathbb{C}$ is an eigenparameter; $0 \le \alpha < \beta \le \pi$; $K \in \mathbb{R}$; $\gamma = \arctan K \in [0, \beta - \alpha)$.

As we all know, Sturm-Liouville problems have been studied by many researchers due to their wide applications in mathematical physics, engineering technology, and other fields. In terms

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of further details, we can see [1], [3], [8], [22], [24], [25], [26], [27] and references cited therein. The investigation of classical Sturm-Liouville problems (1)-(3) without interface conditions has matured in theory and method. However, the classical Sturm-Liouville problems can not be used to describe some practical problems such as string vibration with mass acting at some point, diffraction problems, the heat conduction of thin laminated plates that are made of overlapping materials, etc., which, in general, are associated with non-smooth or discontinuous properties of the medium (see [6], [7], [9], [15], [16] for details). Therefore, when dealing with this kind of problem, we need to consider the correlation relationship between different media, which is the condition across the interface. These conditions are called interface conditions. Sturm-Liouville problems with interface conditions have drawn much attention from an increasing number of authors. Many spectral properties of Sturm-Liouville problems with interface conditions, such as the distribution of eigenvalues, the asymptotic estimations of eigenvalues and eigenfunctions, the finite spectrum, and the completeness of eigenfunction systems, have been widely studied by many authors (see [2], [4], [5], [10], [11], [12], [13], [17], [18], [21], [23] for details).

Among these spectral properties of Sturm-Liouville problems, the oscillation properties of eigenfunctions are of great importance in both theory and practical application. However, less attention has been paid to the oscillation of Sturm-Liouville problems with discontinuity; the major difficulty arises from how to apply classical Sturm-Liouville theory and methods to such problems.

In [19], Wang and Sun studied a class of discontinuous Sturm-Liouville problems as follows:

$$\begin{cases} -y''+q(x)y=\lambda y, \\ y(a)\cos\alpha+y'(a)\sin\alpha=0, \\ y(b)\cos\beta+y'(b)\sin\beta=0, \\ y(c+0)=\tau y(c-0), \\ y'(c+0)=\tau_1 y(c-0)+\tau_2 y'(c-0), \end{cases}$$

where $x\in[a,c)\cup(c,b]$; τ_1, τ_2, τ_3 are real numbers and $\nu=\tau\tau_2>0$. By constructing a new Hilbert space, they got the results of oscillation properties of eigenfunctions for the discontinuous Sturm-Liouville problems in this new Hilbert space. In [20], Wang and Sun considered the following Sturm-Liouville problems with more general transmission conditions

$$\begin{cases} -y''+q(x)y=\lambda y, \\ y(a)\cos\alpha+y'(a)\sin\alpha=0, \\ y(b)\cos\beta+y'(b)\sin\beta=0, \\ y(c+0)=\tau_3 y(c-0)+\tau_4 y'(c-0), \\ y'(c+0)=\tau_5 y(c-0)+\tau_6 y'(c-0), \end{cases}$$

where τ_3, τ_4, τ_5, τ_6 are real numbers and $\nu=\tau_3\tau_6-\tau_4\tau_5>0$. Using a similar method, the results of oscillation properties of eigenfunctions for the Sturm-Liouville problems with transmission conditions are also obtained.

Mukhtarov et al. [14] researched the non-classical Sturm-Liouville problems of the following form

$$\begin{cases} -y''+q(x)y=\lambda y,\\ \zeta y(a)+\eta y'(a)=0,\\ \zeta_1 y(a)+\eta_1 y'(b)=0,\\ \zeta_2 y(c+0)=\eta_2 y(c-0),\\ \zeta_3 y'(c+0)=\eta_3 y'(c-0), \end{cases}$$

where $x\in[a,c)\cup(c,b]$; ζ, ζ_1, ζ_2, ζ_3, η, η_1, η_2 η_3 are real numbers. By giving some restricted conditions, they obtained the eigenfunctions $\varphi(x,\lambda_n)$ corresponding to the eigenvalues λ_n of this problem have precise n zeros on the interval $(a,c)\cup(c,b)$. As is known to all, Prüfer transformation has been used by plenty of authors to study the oscillation of the solutions of classical Sturm-Liouville problems. But compared with that, little work has been done to study Sturm-Liouville problems with discontinuity using Prüfer transformation. As a result, we expect to study the problems (1)-(5) using Prüfer transformation without any assumptions in this paper.

The rest of this paper is organized as follows. In Section 2, we introduce the Prüfer transformation and some essential lemmas. In Section 3, the results of oscillation properties of eigenfunctions to Sturm-Liouville problems (1)-(5) will be presented. Finally, we will show two examples to illustrate our main result.

§2 Preliminaries and lemmas

Let us introduce the phase function $\rho(x)$ and aptitude function $\theta(x)$, which are defined according to the given solution $y(x)$ for the differential equation (1). With these two functions, we can define the Prüfer transformation for the problem (1)-(5) as follows:

$$\begin{cases} y(x,\lambda)=\rho(x,\lambda)\sin\theta(x,\lambda),\\ (py')(x,\lambda)=\rho(x,\lambda)\cos\theta(x,\lambda), \end{cases} \tag{6}$$

where $x\in[a,c)\cup(c,b]$. Applying it to (1), we obtain

$$\begin{cases} \rho'(x,\lambda)=\dfrac{1}{2}\left(\dfrac{1}{p}+q(x,\lambda)-\lambda\right)\rho(x,\lambda)\sin 2\theta(x,\lambda),\\ \theta'(x,\lambda)=\dfrac{1}{p}\cos^2\theta(x,\lambda)+(\lambda-q(x,\lambda))\sin^2\theta(x,\lambda). \end{cases} \tag{7}$$

Thus, using (6) and the interface conditions (4)-(5), we can get

$$\begin{aligned} \tan\theta(c+0)&=\frac{y(c-0)+K(py')(c-0)}{-Ky(c-0)+(py')(c-0)}\\ &=\frac{\frac{y(c-0)}{(py')(c-0)}+K}{1-K\frac{y(c-0)}{(py')(c-0)}}\\ &=\frac{\tan\theta(c-0)+K}{1-K\tan\theta(c-0)}\\ &=\tan(\theta(c-0)+\gamma), \end{aligned}$$

where $\gamma = \arctan K \in [0, \beta-\alpha)$, $K = \tan\gamma$. Furthermore, we have $\theta(c+0) = \theta(c-0)+\gamma$. This equality and the well-known Lemmas to be given below all play an essential role in our proof.

Lemma 2.1. ([8]) *Let θ_1 and θ_2 be the solutions of Cauchy problems*

$$\begin{cases} \theta' = \dfrac{1}{p}\cos^2\theta - q_1(x)\sin^2\theta, \\ \theta(a) = \delta_1. \end{cases} \qquad \begin{cases} \theta' = \dfrac{1}{p}\cos^2\theta - q_2(x)\sin^2\theta, \\ \theta(a) = \delta_2, \end{cases}$$

respectively, where $\delta_1 \le \delta_2$, $q_1(x) > q_2(x)$, $x \in [a,b]$, then

$$\theta_1(x) < \theta_2(x), \quad x \in (a,b].$$

Lemma 2.2. ([8]) *Suppose function $q(x,\lambda) : [a,b]\times\mathbb{R} \to \mathbb{R}$ is continuous, $\lim\limits_{\lambda\to-\infty} q(x,\lambda) = -\infty$, $\lim\limits_{\lambda\to\infty} q(x,\lambda) = \infty$ uniformly hold with respect to $x \in [a,b]$, $\theta(x,\lambda)$ is the solution of the Cauchy problem*

$$\begin{cases} \theta' = \dfrac{1}{p}\cos^2\theta - q(x,\lambda)\sin^2\theta, \quad x \in [a,b], \\ \theta(a) = \alpha \in [0,\pi), \end{cases}$$

then $\theta(x,\lambda) \ge 0$, $\lim\limits_{\lambda\to-\infty}\theta(b,\lambda) = 0$, $\lim\limits_{\lambda\to\infty}\theta(b,\lambda) = \infty$.

Lemma 2.3. *Let θ_3 and θ_4 be the solutions of following Cauchy problems*

$$\begin{cases} \theta' = \dfrac{1}{p}\cos^2\theta - h_1(x)\sin^2\theta, \\ \theta(a) = \delta_3. \end{cases} \qquad \begin{cases} \theta' = \dfrac{1}{p}\cos^2\theta - h_2(x)\sin^2\theta, \\ \theta(a) = \delta_4, \end{cases}$$

respectively, where $\delta_3 \le \delta_4$, $h_1(x) > h_2(x)$, $x \in [a,c)\cup(c,b]$, then

$$\theta_1(x) < \theta_2(x), \quad x \in (a,c)\cup(c,b].$$

Proof. By Lemma 2.1, we obtain $\theta_3(x) < \theta_4(x)$, $x \in (a,c)$. According to the interface conditions (4)-(5), we can get

$$\theta_3(c+0) = \theta_3(c-0)+\gamma, \quad \theta_4(c+0) = \theta_4(c-0)+\gamma,$$

where $\gamma \in [0,M)$, $M := \min\{\pi/2, \beta-\alpha\}$. So, we have $\theta_3(c+0) < \theta_4(c+0)$. By using Lemma 2.1 again, we can get $\theta_3(x) < \theta_4(x)$, $x \in (c,b)$. Given the above, we have

$$\theta_3(x) < \theta_4(x), \quad x \in (a,c)\cup(c,b],$$

which completes the proof. □

Lemma 2.4. *Suppose function $q(x,\lambda) : ([a,c)\cup(c,b])\times\mathbb{R} \to \mathbb{R}$ is continuous, $\lim\limits_{\lambda\to-\infty} q(x,\lambda) = -\infty$, $\lim\limits_{\lambda\to\infty} q(x,\lambda) = \infty$ uniformly hold with respect to $x \in [a,c)\cup(c,b]$, $\theta(x,\lambda)$ is the solution of the Cauchy problem*

$$\begin{cases} \theta' = \dfrac{1}{p}\cos^2\theta - q(x,\lambda)\sin^2\theta, \quad x \in [a,c)\cup(c,b], \\ \theta(a) = \alpha \in [0,\pi), \end{cases}$$

then $\theta(x,\lambda) \ge 0$, $\lim\limits_{\lambda\to-\infty}\theta(b,\lambda) = 0$, $\lim\limits_{\lambda\to\infty}\theta(b,\lambda) = \infty$.

Proof. By Lemma 2.2, we can get $\theta(x) \ge 0$, $x \in [a,c)$. According to the relation $\theta(c+0) = \theta(c-0)+\gamma$, using Lemma 2.2 again, we have $\theta(x) \ge 0$, $x \in [a,c)\cup(c,b]$. Meanwhile, an argument similar to the proof of Lemma 2.2 in [8] shows that

$$\lim_{\lambda\to-\infty}\theta(b,\lambda)=0,\quad \lim_{\lambda\to\infty}\theta(b,\lambda)=\infty.$$

The proof is completed. □

§3 Main results

Theorem 3.1. *There is an infinity of real eigenvalues λ_n, $n=0,1,2,\dots$ of Sturm-Liouville problems (1)-(5), and $\lambda_n\to+\infty$, as $n\to+\infty$, the normalized eigenfunction $\phi(x,\lambda_n)$ corresponding to the eigenvalue λ_n $(n\geq 1)$ has $n-1$ or n zeros on $(a,c)\cup(c,b)$. In addition, the eigenfunction $\phi(x,\lambda_0)$ corresponding to eigenvalue λ_0 that has no zero on the interval $(a,c)\cup(c,b)$.*

Proof. Let $y(x)$ be defined as (6), it follows from (7) that $\theta(x,\lambda)$ is uniquely determined by the initial condition $\theta(a,\lambda)=\alpha$, by Lemma 2.3, it is an increasing function of λ; by Lemma 2.4, it is non-negative, and

$$\lim_{\lambda\to-\infty}\theta(b,\lambda)=0,\quad \lim_{\lambda\to\infty}\theta(b,\lambda)=\infty. \tag{8}$$

Meanwhile,

$$y(x,\lambda)=\rho(x,\lambda)\sin\theta(x,\lambda) \tag{9}$$

satisfies the first boundary condition (2), and the interface conditions (4)-(5). Now, taking $y(x,\lambda)$ into account the second boundary condition (3) yields

$$0=y(b)\cos\beta-(py')(b)\sin\beta=\rho(b,\lambda)\sin\left(\theta(b,\lambda)-\beta\right).$$

If we can find λ_n, such that $\theta(b,\lambda)=\beta+n\pi$, $n=0,1,2,\cdots$, then $y(x,\lambda)$ also satisfies boundary condition (3), therefore λ_n is eigenvalue and $y(x,\lambda_n)$ is the eigenfunction corresponding to λ_n.

In fact, $\theta(b,\lambda)$ is a strictly increasing function of λ, from the relation (8), we obtain $\theta(b,\lambda)=\beta+n\pi$ has a unique solution λ_n, in this way, from the monotonicity and $\lim_{\lambda\to\infty}\theta(b,\lambda)=\infty$, there is an infinity of eigenvalues λ_n satisfying

$$\lambda_0<\lambda_1<\cdots<\lambda_n<\cdots,\ \lim_{n\to\infty}\lambda_n=\infty.$$

For every λ_n, eigenfunction is $y(x,\lambda_n)=\rho(x,\lambda_n)\sin\theta(x,\lambda_n)$. Let $\phi(x,\lambda_n)$ denote the normalize eigenfunction of $y(x,\lambda_n)$. We now consider the zeros of $\phi(x,\lambda_n)$. Since $\rho(x,\lambda)$ has a positive lower bound, the zeros of $\phi(x,\lambda_n)$ are the points such that $\theta(x,\lambda_n)=0(\mathrm{mod}\pi)$ holds, and at these points, it follows from (6) that

$$\phi'(x,\lambda_n)=c_n(py')(x,\lambda_n)=\pm c_n\rho(x,\lambda_n)\neq 0,$$

therefore, zeros are simple. If

$$\theta(x_0,\lambda_n)=k\pi,$$

then

$$\theta'(x_0,\lambda_n)=\frac{1}{p}.$$

So, $\theta(x,\lambda_n)$ is increasing with respect to x at x_0, therefore $\theta(x,\lambda_n)$ can only reach $k\pi$ once. If not, there is an $x_1>x_0$ such that $\theta(x_1,\lambda_n)=k\pi$, then $\theta'(x_1,\lambda_n)<0$, it is impossible. When

$n > 0$,

$$\theta(a, \lambda_n) = \alpha, \quad \theta(b, \lambda_n) = \beta + n\pi.$$

By $\theta(c+0) = \theta(c-0) + \gamma$, $\gamma \in [0, \beta - \alpha)$, we know the graph of $\phi(x, \lambda_n)$ has two cases, as shown in Figs. 1 and 2, respectively.

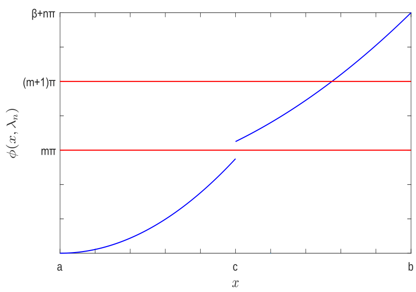

Fig 1. The first case of $\phi(x, \lambda_n)$.

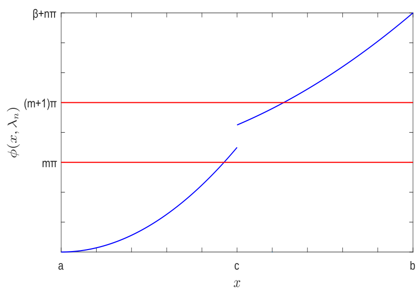

Fig 2. The second case of $\phi(x, \lambda_n)$.

Hence $\phi(x, \lambda_n)$ $(n \geq 1)$ has $n-1$ or n zeros on $(a, c) \cup (c, b)$, in addition, $\phi(x, \lambda_0)$ has no zero on $(a, c) \cup (c, b)$. $\phi(x, \lambda_n)$ can not have another zero. If not, there is an x_1 such that

$$\theta(x_1, \lambda_n) = k\pi, \quad k \geq n+1,$$

so, $\theta(x, \lambda_n)$ is increasing with respect to x at this point, hence

$$(n+1)\pi \geq \beta + n\pi = \theta(b, \lambda_n) > \theta(x_1, \lambda_n) \geq (n+1)\pi,$$

it is a contradiction, so, $\phi(x, \lambda_n)$ $(n \geq 1)$ has n or $n-1$ zeros on $(a, c) \cup (c, b)$. In addition, the eigenfunction $\phi(x, \lambda_0)$ has no zero on $(a, c) \cup (c, b)$. $\square$

§4 Examples

Example 4.1. Consider the following boundary value problem:

$$\begin{cases} -y'' = \lambda y, & (10) \\ y(0) = y(\pi) = 0, & (11) \\ y\left(\frac{\pi}{2}+0\right) = y\left(\frac{\pi}{2}-0\right) + y'\left(\frac{\pi}{2}-0\right), & (12) \\ y'\left(\frac{\pi}{2}+0\right) = -y\left(\frac{\pi}{2}-0\right) + y'\left(\frac{\pi}{2}-0\right), & (13) \end{cases}$$

after a simple calculation, we can obtain the solution that satisfies $y(0) = 0$, (10) and (12)-(13), i.e.,

$$y(x, \lambda) = \begin{cases} \sin\sqrt{\lambda}x, \quad \text{for } x \in \left[0, \frac{\pi}{2}\right), \\ \frac{1}{\sqrt{\lambda}}\left(\sin^2\frac{\sqrt{\lambda}\pi}{2} + \lambda\cos^2\frac{\sqrt{\lambda}\pi}{2}\right)\cos\sqrt{\lambda}x \\ +\left(1 + \frac{1}{2}\left(\sqrt{\lambda} - \frac{1}{\sqrt{\lambda}}\right)\sin\sqrt{\lambda}\pi\right)\sin\sqrt{\lambda}x, \quad \text{for } x \in \left(\frac{\pi}{2}, \pi\right], \end{cases}$$

by $y(\pi)=0$, we can get the critical function

$$\Delta(\lambda):=\sin\sqrt{\lambda}\pi+\left(\frac{\sqrt{\lambda}}{2}+\frac{1}{2\sqrt{\lambda}}\right)\cos\sqrt{\lambda}\pi+\left(\frac{\sqrt{\lambda}}{2}-\frac{1}{2\sqrt{\lambda}}\right)=0.$$

The graphs of critical function on the intervals [0,10] and [0,10000] are shown in Figs. 3 and 4, respectively.

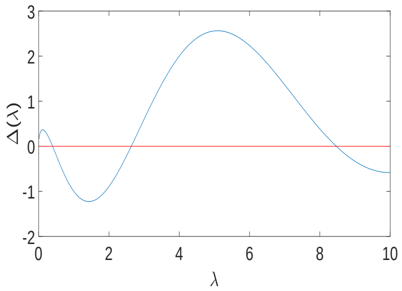

Fig 3. $\Delta(\lambda)$ on the interval [0,10].

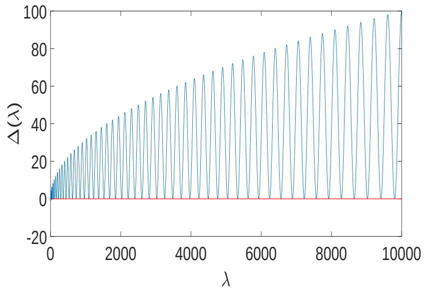

Fig 4. $\Delta(\lambda)$ on the interval [0,10000].

where $\lambda_1\approx 2.63$, $\lambda_2\approx 8.47$, their corresponding eigenfunctions $y(x,\lambda_1)$ and $y(x,\lambda_2)$ are shown in Figs. 5 and 6, respectively.

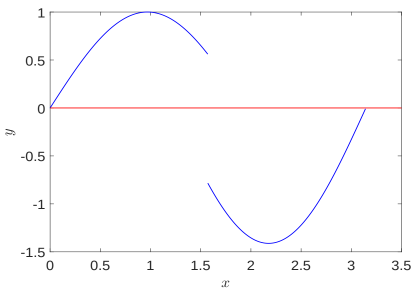

Fig 5. $y(x,\lambda_1)$.

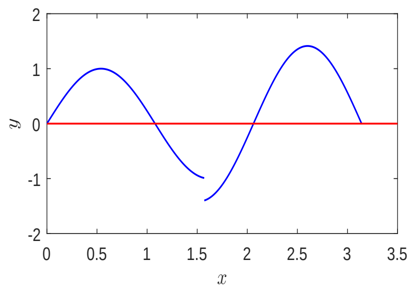

Fig 6. $y(x,\lambda_2)$.

Obviously, the eigenfunction $y(x,\lambda_1)$ has no zero, and $y(x,\lambda_2)$ has two zeros on the interval $\left(0,\frac{\pi}{2}\right)\cup\left(\frac{\pi}{2},\pi\right)$, respectively.

Example 4.2. Consider the following boundary value problem:

$$\begin{cases}-y''=\lambda y, & (14)\\ y(0)=y(\pi)=0, & (15)\\ y\left(\frac{\pi}{4}+0\right)=y\left(\frac{\pi}{4}-0\right)+2y'\left(\frac{\pi}{4}-0\right), & (16)\\ y'\left(\frac{\pi}{4}+0\right)=-2y\left(\frac{\pi}{4}-0\right)+y'\left(\frac{\pi}{4}-0\right), & (17)\end{cases}$$

after a simple calculation, we can obtain the solution that satisfies $y(0)=0$, (14) and (16)-(17), i.e.,

$$y(x,\lambda)=\begin{cases}\sin\sqrt{\lambda}x, \quad \text{for } x\in\left[0,\dfrac{\pi}{4}\right),\\ \dfrac{2}{\sqrt{\lambda}}\left(\sin^2\dfrac{\sqrt{\lambda}\pi}{4}+\lambda\cos^2\dfrac{\sqrt{\lambda}\pi}{4}\right)\cos\sqrt{\lambda}x\\ +\left(1+\left(\sqrt{\lambda}-\dfrac{1}{\sqrt{\lambda}}\right)\sin\dfrac{\sqrt{\lambda}\pi}{2}\right)\sin\sqrt{\lambda}x, \quad \text{for } x\in\left(\dfrac{\pi}{4},\pi\right],\end{cases}$$

by $y(\pi)=0$, we can get the critical function

$$\Delta(\lambda):=\sin\sqrt{\lambda}\pi+\left(\sqrt{\lambda}+\frac{1}{\sqrt{\lambda}}\right)\cos\sqrt{\lambda}\pi+\left(\sqrt{\lambda}-\frac{1}{\sqrt{\lambda}}\right)\cos\frac{\sqrt{\lambda}\pi}{2}=0.$$

The graphs of critical function on the intervals [0,10] and [0,10000] are shown in Figs. 7 and 8, respectively.

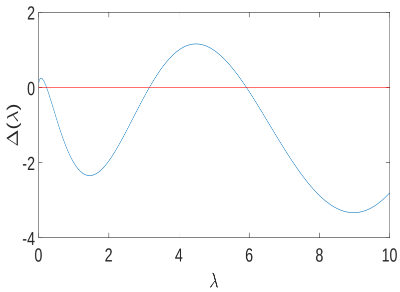

Fig 7. $\Delta(\lambda)$ on the interval [0,10].

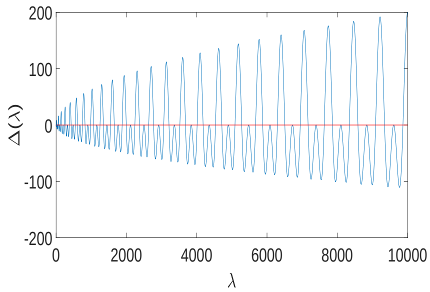

Fig 8. $\Delta(\lambda)$ on the interval [0,10000].

where $\lambda_1\approx 3.16$, $\lambda_2\approx 5.92$, their corresponding eigenfunctions $y(x,\lambda_1)$ and $y(x,\lambda_2)$ are shown in Figs. 9 and 10, respectively.

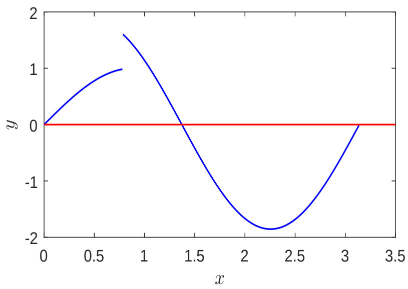

Fig 9. $y(x,\lambda_1)$.

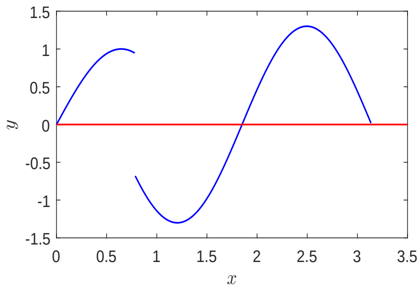

Fig 10. $y(x,\lambda_2.)$

Obviously, the eigenfunctions $y(x,\lambda_1)$ and $y(x,\lambda_2)$ have one zero on the interval $\left(0,\frac{\pi}{4}\right)\cup\left(\frac{\pi}{4},\pi\right)$.

Declarations

Conflict of interest The authors declare no conflict of interest.

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