Appl. Math. J. Chinese Univ.
2024, 39(1): 98-113

Locally and globally uniform approximations for ruin probabilities of a nonstandard bidimensional risk model with subexponential claims

LIU Zai-ming[1] GENG Bing-zhen[1,2] WANG Shi-jie[2,*]

Abstract. Consider a nonstandard continuous-time bidimensional risk model with constant force of interest, in which the two classes of claims with subexponential distributions satisfy a general dependence structure and each pair of the claim-inter-arrival times is arbitrarily dependent. Under some mild conditions, we achieve a locally uniform approximation of the finite-time ruin probability for all time horizon within a finite interval. If we further assume that each pair of the claim-inter-arrival times is negative quadrant dependent and the two classes of claims are consistently-varying-tailed, it shows that the above obtained approximation is also globally uniform for all time horizon within an infinite interval.

§1 Introduction

Consider a continuous-time nonstandard bidimensional risk model with constant force of interest, in which an insurance company launches two classes of insurance business simultaneously. The surplus process is described as

$$\begin{pmatrix} U_1(x_1,t) \\ U_2(x_2,t) \end{pmatrix} = e^{rt}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} \int_{0-}^{t} e^{r(t-s)}C_1(ds) \\ \int_{0-}^{t} e^{r(t-s)}C_2(ds) \end{pmatrix} - \begin{pmatrix} \sum_{i=1}^{N_1(t)} X_i^{(1)} e^{r(t-\tau_i^{(1)})} \\ \sum_{j=1}^{N_2(t)} X_j^{(2)} e^{r(t-\tau_j^{(2)})} \end{pmatrix}, \tag{1}$$

where $\{(U_1(x_1,t),U_2(x_2,t))^\top; t \ge 0\}$ denotes the bidimensional surplus processes, $(x_1,x_2)^\top$ the vector of initial surpluses, $r \ge 0$ the constant force of interest, $\{(C_1(t),C_2(t))^\top;\ t \ge 0\}$ the bidimensional premium processes, $\{(X_i^{(1)},X_j^{(2)})^\top; i,j \ge 1\}$ the sequence of claim size vectors and $\{(\tau_i^{(1)},\tau_j^{(2)})^\top; i,j \ge 1\}$ the sequence of claim arrival-time vectors which drive the corresponding bidimensional renewal counting processes $\{(N_1(t),N_2(t))^\top; t \ge 0\}$.

Received: 2020-07-20. Revised: 2021-04-07.
MR Subject Classification: 62P05, 62E10.
Keywords: bidimensional risk model, asymptotic formula, subexponential distribution, consistently varying tail, ruin probability.
Digital Object Identifier(DOI): https://doi.org/10.1007/s11766-024-4213-6.
Supported by the Natural Science Foundation of China(12071487, 11671404), the Natural Science Foundation of Anhui Province(2208085MA06), the Provincial Natural Science Research Project of Anhui Colleges(KJ2021A0049, KJ2021A0060) and Hunan Provincial Innovation Foundation for Postgraduate(CX20200146).
*Corresponding author.

Throughout this paper, for each $k = 1, 2$, assume that the claim sizes from the k-th line $\{X_i^{(k)}, i \geq 1\}$ constitute a sequence of identically distributed, but not necessarily independent, nonnegative random variables (r.v.s) with a common distribution F_k; the claim arrival-times $\{(\tau_i^{(k)}, i \geq 1\}$ form another sequence of nonnegative r.v.s with $\tau_i^{(k)} = \sum_{j=1}^{i} \theta_j^{(k)},\ i \geq 1$, where the claim inter-arrival-times $\{(\theta^{(1)}, \theta^{(2)}), (\theta_j^{(1)}, \theta_j^{(2)}), j \geq 1\}$ are independent and identically distributed (i.i.d.) nonnegative random vectors, but the two components of each vector are arbitrarily dependent; the premium process $\{(C_k(t), t \geq 0\}$ is a nonnegative and nondecreasing stochastic process with $C_k(0) = 0$ and $C_k(t) < \infty$ almost surely for every $t \geq 0$. To avoid triviality, we assume that either $P(\theta_1^{(1)} > 0) > 0$ or $P(\theta_1^{(2)} > 0) > 0$ holds. As we know, the renewal counting process $\{N_k(t), t \geq 0\}$ forms a nonnegative, nondecreasing, right continuous, and integer-valued stochastic process with mean function $EN_k(t) < \infty$. Define

$$\Lambda_k = \{t > 0, EN_k(t) > 0\} = \{t > 0, P(\tau_1^{(k)} \leq t) > 0\}$$

for later use and write $\Lambda = \bigcap_{k=1}^{2} \Lambda_k$. Moreover, suppose that $\{X_i^{(1)}, i \geq 1\}$, $\{X_i^{(2)}, i \geq 1\}$, $\{N_1(t), N_2(t); t \geq 0\}$ and $\{C_1(t), C_2(t);\ t \geq 0\}$ are mutually independent.

For the above continuous-time bidimensional renewal risk model, denote by

$$\tau_{\max} = \inf\{t > 0 : \max\{U_1(x_1, t), U_2(x_2, t)\} < 0\}$$

the first time when both $U_1(x_1, t)$ and $U_2(x_2, t)$ become negative. Then the corresponding finite-time ruin probability $\psi_{\max}(x_1, x_2; t)$ is defined as

$$P\big(\tau_{\max}(x_1, x_2) \leq t | (U_1(x_1, 0), U_2(x_2, 0))^\top = (x_1, x_2)^\top\big)$$

and further the infinite-time ruin probability is defined as

$$\psi_{\max}(x_1, x_2) = \lim_{t \to \infty} \psi_{\max}(x_1, x_2; t) = P\big(\tau_{\max} < \infty | (U_1(x_1, 0), U_2(x_2, 0))^\top = (x_1, x_2)^\top\big).$$

Historically, bidimensional risk models have been widely investigated by many authors. In general, some of them studied ruin probabilities of bidimensional risk models without constant force of interest. See, for example, Chen et al. (2003), Yuen et al. (2006), Li et al. (2007), Chen et al. (2011), Lu and Zhang (2016), Chen and Yu (2017) and so on. Whereas, others investigated ruin probabilities for bidimensional risk models with constant force of interest or stochastic investments driven by Lévy process. Besides, for the potential practicability of the results, some dependence structures were also imposed among the claim sizes at the same time in some recent papers. For instance, Chen et al. (2013), Gao and Yang (2014), Yang and Li (2014), Yang and Yuen (2016), Li (2018), Chen et al.(2019), Yang et al. (2018, 2019) and references therein.

However, to the best of our knowledge, most of the above-mentioned papers restricted the claims belonging to some smaller heavy-tailed subclass (mainly belonging to the consistently-varying-tailed class or the intersection of dominatedly-varying-tailed class and long-tailed class) when studying ruin probabilities. Also, few works were devoted to investigating the uniformity of asymptotics for both finite-time and infinite-time ruin probabilities. Based on these two considerations, this paper aims to do some jobs on both objects simultaneously. Firstly, after assuming that the claims from the same lines satisfy a general dependence structure and each pair of the inter-arrival times are arbitrarily dependent, we achieve a locally uniform approximation of the finite-time ruin probability for all time horizon within a finite interval. But more than that, if we suppose that the two classes of claims are consistently-varying-tailed, this approximation can be extended to the infinite-time case. Finally, if we further assume that each pair of the inter-arrival times are negative quadrant dependent (NQD), it shows that the above obtained approximation is also globally uniform for all time horizon within an infinite interval. Our obtained results coincide with and substantially extend some existing ones in the literature.

At the end of this section, let us recall two dependence structures which will be used in this paper. The first was proposed in Ko and Tang (2008) and then extended to infinite sequences by Zhang and Cheng (2016); see also Yang et al. (2012) or Gong et al. (2020). Moreover, a weighted version of this dependent structure can be found in Cheng and Cheng (2018), where they called it conditionally linearly wide dependence. It is worth mentioning that Asimit and Jones (2008) first introduced a well-known bivariate dependence structure through copula function. Even as for the bivariate case, the following Assumption 1 obviously covers the dependence structure in Asimit and Jones (2008) when the measurable function g in Assumption 2 of Asimit and Jones (2008) is assumed to be bounded. One can also refer to Remark 2.3 of Ko and Tang (2008) for more details. The second dependent structure named NQD was initiated by Lehmann (1966).

Assumption 1. *Let $\{X_k, k \geq 1\}$ be a sequence of nonnegative r.v.s. There exist some large $x_0 > 0$, irrespective of n, and a sequence of nondecreasing constants $\{m_k, k \geq 1\}$, such that, for all $x \geq x_0$ and every $n \geq 2$, the relation*

$$P\Big(\sum_{i=1}^{n-1} X_i > x - y\Big| X_n = y\Big) \leq m_{n-1} P\Big(\sum_{i=1}^{n-1} X_i > x - y\Big)$$

holds uniformly for all $y \in [x_0, x - x_0]$.

Definition 1.1 *Two r.v.s ξ_1 and ξ_2 are said to be NQD, if for any $x_1, x_2 \in \mathbb{R}$,*

$$P(\xi_1 > x_1, \xi_2 > x_2) \leq P(\xi_1 > x_1) P(\xi_2 > x_2).$$

The rest of this paper consists of five sections. Section 2 prepares some preliminaries on heavy-tailed distributions. Section 3 gives our main results and their proofs are presented in Sections 4, 5 and 6, respectively.

§2 Preliminaries

Throughout this paper, all limit relationships are according to $x_1, x_2 \to \infty$ unless stated otherwise. For two positive functions $f(\cdot)$ and $g(\cdot)$, assume that

$$a = \liminf \frac{f(\cdot)}{g(\cdot)} \leq \limsup \frac{f(\cdot)}{g(\cdot)} = b.$$

Write $f(x) \gtrsim g(x)$ if $a \geq 1$; write $f(x) \lesssim g(x)$ if $b \leq 1$; write $f(x) \asymp g(x)$ if $0 < a \leq b < \infty$; write $f(x) \sim g(x)$ if $a = b = 1$; write $f(x) = o(g(x))$ if $b = 0$; write $f(x) = O(g(x))$ if $b < \infty$. Furthermore, for two positive trivariate functions $a(\cdot,\cdot,\cdot)$ and $b(\cdot,\cdot,\cdot)$, we say that the asymptotic relation $a(x_1, x_2, t) \sim b(x_1, x_2, t)$ holds uniformly over all t in a nonempty set Δ if

$$\lim_{(x_1,x_2)\to(\infty,\infty)} \sup_{t\in\Delta} \Big| \frac{a(x_1, x_2, t)}{b(x_1, x_2, t)} - 1 \Big| = 0.$$

As usual, for a r.v. X, we write $X^+ = \max\{X, 0\}$; for two constants a and b, write $a \vee b = \max\{a, b\}$ and $a \wedge b = \min\{a, b\}$.The indicator function of an event A is denoted by $\mathbf{I}_A$. For any distribution function F, denote its tail by $\overline{F}(x) = 1 - F(x)$.

For convenience of later use, let us recall some definitions and properties of heavy-tailed distribution classes which have been widely applied in many fields such as insurance, financial mathematics, queueing theory and so on. We say that a r.v. X (or its distribution F) is heavy-tailed if it has no finite exponential moments. In the following, we list some important heavy-tailed classes.

We say a distribution function F on $[0,\infty)$ belongs to the subexponential class, denoted by $F \in \mathscr{S}$, if $\lim_{x\to\infty} \overline{F^{*n}}(x)/\overline{F}(x) = n$ holds for all $n \geq 2$, where F^{*n} is the n-fold convolution of F; belongs to the long-tailed class, written as $F \in \mathscr{L}$, if $\lim_{x\to\infty} \overline{F}(x-y)/\overline{F}(x) = 1$ for all $y \in \mathbb{R}$; belongs to the dominatedly-varying-tailed class, written as $F \in \mathscr{D}$, if $\lim_{x\to\infty} \overline{F}(xy)/\overline{F}(x) < \infty$ for some $0 < y < 1$; belongs to the consistently-varying-tailed class, written as $F \in \mathscr{C}$, if $\lim_{y\downarrow 1} \liminf_{x\to\infty} \overline{F}(xy)/\overline{F}(x) = 1$ (or, equivalently, $\lim_{y\uparrow 1} \limsup_{x\to\infty} \overline{F}(xy)/\overline{F}(x) = 1$); belongs to the extended regularly-varying-tailed class, written as $F \in \mathrm{ERV}(-\alpha, -\beta)$ for some $0 \leq \alpha \leq \beta < \infty$, if $y^{-\beta} \leq \liminf_{x\to\infty} \overline{F}(xy)/\overline{F}(x) \leq \limsup_{x\to\infty} \overline{F}(xy)/\overline{F}(x) \leq y^{-\alpha}$, for any $y > 1$. Among the heavy-tailed subclasses mentioned above, by Embrechts et al.(1997), the following inclusion relationships are well-known:

$$\mathrm{ERV} \subset \mathscr{C} \subset \mathscr{D} \cap \mathscr{L} \subset \mathscr{S} \subset \mathscr{L}.$$

Moreover, the long-tailed distribution class has some elementary properties. It was proved that if $F \in \mathscr{L}$, then the function class

$$\mathscr{H}(F) = \big\{ l(\cdot) \text{ on } [0,\infty) : l(x) \uparrow \infty,\ l(x) \leq x/2,\ l(x)/x \downarrow 0 \text{ and } \overline{F}(x - l(x)) \sim \overline{F}(x) \big\}$$

is not empty; see for instance Cline and Samorodnitsky (1994) or Foss et al. (2013). Clearly, for any $l(x) \in \mathscr{H}(F)$, $\overline{F}(x - Kl(x)) \sim \overline{F}(x)$ for any constant $K > 0$.

Furthermore, for a distribution function F with an ultimate right tail, define

$$J_F^+ = \inf\left\{ -\frac{\log \overline{F}_*(y)}{\log y} : y > 1 \right\} \quad \text{and} \quad J_F^- = \sup\left\{ -\frac{\log \overline{F}^*(y)}{\log y} : y > 1 \right\}$$

with

$$\overline{F}_*(y) = \liminf_{x\to\infty} \frac{\overline{F}(xy)}{\overline{F}(x)} \quad \text{and} \quad \overline{F}^*(y) = \limsup_{x\to\infty} \frac{\overline{F}(xy)}{\overline{F}(x)}.$$

Following the terminology of Tang and Tsitsiashvili (2003a), J_F^+ and J_F^- are called the upper and lower Matuszewska index of F, respectively. It is known that $F \in \mathscr{D}$ is equivalent to $J_F^+ < \infty$; see Embrechts et al.(1997). Moreover, from Proposition 2.2.1 of Bingham et al. (1987), for any $p_1 < J_F^-$ and $p_2 > J_F^+$, there exist positive and sufficiently large constants C, D such that

$$C^{-1}\Big(\frac{x}{y}\Big)^{p_1} \leq \frac{\overline{F}(y)}{\overline{F}(x)} \leq C\Big(\frac{x}{y}\Big)^{p_2} \quad \text{for all } x \geq y \geq D. \tag{2}$$

§3 Main results

In this section, let us present our main results. Theorem 3.1 gives a locally uniform asymptotic for the finite-time ruin probability of bidimensional renewal risk models for all time horizon within a finite interval. Under some stronger conditions, Theorem 3.2 derives an asymptotic formula for the infinite-time ruin probability of bidimensional renewal risk models.

Theorem 3.1. *Consider the bidimensional renewal risk model introduced in Section 1 with* $r \geq 0$*, in which, for* $k = 1, 2$*, the claims* $\{X_i^{(k)}; i \geq 1\}$ *with common distribution* $F_k \in \mathscr{S}$ *satisfy Assumption 1. Assume that the claim inter-arrival-times* $\{(\theta^{(1)}, \theta^{(2)}), (\theta_j^{(1)}, \theta_j^{(2)}), j \geq 1\}$ *are i.i.d. nonnegative random vectors, but the two components of each vector are arbitrarily dependent and further* $\{X_i^{(1)}, i \geq 1\}$*,* $\{X_i^{(2)}, i \geq 1\}$*,* $\{(N_1(t), N_2(t)); t \geq 0\}$ *and* $\{(C_1(t), C_2(t));\ t \geq 0\}$ *are mutually independent. If there exists a constant* M *such that* $m_n \leq M$ *for all* $n \geq 1$*, then, for any fixed* $T \in \Lambda$*, it holds uniformly for all* $t \in \Lambda_T =: \Lambda \cap [0, T]$ *that*

$$\psi_{\max}(x_1, x_2; t) \sim \int_{0-}^{t} \int_{0-}^{t} \overline{F_1}(x_1 e^{rs_1}) \overline{F_2}(x_2 e^{rs_2}) dE[N_1(s_1) N_2(s_2)]. \tag{3}$$

Theorem 3.2. *Suppose that all the same conditions in Theorem 3.1 hold. Additionally, if for* $k = 1, 2,$ $F_k \in \mathscr{C}, k = 1, 2$ *with* $J_{F_k}^- > 0$*, then the relation (3) also holds for* $t = \infty$*, i.e.*

$$\psi_{\max}(x_1, x_2) \sim \int_{0-}^{\infty} \int_{0-}^{\infty} \overline{F_1}(x_1 e^{rs_1}) \overline{F_2}(x_2 e^{rs_2}) dE[N_1(s_1) N_2(s_2)].$$

In particular, next we intend to verify the global uniformity of the relation (3) for all $t \in \Lambda$. However, it is mathematically difficult under the assumption that $\theta^{(1)}$ and $\theta^{(2)}$ are arbitrarily dependent. Fortunately, if we further assume that $\theta^{(1)}$ and $\theta^{(2)}$ are NQD, then the global uniformity result of the relation (3) is achieved in the following Theorem 3.3.

Theorem 3.3. *Suppose that all the same conditions in Theorem 3.2 hold. Moreover, if each pair of the claim-inter-arrival times is NQD and* $\{(\theta^{(1)}, \theta^{(2)})^\top, (\theta_i^{(1)}, \theta_i^{(2)})^\top, i \geq 1\}$ *is a sequence of i.i.d. nonnegative random vectors, then the relation (3) holds uniformly for all* $t \in \Lambda$*.*

§4 Proof of Theorem 3.1

In order to prove Theorem 3.1, we need to prepare some lemmas. Lemma 4.1 below is due to Ko and Tang (2008). Lemma 4.2 is a special case of Theorem 2.1 of Gong and Yang (2020). Lemma 4.3 is an asymptotic formula of weighted sums of r.v.s satisfying Assumption 1 which plays a key role for proving the next Lemma 4.4. Certainly, Lemma 4.3 has its own interest since it derives some type of asymptotic formula for weighted sums under Assumption 1. An independent randomly weighted sum version of this result was proposed in Tang and Tsitsiashvili (2003b) and more recently a dependent version can be also found in Proposition 1 of Yang et al. (2020) where the dependence structure used is different from ours. Lemma 4.4 is an analogue of Lemma 4.4 in Yang and Li (2017) which is essentially important for proving Theorem 3.1.

Lemma 4.1. *Let* $\{X_i, i \geq 1\}$ *be a sequence of nonnegative r.v.s satisfying Assumption 1 with a common distribution* $F \in \mathscr{S}$*, then, for any fixed* $n \geq 1$*, it holds that*

$$P\Big(\sum_{i=1}^{n} X_i > x\Big) \sim \sum_{i=1}^{n} P(X_i > x).$$

Lemma 4.2. *Let* $\{X_i, i \geq 1\}$ *be a sequence of nonnegative r.v.s satisfying Assumption 1 with common distribution* $F \in \mathscr{S}$*. If there exists a constant* M *such that* $m_n \leq M$ *for all* $n \geq 1$*, then, for any fixed* $n \geq 1$ *and* $\varepsilon > 0$*, there exist some sufficiently large* $x_0 > 0$ *and some* *constant* $K = K(\varepsilon, M, x_0) > 0$ *such that for all* $x > x_0$,

$$P\Big(\sum_{i=1}^{n} X_i > x\Big) \le K(1+\varepsilon)^n \overline{F}(x).$$

Lemma 4.3. *Let* $\{X_i, i \ge 1\}$ *be a sequence of nonnegative r.v.s satisfying Assumption 1 with a common distribution* $F \in \mathscr{S}$, *then, for any fixed* $n \ge 1$ *and any cosntants* $0 < a \le b < \infty$, *the relation*

$$P\Big(\sum_{i=1}^{n} c_i X_i > x\Big) \sim \sum_{i=1}^{n} P(c_i X_i > x), \tag{4}$$

holds uniformly for all $(c_1, \ldots, c_n) \in [a, b]^n$.

Proof. For simplicity, in what follows, denote by $c_{(n)} =: \max_{1 \le i \le n} c_i$. First, we proceed to use the induction method to prove the asymptotic upper bound of (4). Clearly, this case $n = 1$ is trivial. Suppose that (4) holds for any fixed positive integer $n-1$, we aim to show that (4) also holds for n. To do so, we arbitrarily choose some $l(x) \in \mathscr{H}(F)$ and divide the probability $P\big(\sum_{i=1}^{n} c_i X_i > x\big)$ as the following three parts:

$$\begin{aligned} & P\Big(\sum_{i=1}^{n} c_i X_i > x, c_n X_n \le l(x)\Big) + P\Big(\sum_{i=1}^{n} c_i X_i > x, c_n X_n > x - l(x)\Big) \\ & + P\Big(\sum_{i=1}^{n} c_i X_i > x, l(x) < c_n X_n \le x - l(x)\Big) \\ := \; & I_1(x) + I_2(x) + I_3(x). \end{aligned} \tag{5}$$

For any $\varepsilon > 0$ and some sufficiently large $x_1 > 0$, by induction hypothesis and $F \in \mathscr{S} \subset \mathscr{L}$, it holds for all $x \ge x_1$ and uniformly for all $(c_1, \ldots, c_n) \in [a, b]^n$ that

$$I_1(x) \le (1+\varepsilon) \sum_{i=1}^{n-1} P(c_i X_i > x - l(x)) \le (1+\varepsilon)^2 \sum_{i=1}^{n-1} P(c_i X_i > x) \tag{6}$$

and

$$I_2(x) \le P(c_n X_n > x - l(x)) \le (1+\varepsilon) P(c_n X_n > x). \tag{7}$$

For $I_3(x)$, let X_1^*, X_n^* be two independent copies of X_1. By Assumption 1 and Lemma 4.1, there exits some some sufficiently large $x_2 > x_1$ such that for all $x > x_2$ and uniformly for all $(c_1, \ldots, c_n) \in [a, b]^n$,

$$\begin{aligned} I_3(x) & \le \int_{\frac{l(x)}{c_n}}^{\frac{x-l(x)}{c_n}} P\Big(\sum_{i=1}^{n-1} X_i > \frac{x - c_n t}{c_{(n-1)}} \Big| X_n = t\Big) P(X_n \in dt) \\ & \le m_{n-1} \int_{\frac{l(x)}{c_n}}^{\frac{x-l(x)}{c_n}} P\Big(\sum_{i=1}^{n-1} X_i > \frac{x - c_n t}{c_{(n-1)}}\Big) P(X_n \in dt) \\ & \le (1+\varepsilon) m_{n-1} \sum_{i=1}^{n-1} \int_{\frac{l(x)}{c_n}}^{\frac{x-l(x)}{c_n}} P\Big(X_i > \frac{x - c_n t}{c_{(n-1)}}\Big) P(X_n \in dt) \\ & = (1+\varepsilon) m_{n-1} (n-1) P\big(c_{(n-1)} X_1^* + c_n X_n^* > x, l(x) < c_n X_n^* \le x - l(x)\big) \\ & \le \varepsilon (1+\varepsilon) m_{n-1} (n-1) (P(c_{(n-1)} X_1^* > x) + P(c_n X_n^* > x)) \end{aligned}$$

$$\begin{aligned}
&\leq \varepsilon(1+\varepsilon)m_{n-1}(n-1)\Big(\sum_{i=1}^{n-1}P(c_iX_i>x)+P(c_nX_n>x)\Big)\\
&= \varepsilon(1+\varepsilon)m_{n-1}(n-1)\sum_{i=1}^{n}P(c_iX_i>x), \qquad (8)
\end{aligned}$$

where in the fifth step we used Lemma 3.2 in Gong et al. (2020). Plugging (6)-(8) into (5) and letting $\varepsilon \downarrow 0$, the asymptotic upper bound of (4) is derived.

Next, we turn to verity the asymptotic lower bound of (4). Indeed, since the nonnegativity of $X_1, \dots, X_n$, it holds that

$$P\Big(\sum_{i=1}^{n}c_iX_i>x\Big) \geq \sum_{i=1}^{n}P(c_iX_i>x) - \sum_{1\leq i<j\leq n}P(c_iX_i>x, c_jX_j>x). \qquad (9)$$

By applying Assumption 1 and Lemma 4.1, there exists some sufficiently large x_3 such that for all $x > x_3$,

$$\begin{aligned}
& P(c_iX_i>x, c_jX_j>x)\\
\leq\ & \int_{\frac{x}{c_j}}^{\infty}P\Big(\sum_{i=1}^{j-1}X_i>\frac{x}{c_{(j-1)}}\Big|X_j=t\Big)P(X_j\in dt)\\
\leq\ & P\Big(\sum_{i=1}^{j-1}X_i>\frac{x}{c_{(j-1)}}\Big)P(c_jX_j>x)\sup_{t>x/c_j}\frac{P\Big(\sum_{i=1}^{j-1}X_i>\frac{x}{c_{(j-1)}}+t-t\Big|X_j=t\Big)}{P\Big(\sum_{i=1}^{j-1}X_i>\frac{x}{c_{(j-1)}}+t-t\Big)}\\
\leq\ & MP\Big(\sum_{i=1}^{j-1}X_i>\frac{x}{c_{(j-1)}}\Big)P(c_jX_j>x)\\
\leq\ & (1+\varepsilon)M\sum_{i=1}^{j-1}P\Big(X_i>\frac{x}{(j-1)c_{(j-1)}}\Big)P(c_jX_j>x)\\
=\ & o(1)P(c_jX_j>x). \qquad (10)
\end{aligned}$$

Thus, letting $\varepsilon \downarrow 0$, (10) together with (9) yields the desired asymptotic lower bound of (4) and this ends the proof.

Lemma 4.4. *Under the conditions of Theorem 3.1, it holds uniformly for $t \in \Lambda_T$ that*

$$\begin{aligned}
& P\Big(\sum_{i=1}^{N_1(t)}X_i^{(1)}e^{-r\tau_i^{(1)}}>x_1, \sum_{j=1}^{N_2(t)}X_j^{(2)}e^{-r\tau_j^{(2)}}>x_2\Big)\\
\sim & \int_{0-}^{t}\int_{0-}^{t}\overline{F}_1(x_1e^{rs_1})\overline{F}_2(x_2e^{rs_2})dE[N_1(s_1)N_2(s_2)].
\end{aligned}$$

Proof. Following the same method used as in Lemma 4.4 in Yang and Li (2017) by Lemmas 4.2 and 4.3 instead of Lemmas 4.1 and 4.2 in their paper, one can easily derive this lemma. To save space, we omit it here.

Proof of Theorem 3.1. Following the proof method of Theorem 2.1 in Yang and Li (2014), Theorem 3.1 can be easily achieved according to Lemma 4.4. For simplicity, we also omit it here.

§5 Proof of Theorem 3.2

In this section, we also present some lemmas for proving Theorem 3.2. Lemmas 5.1 and 5.2 are similar results as Lemma 5.2 in Yang and Li (2017), but it is worth mentioning that the dependence structure appeared in their paper is quite different from ours. Hence the treatment should be adjusted.

Lemma 5.1. *Under the conditions of Theorem 3.2, it holds that*

$$\lim_{N\to\infty}\limsup_{(x_1,x_2)\to(\infty,\infty)}\frac{P\big(\sum_{i=N}^{\infty}X_i^{(1)}e^{-r\tau_i^{(1)}}>x_1,\sum_{j=1}^{\infty}X_j^{(2)}e^{-r\tau_j^{(2)}}>x_2\big)}{\overline{F_1}(x_1)\overline{F_2}(x_2)}=0, \tag{11}$$

and

$$\lim_{N\to\infty}\limsup_{(x_1,x_2)\to(\infty,\infty)}\frac{P\big(\sum_{i=1}^{\infty}X_i^{(1)}e^{-r\tau_i^{(1)}}>x_1,\sum_{j=N}^{\infty}X_j^{(2)}e^{-r\tau_j^{(2)}}>x_2\big)}{\overline{F_1}(x_1)\overline{F_2}(x_2)}=0. \tag{12}$$

Proof. The proof idea of this lemma is due to Lemma 5.2 in Yang and Li (2017). It suffices to prove (11) since the proof of (12) is fully similar. Firstly, arbitrarily choose some large M such that $\sum_{i=1}^{\infty}1/i^2<M$. Moreover, noting that $F_1,F_2\in\mathscr{C}\subset\mathscr{D}$, thus, we can two positive constant p_1 and p_2 satisfying $0<p_1<J_{F_1}^-\wedge J_{F_2}^-\le J_{F_1}^+\vee J_{F_2}^+<p_2<\infty$. It is obvious that

$$\begin{aligned}
&P\Big(\sum_{i=N}^{\infty}X_i^{(1)}e^{-r\tau_i^{(1)}}>x_1,\sum_{j=1}^{\infty}X_j^{(2)}e^{-r\tau_j^{(2)}}>x_2\Big)\\
\le\ & P\Big(\sum_{i=N}^{\infty}X_i^{(1)}e^{-r\tau_i^{(1)}}>\sum_{i=N}^{\infty}\frac{x_1}{i^2M},\sum_{j=1}^{\infty}X_j^{(2)}e^{-r\tau_j^{(2)}}>\sum_{j=1}^{\infty}\frac{x_2}{j^2M}\Big)\\
\le\ & P\Big(\bigcup_{i=N}^{\infty}\Big\{X_i^{(1)}e^{-r\tau_i^{(1)}}>\frac{x_1}{i^2M}\Big\},\bigcup_{j=1}^{\infty}\Big\{X_j^{(2)}e^{-r\tau_j^{(2)}}>\frac{x_2}{j^2M}\Big\}\Big)\\
\le\ & \sum_{i=N}^{\infty}\sum_{j=1}^{\infty}P\Big(X_i^{(1)}e^{-r\tau_i^{(1)}}>\frac{x_1}{i^2M},X_j^{(2)}e^{-r\tau_j^{(2)}}>\frac{x_2}{j^2M}\Big)\\
=\ & \sum_{i=N}^{\infty}\sum_{j=1}^{\infty}\Big(\int_1^{i^2M}\int_1^{j^2M}+\int_{i^2M}^{\infty}\int_1^{j^2M}+\int_1^{i^2M}\int_{j^2M}^{\infty}+\int_{i^2M}^{\infty}\int_{j^2M}^{\infty}\Big)\\
&\times\overline{F_1}\Big(\frac{x_1u}{i^2M}\Big)\overline{F_2}\Big(\frac{x_2v}{j^2M}\Big)P(e^{r\tau_i^{(1)}}\in du,e^{r\tau_j^{(2)}}\in dv)\\
:=\ & I_1(x_1,x_2,N)+I_2(x_1,x_2,N)+I_3(x_1,x_2,N)+I_4(x_1,x_2,N).
\end{aligned}\tag{13}$$

For $I_1(x_1,x_2,N)$, noting that $F_i\in\mathscr{C}\subset\mathscr{D},i=1,2$, by the second inequality in (2), for sufficiently large x_1,x_2, the quantity $I_1(x_1,x_2,N)$ is not large than

$$\begin{aligned}
&\sum_{i=N}^{\infty}\sum_{j=1}^{\infty}C^2\overline{F_1}(x_1)\overline{F_2}(x_2)\int_1^{i^2M}\int_1^{j^2M}\Big(\frac{i^2M}{u}\frac{j^2M}{v}\Big)^{p_2}P(e^{r\tau_i^{(1)}}\in du,e^{r\tau_j^{(2)}}\in dv)\\
\le\ & C^2M^{2p_2}\overline{F_1}(x_1)\overline{F_2}(x_2)\sum_{i=N}^{\infty}\sum_{j=1}^{\infty}i^{2p_2}j^{2p_2}E(e^{-rp_2\tau_i^{(1)}}e^{-rp_2\tau_j^{(2)}})\\
\le\ & C^2M^{2p_2}\overline{F_1}(x_1)\overline{F_2}(x_2)\sum_{i=N}^{\infty}i^{2p_2}\{E(e^{-2rp_2\tau_1^{(1)}})\}^{i/2}\sum_{j=1}^{\infty}j^{2p_2}\{E(e^{-2rp_2\tau_1^{(2)}})\}^{j/2},
\end{aligned}$$

where the last step holds due to the Hölder's inequality. Noting the convergence of the series appearing in the above relation, we have

$$\lim_{N\to\infty}\limsup_{(x_1,x_2)\to(\infty,\infty)}\frac{I_1(x_1,x_2,N)}{\overline{F_1}(x_1)\overline{F_2}(x_2)}=0. \tag{14}$$

By the same arguments as above and applying the first inequality in (2), one can also obtain

$$\lim_{N\to\infty}\limsup_{(x_1,x_2)\to(\infty,\infty)}\frac{I_i(x_1,x_2,N)}{\overline{F_1}(x_1)\overline{F_2}(x_2)}=0,\quad i=2,3,4.$$

These together with (13) and (14) yield the desired (11) and this ends the proof.

Lemma 5.2. *Under the conditions of Theorem 3.2, it holds that*

$$\lim_{N\to\infty}\limsup_{(x_1,x_2)\to(\infty,\infty)}\sum_{i=N}^{\infty}\sum_{j=1}^{\infty}\frac{P(X_i^{(1)}e^{-r\tau_i^{(1)}}>x_1,X_j^{(2)}e^{-r\tau_j^{(2)}}>x_2)}{\overline{F_1}(x_1)\overline{F_2}(x_2)}=0, \tag{15}$$

and

$$\lim_{N\to\infty}\limsup_{(x_1,x_2)\to(\infty,\infty)}\sum_{i=1}^{\infty}\sum_{j=N}^{\infty}\frac{P(X_i^{(1)}e^{-r\tau_i^{(1)}}>x_1,X_j^{(2)}e^{-r\tau_j^{(2)}}>x_2)}{\overline{F_1}(x_1)\overline{F_2}(x_2)}=0. \tag{16}$$

Proof. It suffices to prove (15). Indeed, for sufficiently large x, y, it follows from (2) that

$$\begin{aligned}
&\sum_{i=N}^{\infty}\sum_{j=1}^{\infty}\frac{P(X_i^{(1)}e^{-r\tau_i^{(1)}}>x_1,X_j^{(2)}e^{-r\tau_j^{(2)}}>x_2)}{\overline{F_1}(x_1)\overline{F_2}(x_2)}\\
\le\ &\sum_{i=N}^{\infty}\sum_{j=1}^{\infty}\int_1^{\infty}\int_1^{\infty}\frac{\overline{F_1}(x_1u)\overline{F_2}(x_2v)}{\overline{F_1}(x_1)\overline{F_2}(x_2)}P(e^{r\tau_i^{(1)}}\in du,e^{r\tau_j^{(2)}}\in dv)\\
\le\ &C^2\sum_{i=N}^{\infty}\sum_{j=1}^{\infty}\int_1^{\infty}\int_1^{\infty}(uv)^{-p_1}P(e^{r\tau_i^{(1)}}\in du,e^{r\tau_j^{(2)}}\in dv)\\
\le\ &C^2\sum_{i=N}^{\infty}(E(e^{-2rp_1\tau_1^{(1)}}))^{i/2}\sum_{j=1}^{\infty}(E(e^{-2rp_1\tau_1^{(2)}}))^{j/2},
\end{aligned}$$

which implies the desired (15) by letting $N\uparrow\infty$ and this ends the proof.

Proof of Theorem 3.2. It suffices to prove the following fact, then the remained proof can fully be followed from the method used in Theorem 2.2 in Yang and Li (2017) and Theorem 3.2 can be easily derived.

$$\begin{aligned}
&P\Big(\sum_{i=1}^{\infty}X_i^{(1)}e^{-r\tau_i^{(1)}}>x_1,\sum_{j=1}^{\infty}X_j^{(2)}e^{-r\tau_j^{(2)}}>x_2\Big)\\
\sim&\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}P(X_i^{(1)}e^{-r\tau_i^{(1)}}>x_1,X_j^{(2)}e^{-r\tau_j^{(2)}}>x_2).
\end{aligned}$$

In fact, the proof of this lemma is routine. For any $0<\delta<1/2$ and each fixed $m,n\ge 1$, we first consider the asymptotic upper bound. Note that

$$P\Big(\sum_{i=1}^{\infty}X_i^{(1)}e^{-r\tau_i^{(1)}}>x_1,\sum_{j=1}^{\infty}X_j^{(2)}e^{-r\tau_j^{(2)}}>x_2\Big)\le J_1+J_2+J_3+J_4$$

with

$$J_1 = P\Big(\sum_{i=1}^{n} X_i^{(1)} e^{-r\tau_i^{(1)}} > (1-\delta)x_1, \sum_{j=1}^{m} X_j^{(2)} e^{-r\tau_j^{(2)}} > (1-\delta)x_2\Big),$$

$$J_2 = P\Big(\sum_{i=1}^{n} X_i^{(1)} e^{-r\tau_i^{(1)}} > (1-\delta)x_1, \sum_{j=m+1}^{\infty} X_j^{(2)} e^{-r\tau_j^{(2)}} > \delta x_2\Big),$$

$$J_3 = P\Big(\sum_{i=n+1}^{\infty} X_i^{(1)} e^{-r\tau_i^{(1)}} > \delta x_1, \sum_{j=1}^{m} X_j^{(2)} e^{-r\tau_j^{(2)}} > (1-\delta)x_2\Big),$$

and

$$J_4 = P\Big(\sum_{i=n+1}^{\infty} X_i^{(1)} e^{-r\tau_i^{(1)}} > \delta x_1, \sum_{j=m+1}^{\infty} X_j^{(2)} e^{-r\tau_j^{(2)}} > \delta x_2\Big).$$

We first estimate J_1. For any $\varepsilon > 0$ and sufficiently large x_1, x_2, it follows from Lemma 4.3 and (2) that

$$\begin{aligned}
J_1 &\leq (1+\varepsilon)\sum_{i=1}^{n}\sum_{j=1}^{m} P\big(X_i^{(1)} e^{-r\tau_i^{(1)}} > (1-\delta)x_1, X_j^{(2)} e^{-r\tau_j^{(2)}} > (1-\delta)x_2\big) \\
&= (1+\varepsilon)\sum_{i=1}^{n}\sum_{j=1}^{m} \int_1^\infty \int_1^\infty \overline{F_1}((1-\delta)x_1 u)\overline{F_2}((1-\delta)x_2 v) P(e^{r\tau_i^{(1)}} \in du, e^{r\tau_j^{(2)}} \in dv) \\
&\leq (1+\varepsilon) \sup_{u\in(1,\infty)} \frac{\overline{F_1}((1-\delta)x_1 u)}{\overline{F_1}(x_1 u)} \cdot \sup_{v\in(1,\infty)} \frac{\overline{F_2}((1-\delta)x_2 v)}{\overline{F_2}(x_2 v)} \\
&\quad \times \sum_{i=1}^{n}\sum_{j=1}^{m} \int_1^\infty \int_1^\infty \overline{F_1}(x_1 u)\overline{F_2}(x_2 v) P(e^{r\tau_i^{(1)}} \in du, e^{r\tau_j^{(2)}} \in dv).
\end{aligned}$$

Noting that $F_1, F_2 \in \mathscr{C}$ and the arbitrariness of ε, we conclude

$$J_1 \lesssim \sum_{i=1}^{n}\sum_{j=1}^{m} P\big(X_i^{(1)} e^{-r\tau_i^{(1)}} > x_1, X_j^{(2)} e^{-r\tau_j^{(2)}} > x_2\big).$$

Next we deal with J_2. By Lemmas 5.1 and 4.3 and (2), it holds for large enough x_1, x_2, m and n that

$$\begin{aligned}
J_2 &\leq P\Big(\sum_{i=1}^{\infty} X_i^{(1)} e^{-r\tau_i^{(1)}} > (1-\delta)x_1, \sum_{j=m+1}^{\infty} X_j^{(2)} e^{-r\tau_j^{(2)}} > \delta x_2\Big) \\
&\leq \varepsilon \overline{F_1}((1-\delta)x_1)\overline{F_2}(\delta x_2) \\
&\leq \frac{\varepsilon}{1-\varepsilon} \cdot \frac{\overline{F_1}((1-\delta)x_1)\overline{F_2}(\delta x_2)}{\sum_{i=1}^n \sum_{j=1}^m P\big(X_i^{(1)} e^{-r\tau_i^{(1)}} > (1-\delta)x_1, X_j^{(2)} e^{-r\tau_j^{(2)}} > (1-\delta)x_2\big)} \cdot J_1 \\
&\leq \frac{\varepsilon}{1-\varepsilon} \cdot \frac{\overline{F_1}((1-\delta)x_1)\overline{F_2}(\delta x_2)}{P\big(X_1^{(1)} e^{-r\tau_1^{(1)}} > (1-\delta)x_1, X_1^{(2)} e^{-r\tau_1^{(2)}} > (1-\delta)x_2\big)} \cdot J_1 \\
&= \frac{\varepsilon}{1-\varepsilon} \cdot \frac{\overline{F_1}((1-\delta)x_1)\overline{F_2}(\delta x_2)}{\int_1^\infty \int_1^\infty \overline{F_1}((1-\delta)x_1 u)\overline{F_2}((1-\delta)x_2 v) P(e^{r\tau_1^{(1)}} \in du, e^{r\tau_1^{(2)}} \in dv)} \cdot J_1 \\
&\leq \frac{\varepsilon}{1-\varepsilon} \cdot \frac{1}{C^{-2}\int_1^\infty \int_1^\infty u^{-p_2}\big(\frac{(1-\delta)v}{\delta}\big)^{-p_2} P(e^{r\tau_1^{(1)}} \in du, e^{r\tau_1^{(2)}} \in dv)} \cdot J_1
\end{aligned}$$

$$= \frac{\varepsilon C^2 \delta^{p_2}}{(1-\varepsilon)(1-\delta)^{p_2} E e^{-rp_2(\tau_1^{(1)}+\tau_1^{(2)})}} \cdot J_1, \tag{17}$$

this implies $J_2 = o(J_1)$ due to $Ee^{-rp_2(\tau_1^{(1)}+\tau_1^{(2)})} > 0$ and the arbitrariness of ε. For J_3 and J_4, one can get the similar results by using the same arguments as above. Plugging all these relations back and letting $m, n \to \infty$ lead to

$$\begin{aligned} & P\Big(\sum_{i=1}^{\infty} X_i^{(1)} e^{-r\tau_i^{(1)}} > x_1, \sum_{j=1}^{\infty} X_j^{(2)} e^{-r\tau_j^{(2)}} > x_2\Big) \\ \lesssim & \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} P\big(X_i^{(1)} e^{-r\tau_i^{(1)}} > x_1, X_j^{(2)} e^{-r\tau_j^{(2)}} > x_2\big). \end{aligned}$$

This ends the proof of asymptotic upper bound. Now we turn to the proof of asymptotic lower bound. Indeed, by Lemma 4.3, it holds that

$$\begin{aligned} & P\Big(\sum_{i=1}^{\infty} X_i^{(1)} e^{-r\tau_i^{(1)}} > x_1, \sum_{j=1}^{\infty} X_j^{(2)} e^{-r\tau_j^{(2)}} > x_2\Big) \\ \geq\ & P\Big(\sum_{i=1}^{n} X_i^{(1)} e^{-r\tau_i^{(1)}} > x_1, \sum_{j=1}^{m} X_j^{(2)} e^{-r\tau_j^{(2)}} > x_2\Big) \\ \sim\ & \sum_{i=1}^{n}\sum_{j=1}^{m} P(X_i^{(1)} e^{-r\tau_i^{(1)}} > x_1, X_j^{(2)} e^{-r\tau_j^{(2)}} > x_2) \\ =\ & \Big[\sum_{i=1}^{\infty}\sum_{j=1}^{\infty} - \big(\sum_{i=1}^{\infty}\sum_{j=1}^{\infty} - \sum_{i=1}^{n}\sum_{j=1}^{m}\big)\Big] P(X_i^{(1)} e^{-r\tau_i^{(1)}} > x_1, X_j^{(2)} e^{-r\tau_j^{(2)}} > x_2\big). \end{aligned} \tag{18}$$

Since

$$\begin{aligned} & \Big(\sum_{i=1}^{\infty}\sum_{j=1}^{\infty} - \sum_{i=1}^{n}\sum_{j=1}^{m}\Big) P(X_i^{(1)} e^{-r\tau_i^{(1)}} > x_1, X_j^{(2)} e^{-r\tau_j^{(2)}} > x_2\big) \\ \leq\ & \Big(\sum_{i=1}^{\infty}\sum_{j=m+1}^{\infty} + \sum_{i=n+1}^{\infty}\sum_{j=1}^{\infty}\Big) P(X_i^{(1)} e^{-r\tau_i^{(1)}} > x_1, X_j^{(2)} e^{-r\tau_j^{(2)}} > x_2\big) =: L_1 + L_2, \end{aligned}$$

applying Lemma 5.2 and using the same treatment as in (17) yield that as $x_1, x_2, m, n \to \infty$,

$$L_k = o(1)\overline{F_1}(x_1)\overline{F_2}(y) = o(1)\sum_{i=1}^{n}\sum_{j=1}^{m} P(X_i^{(1)} e^{-r\tau_i^{(1)}} > x_1, X_j^{(2)} e^{-r\tau_j^{(2)}} > x_2), \quad k = 1, 2. \tag{19}$$

(18) and (19) imply that

$$\begin{aligned} & P\Big(\sum_{i=1}^{\infty} X_i^{(1)} e^{-r\tau_i^{(1)}} > x_1, \sum_{j=1}^{\infty} X_j^{(2)} e^{-r\tau_j^{(2)}} > x_2\Big) \\ \gtrsim & \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} P(X_i^{(1)} e^{-r\tau_i^{(1)}} > x_1, X_j^{(2)} e^{-r\tau_j^{(2)}} > x_2\big). \end{aligned}$$

Thus, the proof is now completed.

§6 Proof of Theorem 3.3

In this section, we aim to prove Theorem 3.3. Let us prepare two lemmas for later use. Lemma 6.1 is a property of renewal process. Lemma 6.2 plays a key role in proving Theorem 3.3.

Lemma 6.1. *Let $\{(\theta^{(1)},\theta^{(2)}),(\theta_i^{(1)},\theta_i^{(2)}),i\geq 1\}$ be a sequence of i.i.d. nonnegative random vectors. For $k=1,2$, suppose that $\{\theta_i^{(k)},i\geq 1\}$ drive a renewal process $\{N_k(t),t\geq 0\}$. If $\theta^{(1)}$ and $\theta^{(2)}$ are NQD, then, for all $t\in\Lambda$, it holds that*

$$E[N_1(t)N_2(t)]\leq EN_1(t)EN_2(t).$$

Proof. For $k=1,2$, let $\{\theta_i^{*(k)},i\geq 1\}$ be independent copies of $\{\theta_i^{(k)},i\geq 1\}$ and satisfy $\{\theta_i^{*(1)},i\geq 1\}$ and $\{\theta_i^{*(2)},i\geq 1\}$ are independent. Note that

$$\begin{aligned}E[N_1(t)N_2(t)] &= \sum_{n=1}^{\infty}\sum_{m=1}^{\infty}P(\tau_n^{(1)}\leq t,\tau_m^{(2)}\leq t)\\ &= \Big(\sum_{n=1}^{\infty}\sum_{m=1}^{n}+\sum_{n=1}^{\infty}\sum_{m=n+1}^{\infty}\Big)P\Big(\sum_{i=1}^{n}\theta_i^{(1)}\leq t,\sum_{j=1}^{m}\theta_j^{(2)}\leq t\Big)\\ &:= Q_1(t)+Q_2(t). \end{aligned}\tag{20}$$

Since $\theta^{(1)}$ and $\theta^{(2)}$ are NQD, it is equivalent to $P(\theta^{(1)}\leq x,\theta^{(2)}\leq y)\leq P(\theta^{(1)}\leq x)P(\theta^{(2)}\leq y)$ for any $x,y\in\mathbb{R}$. Thus, for $Q_1(t)$, we have

$$\begin{aligned}Q_1(t) &= \sum_{n=1}^{\infty}\sum_{m=1}^{n}\int_{0-}^{t}\int_{0-}^{t}P\big(\theta_1^{(1)}\leq t-s^{(1)},\theta_1^{(2)}\leq t-s^{(2)}\big)\\ &\quad\times P\Big(\sum_{i=2}^{n}\theta_i^{(1)}\in ds^{(1)},\sum_{j=2}^{m}\theta_j^{(2)}\in ds^{(2)}\Big)\\ &\leq \sum_{n=1}^{\infty}\sum_{m=1}^{n}\int_{0-}^{t}\int_{0-}^{t}P(\theta_1^{(1)}\leq t-s^{(1)})P(\theta_1^{(2)}\leq t-s^{(2)})\\ &\quad\times P\Big(\sum_{i=2}^{n}\theta_i^{(1)}\in ds^{(1)},\sum_{j=2}^{m}\theta_j^{(2)}\in ds^{(2)}\Big)\\ &\leq \sum_{n=1}^{\infty}\sum_{m=1}^{n}P\Big(\theta_1^{*(1)}+\sum_{i=2}^{n}\theta_i^{(1)}\leq t,\theta_1^{*(2)}+\sum_{j=2}^{m}\theta_j^{(2)}\leq t\Big).\end{aligned}$$

Repeating the above steps for a limited number of times leads to

$$\begin{aligned}Q_1(t) &\leq \sum_{n=1}^{\infty}\sum_{m=1}^{n}P\Big(\sum_{i=1}^{m}\theta_i^{*(1)}+\sum_{i=m+1}^{n}\theta_i^{(1)}\leq t\Big)P\Big(\sum_{j=1}^{m}\theta_j^{*(2)}\leq t\Big)\\ &= \sum_{n=1}^{\infty}\sum_{m=1}^{n}P\Big(\sum_{i=1}^{n}\theta_i^{*(1)}\leq t\Big)P\Big(\sum_{j=1}^{m}\theta_j^{*(2)}\leq t\Big).\end{aligned}\tag{21}$$

The same treatment applying to $Q_2(t)$ also yields that

$$Q_2(t)\leq \sum_{n=1}^{\infty}\sum_{m=n+1}^{\infty}P\Big(\sum_{i=1}^{n}\theta_i^{*(1)}\leq t\Big)P\Big(\sum_{j=1}^{m}\theta_j^{*(2)}\leq t\Big).\tag{22}$$

Plugging (21) and (22) back into (20), the desired result is achieved immediately.

Lemma 6.2. *Under the conditions of Theorem 3.3, it holds that*

$$\lim_{t\to\infty}\lim_{(x_1,x_2)\to(\infty,\infty)}\int_{0-}^{t}\int_{0-}^{t}\overline{F_1}(x_1e^{rs_1})\overline{F_2}(x_2e^{rs_2})dE[N_1(s_1)N_2(s_2)]\asymp\overline{F_1}(x)\overline{F_2}(y) \tag{23}$$

and

$$\begin{aligned}&\lim_{t\to\infty}\lim_{(x_1,x_2)\to(\infty,\infty)}\frac{\int_t^\infty\int_{0-}^\infty\overline{F_1}(x_1e^{rs_1})\overline{F_2}(x_2e^{rs_2})dE[N_1(s_1)N_2(s_2)]}{\overline{F_1}(x_1)\overline{F_2}(x_2)}\\ =\ &\lim_{t\to\infty}\lim_{(x_1,x_2)\to(\infty,\infty)}\frac{\int_{0-}^\infty\int_t^\infty\overline{F_1}(x_1e^{rs_1})\overline{F_2}(x_2e^{rs_2})dE[N_1(s_1)N_2(s_2)]}{\overline{F_1}(x_1)\overline{F_2}(x_2)}=0.\end{aligned} \tag{24}$$

Proof. Arbitrarily choose two positive constant p_1 and p_2 satisfying $0<p_1<J_{F_1}^-\wedge J_{F_2}^-\le J_{F_1}^+\vee J_{F_2}^+<p_2<\infty$. Then, for large enough x_1,x_2, it follows from Theorem 3.2 and (2) that

$$\begin{aligned}&\limsup_{t\to\infty}\lim_{(x_1,x_2)\to(\infty,\infty)}\frac{\int_{0-}^t\int_{0-}^t\overline{F_1}(x_1e^{rs_1})\overline{F_2}(x_2e^{rs_2})dE[N_1(s_1)N_2(s_2)]}{\overline{F_1}(x_1)\overline{F_2}(x_2)}\\ =\ &\lim_{(x_1,x_2)\to(\infty,\infty)}\frac{\sum_{i=1}^\infty\sum_{j=1}^\infty P(X_i^{(1)}e^{-r\tau_i^{(1)}}>x_1,X_j^{(2)}e^{-r\tau_j^{(2)}}>x_2)}{\overline{F_1}(x_1)\overline{F_2}(x_2)}\\ \le\ &\lim_{(x_1,x_2)\to(\infty,\infty)}\sum_{i=1}^\infty\sum_{j=1}^\infty\int_1^\infty\int_1^\infty\frac{\overline{F_1}(x_1u)\overline{F_2}(x_2v)}{\overline{F_1}(x_1)\overline{F_2}(x_2)}P(e^{r\tau_i^{(1)}}\in du,e^{r\tau_j^{(2)}}\in dv)\\ \le\ &C^2\sum_{i=1}^\infty\sum_{j=1}^\infty\int_1^\infty\int_1^\infty(uv)^{-p_1}P(e^{r\tau_i^{(1)}}\in du,e^{r\tau_j^{(2)}}\in dv)\\ \le\ &C^2\sum_{i=1}^\infty(E(e^{-2rp_1\tau_1^{(1)}}))^{i/2}\sum_{j=1}^\infty(E(e^{-2rp_1\tau_1^{(2)}}))^{j/2}\\ <\ &\infty.\end{aligned}$$

Conversely,

$$\begin{aligned}&\limsup_{t\to\infty}\lim_{(x_1,x_2)\to(\infty,\infty)}\frac{\overline{F_1}(x_1)\overline{F_2}(x_2)}{\int_{0-}^t\int_{0-}^t\overline{F_1}(x_1e^{rs_1})\overline{F_2}(x_2e^{rs_2})dE[N_1(s_1)N_2(s_2)]}\\ =\ &\lim_{(x_1,x_2)\to(\infty,\infty)}\frac{\overline{F_1}(x_1)\overline{F_2}(x_2)}{\sum_{i=1}^\infty\sum_{j=1}^\infty P(X_i^{(1)}e^{-r\tau_i^{(1)}}>x_1,X_j^{(2)}e^{-r\tau_j^{(2)}}>x_2)}\\ \le\ &\lim_{(x_1,x_2)\to(\infty,\infty)}\frac{\overline{F_1}(x_1)\overline{F_2}(x_2)}{\int_1^\infty\int_1^\infty\overline{F_1}(x_1u)\overline{F_2}(x_2v)P(e^{r\tau_1^{(1)}}\in du,e^{r\tau_1^{(2)}}\in dv)}\\ \le\ &\lim_{(x_1,x_2)\to(\infty,\infty)}\frac{1}{C^{-2}\int_1^\infty\int_1^\infty u^{-p_2}v^{-p_2}P(e^{r\tau_1^{(1)}}\in du,e^{r\tau_1^{(2)}}\in dv)}\\ \le\ &\frac{C^2}{Ee^{-rp_2(\tau_1^{(1)}+\tau_1^{(2)})}}\\ <\ &\infty.\end{aligned}$$

Thus, the desired (23) is verified.

Next, we turn to prove (24). Similarly done as above, by Lemma 6.1, it holds that

$$\begin{aligned}
&\frac{\int_t^\infty\int_{0-}^\infty \overline{F_1}(x_1e^{rs_1})\overline{F_2}(x_2e^{rs_2})dE[N_1(s_1)N_2(s_2)]}{\overline{F_1}(x_1)\overline{F_2}(x_2)}\\
\leq\ &\frac{\int_t^\infty\int_{0-}^\infty \overline{F_1}(x_1e^{rs_1})\overline{F_2}(x_2e^{rs_2})dEN_1(s_1)dEN_2(s_2)}{\overline{F_1}(x_1)\overline{F_2}(x_2)}\\
\leq\ &C^{-2}\int_t^\infty e^{-rp_1s_1}dEN_1(s_1)\int_{0-}^\infty e^{-rp_1s_2}dEN_2(s_2). \qquad (25)
\end{aligned}$$

Note that, for $k=1,2$,

$$\int_{0-}^\infty e^{-rp_1t}dEN_k(t)=\sum_{i=1}^\infty\int_{0-}^\infty e^{-rp_1t}dP(\tau_i^{(k)}\in dt)=\sum_{i=1}^\infty Ee^{-rp_1\tau_i^{(k)}}=\sum_{i=1}^\infty\left(Ee^{-rp_1\theta_1^{(k)}}\right)^i<\infty,$$

which together with (25) implies the desired (24) and this ends the proof.

Proof of Theorem 3.3. Following the proof method of Theorem 2.2 in Hao and Tang (2008), in order to prove Theorem 3.3, it suffices to prove the relation (3) holds uniformly for all $t\in(T,\infty]$ for some fixed and sufficiently large $T\in\Lambda$ since the relation (3) holds uniformly for all $t\in\Lambda_T$ due to Theorem 3.1. First, according to Lemma 6.2, we easily see that

$$\lim_{T\to\infty}\lim_{(x_1,x_2)\to(\infty,\infty)}\frac{\int_T^\infty\int_{0-}^\infty \overline{F_1}(x_1e^{rs_1})\overline{F_2}(x_2e^{rs_2})dE[N_1(s_1)N_2(s_2)]}{\int_{0-}^T\int_{0-}^T \overline{F_1}(x_1e^{rs_1})\overline{F_2}(x_2e^{rs_2})dE[N_1(s_1)N_2(s_2)]}=0, \qquad (26)$$

and

$$\lim_{T\to\infty}\lim_{(x_1,x_2)\to(\infty,\infty)}\frac{\int_{0-}^\infty\int_T^\infty \overline{F_1}(x_1e^{rs_1})\overline{F_2}(x_2e^{rs_2})dE[N_1(s_1)N_2(s_2)]}{\int_{0-}^T\int_{0-}^T \overline{F_1}(x_1e^{rs_1})\overline{F_2}(x_2e^{rs_2})dE[N_1(s_1)N_2(s_2)]}=0. \qquad (27)$$

For any $\varepsilon>0$, $t\in(T,\infty]$ and sufficiently large x_1,x_2, by Theorem 3.2 and the relations (26) and (27), we obtain

$$\begin{aligned}
\psi_{\max}(x_1,x_2;t)\ &\leq\ \psi_{\max}(x_1,x_2)\\
&\sim\ \int_{0-}^\infty\int_{0-}^\infty \overline{F_1}(x_1e^{rs_1})\overline{F_2}(x_2e^{rs_2})dE[N_1(s_1)N_2(s_2)]\\
&\leq\ \Big(\int_{0-}^t\int_{0-}^t+\int_T^\infty\int_{0-}^\infty+\int_{0-}^\infty\int_T^\infty\Big)\overline{F_1}(x_1e^{rs_1})\overline{F_2}(x_2e^{rs_2})dE[N_1(s_1)N_2(s_2)]\\
&\leq\ (1+2\varepsilon)\int_{0-}^t\int_{0-}^t \overline{F_1}(x_1e^{rs_1})\overline{F_2}(x_2e^{rs_2})dE[N_1(s_1)N_2(s_2)], \qquad (28)
\end{aligned}$$

and

$$\begin{aligned}
\psi_{\max}(x_1,x_2;t)\ &\geq\ \psi_{\max}(x_1,x_2;T)\\
&\sim\ \int_{0-}^T\int_{0-}^T \overline{F_1}(x_1e^{rs_1})\overline{F_2}(x_2e^{rs_2})dE[N_1(s_1)N_2(s_2)]\\
&\geq\ \Big(\int_{0-}^t\int_{0-}^t-\int_T^\infty\int_{0-}^\infty-\int_{0-}^\infty\int_T^\infty\Big)\overline{F_1}(x_1e^{rs_1})\overline{F_2}(x_2e^{rs_2})dE[N_1(s_1)N_2(s_2)]\\
&\leq\ (1-2\varepsilon)\int_{0-}^t\int_{0-}^t \overline{F_1}(x_1e^{rs_1})\overline{F_2}(x_2e^{rs_2})dE[N_1(s_1)N_2(s_2)]. \qquad (29)
\end{aligned}$$

Therefore, by (28), (29) and the arbitrariness of ε, we obtain the uniformity of (3) for all $t\in(T,\infty]$. The proof of Theorem 3.3 is now complete.

Acknowledgements

The authors would like to thank the anonymous referees for their insightful suggestions which have helped us improve the paper greatly.

Declarations

Conflict of interest The authors declare no conflict of interest.

References

[1] A V Asimit, B L Jones. *Dependence and the asymptotic behavior of large claims reinsurance*, Insur Math Econom, 2008, 43: 407-411.

[2] N H Bingham, C M Goldie, J L Teugel. *Regular Variation*, Cambridge University Press, Cambridge, 1987.

[3] Chan W, Yang H, Zhang L. *Some results on ruin probabilities in a two-dimensional risk model*, Insur Math Econom, 2003, 32: 345-358.

[4] F Cheng, D Cheng. *Randomly weighted sums of dependent subexponential random variables with applications to risk theory*, 2018, Scand Actuar J, 2018, 3: 191-202.

[5] Y Chen, L Wang, Y Wang. *Uniform asymptotics for the finite-time ruin probabilities of two kinds of nonstandard bidimensional risk models*, J Math Anal Appl, 2013, 401: 114-129.

[6] D Cheng, C Yu. *Asymptotics for the ruin probabilities of a two-dimensional renewal risk model*, Dynamic System Appl, 2017, 26: 517-534.

[7] Y Chen, Y Yang, T Jiang. *Uniform asymptotics for finite-time ruin probability of a bidimensional risk model*, J Math Anal Appl, 2019, 469: 525-536.

[8] Y Chen, K Yuen, K W Ng. *Asymptotics for the ruin probabilities of a two-dimensional renewal risk model with heavy-tailed claim*, Appl Stoch Models Bus Ind, 2011, 27: 290-300.

[9] D B H Cline, G Samorodnitsky. *Subexponentiality of product of of independent random variables*, Stoch Proc Appl, 1994, 49: 75-98.

[10] Embrechts P, C Klüppelberg, T Mikosch. *Modelling extremal events for insurance and finance*, Berlin: Springer, 1997.

[11] S Foss, D Korshunov, S Zachary. *An introduction to heavy-tailed and subexponential distributions (second edition)*, New York: Springer, 2013.

[12] Y Gong, Y Yang. *On the Kesten-type inequality for randomly weighted sums with applications to an operational risk model*, to appear in Filomat, 2020.

[13] X Hao, Q Tang. *A uniform asymptotic estimate for discounted aggregate claims with subexpoential tails*, Insur Math Econom, 2008, 43: 116-20.

[14] E Lehmann. *Some concepts of dependence*, Ann Math Stat, 1966, 37: 1137-1153.

[15] J Li. *A revisit to asymptotic ruin probabilities for a bidimensional renewal risk model*, Statist Probab Lett, 2018, 140: 23-32.

[16] D Lu, B Zhang. *Some asymptotic results of the ruin probabilities in a two-dimensional renewal risk model with some strongly subexponential claims*, Statist Probab Lett, 2016, 114: 20-29.

[17] J Li, Z Liu, Q Tang. *On the ruin probabilities of a bidimensional perturbed risk model*, Insur Math Econom, 2007, 41: 85-195.

[18] B Ko, Q Tang. *Sums of dependent nonnegative random variables with subexponential tail*, J Appl Probab, 2008, 45: 5-95.

[19] Q Tang, G Tsitsiashvili. *Precise estimates for the ruin probability in finite horizon in a discrete-time model with heavy-tailed insurance and financial risks*, Stoch Proc Appl, 2003, 108(2): 299-325.

[19] Q Tang, G Tsitsiashvili. *Randomly weighted sums of subexponential random variables with application to ruin theory*, Extremes, 2003, 6: 171-188.

[20] K Wang, Y Wang, Q Gao. *Uniform asymptotics for the finite-time ruin probability of a dependent risk model with a constant interest rate*, Methodol Comput Appl Probab, 2013, 15: 109-124.

[21] H Yang, J Li. *Asymptotic finite-time ruin probability for a bidimensional renewal risk model with constant interest force and dependent subexponential claims*, Insur Math Econom, 2014, 58: 185-192.

[22] H Yang, J Li. *Asymptotic ruin probabilities for a bidimensional renewal risk model*, Stoch, 2017, 89: 687-708.

[23] Y Yang, J Lin, C Huang, X Ma. *The finite-time ruin probability in two non-standard renewal risk models with constant interest rate and dependent subexponential claims*, J Korean Stat Soc, 2012, 41(2): 213-224.

[24] Y Yang , K Wang, J Liu, Z Zhang. *Asymptotics for a bidimensional risk model with two geometric Lévy price processes*, J Ind Manag Optim, 2019, 15(2): 481-505.

[25] Y Yang, X Wang, Z Zhang. *Finite-time ruin probability of a perturbed risk model with dependent main and delayed claims*, Nonlinear Analysis Modelling and Control, 2021, 26(5):801-820.

[26] Y Yang, K C Yuen. *Finite-time and infinite-time ruin probabilities in a two-dimensional delayed renewal risk model with Sarmanov dependent claims*, J Math Anal Appl, 2016, 442(2): 600-626.

[27] Y Yang, K C Yuen, J Liu. *Asymptotics for ruin probabilities in Lévy-driven risk models with heavy-tailed claims*, J Ind Manag Optim, 2018, 14(1): 231-247.

[28] K C Yuen, J Guo, X Wu. *On the first time ruin in bivariate compound Poisson model*, Insur Math Econom, 2006, 38: 298-308.

[29] Y Zhang, F Cheng. *Asymptotic tail behavior of a random sum with conditionally dependent subexponential summands*, Comm Statist Theory Methods, 2016, 46(12): 5888-5895.

[1]School of Mathematics and Statistics, Central South University, Changsha 410083, China.

[2]School of Big Data and Statistics, Anhui University, Hefei 230601, China.

Email: ahuwsj@126.com