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Equivalence between the internal observability and exponential decay for the Moore-Gibson-Thompson equation

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Abstract. This paper is concerned with a third order in time linear Moore-Gibson-Thompson equation which describes the acoustic velocity potential in ultrasound wave program. Influenced by the work of Kaltenbacher, Lasiecka and Marchand (Control Cybernet. 2011, 40: 971-988), we establish an observability inequality of the conservative problem, and then discuss the equivalence between the exponential stabilization of a dissipative system and the internal observational inequality of the corresponding conservative system.

§1 Introduction

The Moore-Gibson-Thompson (MGT) equation is known as the linearization of the Jordan-Moore-Gibson-Thompson equation, which is driven by a wide range of applications such as the medical and industrial use of high-intensity ultrasound in lithotripsy, thermotherapy, ultrasound cleaning and sonochemistry. The original derivation dates back to Jordan [10], Stokes [25], Moore and Gibson [18] and Thompson [27]. We refer to Lasiecka et al. [1,3,12–14] for a helpful background on the subject and a deep list of references for the physical motivations of MGT models.

Let $(H, (\cdot,\cdot), \|\cdot\|)$ be a real Hilbert space and let $\mathcal{A}$ be a strictly positive self-adjoint linear operator on H with a dense domain $\mathcal{D}(\mathcal{A}) \subset H$. We consider the following abstract MGT equation

$$\tau u_{ttt} + \alpha u_{tt} + c^2 \mathcal{A} u + b\mathcal{A} u_t = 0, \tag{1.1}$$

with the initial conditions

$$u(0) = u_0,\ u_t(0) = u_1,\ u_{tt}(0) = u_2, \tag{1.2}$$

where τ, α, c, b are strictly positive physical parameters inherited from modeling process, functions u_0, u_1 and u_2 are prescribed data.

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In 2011, Kaltenbacher, Lasiecka and Marchand [11] first discussed MGT equation (1.1). Setting the parameter $\gamma := \alpha - \frac{c^2\tau}{b} \geq 0$ and defining the energies

$$E_0(t) = \frac{\alpha}{2}\|u_t\|^2 + \frac{c^2}{2}\left\|\mathcal{A}^{\frac{1}{2}}u\right\|^2,$$

$$E(t) = \frac{\tau}{2}\left\|u_{tt} + \frac{c^2}{b}u_t\right\|^2 + \frac{b}{2}\left\|\mathcal{A}^{\frac{1}{2}}\left(u_t + \frac{c^2}{b}u\right)\right\|^2 + \frac{c^2}{2b}\gamma\|u_t\|^2,$$

they showed that when $\gamma > 0$, problem (1.1)-(1.2) is well-posed and the total energy $\hat{E} = E + E_0$ is exponentially stable; while when $\gamma = 0$, $E(u, u_t, u_{tt}; t)$ is conserved. After this seminal work, an increasing interest has been developed to study the MGT equation [2,4,5,14–17,22] or other related equations [6–8,19–21].

In 2016, Lasiecka and Wang [13, Section 2] gave a slightly different proof of the above exponential stable result. For $\frac{c^2}{b} < k < \frac{\alpha}{\tau}$, they defined the energy of problem (1.1)-(1.2) as

$$\begin{aligned}\mathcal{E}(t) =& \frac{\tau}{2}\left\|u_{tt} + ku_t\right\|^2 + \frac{b}{2}\left\|\mathcal{A}^{\frac{1}{2}}\left(u_t + \frac{c^2}{b}u\right)\right\|^2 \\ &+ \frac{k\tau}{2}\left(\frac{\alpha}{\tau} - k\right)\|u_t\|^2 + \frac{c^2}{2}\left(k - \frac{c^2}{b}\right)\left\|\mathcal{A}^{\frac{1}{2}}u\right\|^2,\end{aligned} \tag{1.3}$$

and showed that

$$\mathcal{E}(t) \sim F(t) := \|u_{tt}\|^2 + \left\|\mathcal{A}^{\frac{1}{2}}u_t\right\|^2 + \left\|\mathcal{A}^{\frac{1}{2}}u\right\|^2, \tag{1.4}$$

where $F(t)$ is the norm generated by phase space $\mathcal{H}$ which will be defined in Section 2. Consequently, we can get from [11,13] that

$$\mathcal{E}(t) \sim F(t) \sim \hat{E}(t)$$

and there exist $\omega > 0$, $M > 0$ such that $\mathcal{E}(u, u_t, u_{tt}; t)$ satisfies, for all $t > 0$,

$$\mathcal{E}(t) \leq Me^{-\omega t}\mathcal{E}(0). \tag{1.5}$$

Motivated by the stabilization of problem (1.1)-(1.2) and some pioneer works of Haraux [9], Tebou [26] and Ramos et al. [23,24], we are interested in considering an internal observability inequality and establishing the equivalence between exponential stabilization and observability of MGT equation. Although the model discussed in this paper is the Moore-Gibson-Thompson equation, we believe that this method can be extended to other models with appropriate modifications.

The plan of this paper is as follows. In the next section, we recall some basic assumptions and state our main result Theorem 2.1. In Section 3, we prove the inequality of observability by the multiplier method. The proof of our main result is established in Section 4.

§2 Preliminary and main result

In this section, we shall present some assumptions and state the main result of the problem on a phase space $\mathcal{H}$ which will be defined below. Throughout this paper, we use C to denote generic positive constants, the values of which may change from one line to the next, unless we give a special declaration. To the solution trajectory, we define the phase space

$$\mathcal{H} = \mathcal{D}\left(\mathcal{A}^{\frac{1}{2}}\right) \times \mathcal{D}\left(\mathcal{A}^{\frac{1}{2}}\right) \times H.$$

In order to study problem (1.1)-(1.2), we make the following assumptions:

(A1) The parameters in the equation satisfy

$$\frac{c^2}{b} \leq \frac{\alpha}{\tau}.$$

(A2) There exists $\lambda_0 > 0$ such that

$$\|w\|^2 \leq \lambda_0 \|\mathcal{A}^{\frac{1}{2}} w\|^2, \quad \forall\ w \in H.$$

If $\frac{c^2}{b} < \frac{\alpha}{\tau}$, we can pick up a constant k such that $\frac{c^2}{b} < k < \frac{\alpha}{\tau}$ and set the energy of problem (1.1)-(1.2) as

$$\begin{aligned}\mathcal{E}_u(t) =& \frac{\tau}{2} \|u_{tt} + k u_t\|^2 + \frac{b}{2} \left\| \mathcal{A}^{\frac{1}{2}} \left(u_t + \frac{c^2}{b} u \right) \right\|^2 \\ &+ \frac{k\tau}{2} \left(\frac{\alpha}{\tau} - k \right) \|u_t\|^2 + \frac{c^2}{2} \left(k - \frac{c^2}{b} \right) \left\| \mathcal{A}^{\frac{1}{2}} u \right\|^2,\end{aligned} \tag{2.1}$$

which satisfies

$$\frac{\mathrm{d}}{\mathrm{d}t} \mathcal{E}_u(t) = -\tau \left(\frac{\alpha}{\tau} - k \right) \|u_{tt}\|^2 - b \left(k - \frac{c^2}{b} \right) \left\| \mathcal{A}^{\frac{1}{2}} u_t \right\|^2. \tag{2.2}$$

Inspired by Ramos et al. [23], we consider the following conservative problem

$$\tau v_{ttt} + \frac{c^2}{b} \tau v_{tt} + c^2 \mathcal{A} v + b \mathcal{A} v_t = 0, \tag{2.3}$$

with the initial conditions

$$v(0) = v_0,\ v_t(0) = v_1,\ v_{tt}(0) = v_2, \tag{2.4}$$

when $\frac{c^2}{b} = \frac{\alpha}{\tau}$. In this case, the energy of conservative problem is given by

$$\mathcal{E}_v(t) = \frac{\tau}{2} \left\| v_{tt} + \frac{c^2}{b} v_t \right\|^2 + \frac{b}{2} \left\| \mathcal{A}^{\frac{1}{2}} \left(v_t + \frac{c^2}{b} v \right) \right\|^2, \tag{2.5}$$

which satisfies the conservative law

$$\frac{\mathrm{d}}{\mathrm{d}t} \mathcal{E}_v(t) = 0. \tag{2.6}$$

Now we consider $z(x,t) = u(x,t) - v(x,t)$, where $z(x,t)$ is the solution of the auxiliary problem

$$\tau z_{ttt} + \left(\alpha - \frac{c^2}{b} \tau \right) u_{tt} + \frac{c^2}{b} \tau z_{tt} + c^2 \mathcal{A} z + b \mathcal{A} z_t = 0, \tag{2.7}$$

with the initial conditions

$$z(0) = z_t(0) = z_{tt}(0) = 0, \tag{2.8}$$

for the case $\frac{c^2}{b} < \frac{\alpha}{\tau}$. It is easy to see that the energy associated with system (2.7)-(2.8) is in the form of

$$\mathcal{E}_z(t) = \frac{\tau}{2} \left\| z_{tt} + \frac{c^2}{b} z_t \right\|^2 + \frac{b}{2} \left\| \mathcal{A}^{\frac{1}{2}} \left(z_t + \frac{c^2}{b} z \right) \right\|^2, \tag{2.9}$$

which gives

$$\frac{\mathrm{d}}{\mathrm{d}t} \mathcal{E}_z(t) = -\left(\alpha - \frac{c^2 \tau}{b} \right) \left(u_{tt}, z_{tt} + \frac{c^2}{b} z_t \right). \tag{2.10}$$

Our main result reads as follows.

Theorem 2.1. *(Equivalence between stabilization and observability). The following estimates are equivalent.*

(i) Suppose that $\frac{c^2}{b} = \frac{\alpha}{\tau}$. There exist $C(T) > 0$ and $T > T_0$ such that for all $(v_0, v_1, v_2) \in \mathcal{H}$, we have

$$\mathcal{E}_v(0) \leq C(T) \int_0^T \left(\|v_{tt}(t)\|^2 + \left\| \mathcal{A}^{\frac{1}{2}} v_t \right\|^2 \right) \mathrm{d}t. \tag{2.11}$$

(ii) Suppose that $\frac{c^2}{b} < \frac{\alpha}{\tau}$. *There exist* $M > 0$ *and* $\omega > 0$ *such that for all* $(u_0, u_1, u_2) \in \mathcal{H}$, *we have*

$$\mathcal{E}_u(t) \le Me^{-\omega t}\mathcal{E}_u(0), \quad \forall\ t > 0. \tag{2.12}$$

§3 Internal observability

In this section, we will prove the inequality of internal observability of system (2.3)-(2.4) by using the multiplier method.

Theorem 3.1. *Suppose that* $\frac{c^2}{b} = \frac{\alpha}{\tau}$ *and let* $(v_0, v_1, v_2) \in \mathcal{H}$. *Then for all* $T > 3\sigma_1$, *there exists* $C(T) > 0$ *such that the following estimate holds true*

$$\mathcal{E}_v(0) \le C(T) \int_0^T \left(\|v_{tt}(t)\|^2 + \left\| \mathcal{A}^{\frac{1}{2}} v_t(t) \right\|^2 \right) \mathrm{d}t, \tag{3.1}$$

where $C(T) = \frac{\sigma_2}{2(T-3\sigma_1)}$, $\sigma_1 = \max\left\{ \frac{c^2\tau b + 3c^4\tau}{b^3}\lambda_0, \frac{2\lambda_0\tau + b}{b} \right\}$ *and* $\sigma_2 = \max\left\{ \tau, b + \frac{c^4\tau\lambda_0}{b^2} \right\}$.

Proof. Multiplying (2.3) by $\frac{c^2}{b}v$ and integrating by parts on $(0, T) \times H$, we have

$$\begin{aligned}
&\frac{c^2}{b}\left[\tau(v_{tt}, v) - \frac{\tau}{2}\|v_t\|^2 + \frac{c^2\tau}{b}(v_t, v) \right]_0^T \\
&+ \int_0^T \left(\frac{c^4}{b} \left\| \mathcal{A}^{\frac{1}{2}} v(t)) \right\|^2 + b\left(\mathcal{A}^{\frac{1}{2}} v_t(t), \frac{c^2}{b}\mathcal{A}^{\frac{1}{2}} v(t) \right) \right) \mathrm{d}t \\
=&\frac{c^4\tau}{b^2} \int_0^T \|v_t(t)\|^2 \mathrm{d}t,
\end{aligned}$$

then we rewrite this equality as

$$\begin{aligned}
&\frac{c^2}{b}\left[\tau(v_{tt}, v) + \frac{c^2\tau}{b}(v_t, v) - \left(\frac{\tau}{2} + \frac{c^2\tau}{b} \right) \|v_t\|^2 - \frac{c^2}{2}\left\| \mathcal{A}^{\frac{1}{2}} v \right\|^2 \right]_0^T \\
&+ \int_0^T \left(\tau \left\| v_{tt}(t) + \frac{c^2}{b}v_t(t) \right\|^2 + b\left\| \mathcal{A}^{\frac{1}{2}} \left(v_t(t) + \frac{c^2}{b}v(t) \right) \right\|^2 \right) \mathrm{d}t \\
=& \int_0^T \left(\tau \|v_{tt}(t)\|^2 + b\left\| \mathcal{A}^{\frac{1}{2}} v_t(t) \right\|^2 + \left(\frac{c^4\tau}{b^2} + \frac{c^4\tau}{b^2} \right) \|v_t(t)\|^2 \right) \mathrm{d}t.
\end{aligned} \tag{3.2}$$

From (3.2), we can get

$$\begin{aligned}
&\frac{c^2}{b}\left[\tau(v_{tt}, v) + \frac{c^2\tau}{b}(v_t, v) - \frac{\tau b + 2c^2\tau}{2b}\|v_t\|^2 - \frac{c^2}{2}\left\| \mathcal{A}^{\frac{1}{2}} v \right\|^2 \right]_0^T + 2\int_0^T \mathcal{E}_v(t)\mathrm{d}t \\
=& \int_0^T \left(\tau \|v_{tt}(t)\|^2 + b\left\| \mathcal{A}^{\frac{1}{2}} v_t(t) \right\|^2 + \frac{2c^4\tau}{b^2}\|v_t(t)\|^2 \right) \mathrm{d}t.
\end{aligned} \tag{3.3}$$

For the first term in the left-hand side of (3.3), we apply Young's inequality and (A2) to get

$$\frac{c^2}{b}\left| \tau(v_{tt}, v) + \frac{c^2\tau}{b}(v_t, v) - \frac{\tau b + 2c^2\tau}{2b}\|v_t\|^2 - \frac{c^2}{2}\left\| \mathcal{A}^{\frac{1}{2}} v \right\|^2 \right|$$

$$
\begin{aligned}
&\le \frac{c^2}{b}\left[\frac{\tau}{2}\frac{b}{c^2}\|v_{tt}\|^2+\frac{c^2\tau}{2b}\|v\|^2+\frac{c^2\tau}{2b}\|v_t\|^2+\frac{c^2\tau}{2b}\|v\|^2+\frac{\tau b+2c^2\tau}{2b}\|v_t\|^2+\frac{c^2}{2}\left\|\mathcal{A}^{\frac{1}{2}}v\right\|^2\right]\\
&\le \frac{c^2}{b}\left[\frac{\tau}{2}\frac{b}{c^2}\|v_{tt}\|^2+\frac{\tau b+3c^2\tau}{2b}\lambda_0\|\mathcal{A}^{\frac{1}{2}}v_t\|^2+\frac{2\lambda_0c^2\tau+c^2b}{2b}\left\|\mathcal{A}^{\frac{1}{2}}v\right\|^2\right] \qquad (3.4)\\
&=\frac{\tau}{2}\|v_{tt}\|^2+\frac{b}{2}\frac{c^2\tau b+3c^4\tau}{b^3}\lambda_0\left\|\mathcal{A}^{\frac{1}{2}}v_t\right\|^2+\frac{c^4}{2b}\frac{2\lambda_0\tau+b}{b}\left\|\mathcal{A}^{\frac{1}{2}}v\right\|^2,
\end{aligned}
$$

then it is easy to see that

$$
\begin{aligned}
&\frac{c^2}{b}\left|\tau(v_{tt},v)+\frac{c^2\tau}{b}(v_t,v)-\frac{\tau b+2c^2\tau}{2b}\|v_t\|^2-\frac{c^2}{2}\left\|\mathcal{A}^{\frac{1}{2}}v\right\|^2\right|\\
\le&3\max\left\{1,\frac{c^2\tau b+3c^4\tau}{b^3}\lambda_0,\frac{2\lambda_0\tau+b}{b}\right\}\mathcal{E}_v(t) \qquad (3.5)\\
=&3\max\left\{\frac{c^2\tau b+3c^4\tau}{b^3}\lambda_0,\frac{2\lambda_0\tau+b}{b}\right\}\mathcal{E}_v(t).
\end{aligned}
$$

Noting that $\mathcal{E}_v(t)=\mathcal{E}_v(0)$, we arrive at

$$
\frac{c^2}{b}\left[\tau(v_{tt},v)+\frac{c^2\tau}{b}(v_t,v)-\frac{\tau b+2c^2\tau}{2b}\|v_t\|^2-\frac{c^2}{2}\left\|\mathcal{A}^{\frac{1}{2}}v\right\|^2\right]_0^T\ge -6\sigma_1\mathcal{E}_v(0), \qquad (3.6)
$$

with $\sigma_1=\max\left\{\frac{c^2\tau b+3c^4\tau}{b^3}\lambda_0,\frac{2\lambda_0\tau+b}{b}\right\}$. If we now combine (3.3) and (3.6), then we obtain

$$
2\left(T-3\sigma_1\right)\mathcal{E}_v(0)\le\int_0^T\left(\tau\left\|v_{tt}(t)\right\|^2+\left(b+\frac{c^4\tau\lambda_0}{b^2}\right)\left\|\mathcal{A}^{\frac{1}{2}}v_t(t)\right\|^2\right)\mathrm{d}t. \qquad (3.7)
$$

Thus, it is easy to show that

$$
\mathcal{E}_v(0)\le\frac{\sigma_2}{2(T-3\sigma_1)}\int_0^T\left(\|v_{tt}(t)\|^2+\left\|\mathcal{A}^{\frac{1}{2}}v_t(t)\right\|^2\right)\mathrm{d}t, \qquad (3.8)
$$

where $\sigma_2=\max\left\{\tau,b+\frac{c^4\tau\lambda_0}{b^2}\right\}$. □

§4 Proof of Theorem 2.1

In this section, we will prove an equivalence between the stabilization of system (1.1)-(1.2) and the observability of the corresponding conservative system. The proof is based on appropriate decomposition of the solution and the energy method.

Proof of Theorem 2.1. *(i)* ⇒ *(ii)*. Noting that $u=v+z$, $(u_0,u_1,u_2)=(v_0,v_1,v_2)$ and $z(0)=z_t(0)=z_{tt}(0)=0$, we obtain

$$
\begin{aligned}
\mathcal{E}_z(t)=&-\left(\alpha-\frac{c^2\tau}{b}\right)\int_0^t\left(u_{tt}(s),z_{tt}(s)+\frac{c^2}{b}z_t(s)\right)\mathrm{d}s\\
\le&\frac{1}{2}\left(\alpha-\frac{c^2\tau}{b}\right)\int_0^t\left[\|u_{tt}(s)\|^2+\left\|v_{tt}(s)+\frac{c^2}{b}v_t(s)\right\|^2\right]\mathrm{d}s\\
&+\frac{1}{2}\left(\alpha-\frac{c^2\tau}{b}\right)\int_0^t\left[\|u_{tt}(s)\|^2+\left\|u_{tt}(s)+\frac{c^2}{b}u_t(s)\right\|^2\right]\mathrm{d}s
\end{aligned}
$$

$$\begin{aligned}
&\le\left(\alpha-\frac{c^2\tau}{b}\right)\int_0^t\left\|z_{tt}(s)+\frac{c^2}{b}z_t(s)\right\|^2\mathrm{d}s\\
&\quad+\left(\alpha-\frac{c^2\tau}{b}\right)\int_0^t\left[\|u_{tt}(s)\|^2+\frac{3}{2}\left\|u_{tt}(s)+\frac{c^2}{b}u_t(s)\right\|^2\right]\mathrm{d}s\\
&\le\frac{2}{\tau}\left(\alpha-\frac{c^2\tau}{b}\right)\int_0^t\mathcal{E}_z(s)\mathrm{d}s\\
&\quad+\frac{4b^2+3\lambda_0c^4}{b^2}\left(\alpha-\frac{c^2\tau}{b}\right)\int_0^t\left(\|u_{tt}(s)\|^2+\left\|\mathcal{A}^{\frac{1}{2}}u_t(s)\right\|^2\right)\mathrm{d}s,
\end{aligned}\tag{4.1}$$

where we have used (2.10), Young's inequality and (A2). Applying Gronwall's lemma to (4.1), we have, for all $0\le t\le T_0$,

$$\mathcal{E}_z(t)\le M_0\exp\left(\frac{2}{\tau}\left(\alpha-\frac{c^2\tau}{b}\right)T_0\right)\int_0^{T_0}\left(\|u_{tt}(s)\|^2+\left\|\mathcal{A}^{\frac{1}{2}}u_t(s)\right\|^2\right)\mathrm{d}s,\tag{4.2}$$

where $M_0=\frac{4b^2+3\lambda_0c^4}{b^2}\left(\alpha-\frac{c^2\tau}{b}\right)$. From (2.1), (2.5), (2.11) and (4.2), we arrive at, for some positive constant C_1,

$$\begin{aligned}
\mathcal{E}_u(0)&\le\left(1+\frac{bk}{c^2}\right)\mathcal{E}_v(0)\\
&\le\left(1+\frac{bk}{c^2}\right)C(T)\int_0^{T_0}\left(\|v_{tt}(t)\|^2+\left\|\mathcal{A}^{\frac{1}{2}}v_t(t)\right\|^2\right)\mathrm{d}t\\
&\le 2\left(1+\frac{bk}{c^2}\right)C(T)\int_0^{T_0}\left(\|u_{tt}(t)\|^2+\left\|\mathcal{A}^{\frac{1}{2}}u_t(t)\right\|^2+\|z_{tt}(t)\|^2+\left\|\mathcal{A}^{\frac{1}{2}}z_t(t)\right\|^2\right)\mathrm{d}t\\
&\le C_1\int_0^{T_0}\left(\|u_{tt}(t)\|^2+\left\|\mathcal{A}^{\frac{1}{2}}u_t(t)\right\|^2\right)\mathrm{d}t
\end{aligned}\tag{4.3}$$

where we have used the fact that $v=u-z$ together with Young's inequality.

On the other hand, from (2.2) we have

$$\begin{aligned}
&\tau\left(\frac{\alpha}{\tau}-k\right)\int_0^{T_0}\|u_{tt}(t)\|^2\mathrm{d}t+b\left(k-\frac{c^2}{b}\right)\int_0^{T_0}\left\|\mathcal{A}^{\frac{1}{2}}u_t(t)\right\|^2\mathrm{d}t\\
=&\mathcal{E}_u(0)-\mathcal{E}_u(T_0).
\end{aligned}$$

Then there exists some positive constant C_2 such that

$$\int_0^{T_0}\left(\|u_{tt}(t)\|^2+\left\|\mathcal{A}^{\frac{1}{2}}u_t(t)\right\|^2\right)\mathrm{d}t\le C_2\left(\mathcal{E}_u(0)-\mathcal{E}_u(T_0)\right).\tag{4.4}$$

Combining (4.3) and (4.4), we get that

$$\mathcal{E}_u(T_0)\le\mathcal{E}_u(0)\le C_1C_2\left(\mathcal{E}_u(0)-\mathcal{E}_u(T_0)\right).\tag{4.5}$$

Thereafter

$$\mathcal{E}_u(T_0)\le\frac{C_1C_2}{1+C_1C_2}\mathcal{E}_u(0).$$

Then as in [28], we can get the decay result.

$(ii)\Rightarrow(i)$. In view of (2.2) and (ii), we get, for $T_1>T^*=\ln\frac{2M\sigma_3}{\omega}$, $\sigma_3=1+\frac{bk}{c^2}$,

$$\begin{aligned}
\mathcal{E}_u(T_1)=&\mathcal{E}_u(0)-\tau\left(\frac{\alpha}{\tau}-k\right)\int_0^{T_1}\|u_{tt}(t)\|^2\mathrm{d}t\\
&-b\left(k-\frac{c^2}{b}\right)\int_0^{T_1}\left\|\mathcal{A}^{\frac{1}{2}}u_t(t)\right\|^2\mathrm{d}t
\end{aligned}$$

and

$$\mathcal{E}_u(T_1) \leq \frac{1}{2\sigma_3}\mathcal{E}_u(0) \leq \frac{1}{2}\mathcal{E}_v(0).$$

Consequently, we can arrive at

$$\begin{aligned}
\mathcal{E}_v(0) \leq& 2\mathcal{E}_u(0) - \mathcal{E}_v(0)\\
\leq& 2\tau\left(\frac{\alpha}{\tau} - k\right)\int_0^{T_1} \|u_{tt}(t)\|^2 \mathrm{d}t + 2b\left(k - \frac{c^2}{b}\right)\int_0^{T_1} \left\|\mathcal{A}^{\frac{1}{2}} u_t(t)\right\|^2 \mathrm{d}t\\
\leq& 2\left(\alpha - \tau k + bk - c^2\right)\int_0^{T_1} \left(\|u_{tt}(t)\|^2 + \left\|\mathcal{A}^{\frac{1}{2}} u_t(t)\right\|^2\right) \mathrm{d}t.
\end{aligned} \tag{4.6}$$

Let us apply $u = v + z$ to (4.6), which we rewrite as

$$\begin{aligned}
\mathcal{E}_v(0) \leq& 2\left(\alpha - \tau k + bk - c^2\right)\int_0^{T_1} \left(\|v_{tt}(t)\|^2 + \left\|\mathcal{A}^{\frac{1}{2}} v_t(t)\right\|^2\right) \mathrm{d}t\\
&+ 2\left(\alpha - \tau k + bk - c^2\right)\int_0^{T_1} \left(\|z_{tt}(t)\|^2 + \left\|\mathcal{A}^{\frac{1}{2}} z_t(t)\right\|^2\right) \mathrm{d}t.
\end{aligned} \tag{4.7}$$

As in the previous computations in (4.1), we can deduce that

$$\begin{aligned}
\mathcal{E}_z(t) \leq& \left(\alpha - \frac{c^2\tau}{b}\right)\int_0^t \left[\|u_{tt}(s)\|^2 + \frac{1}{2}\left\|z_{tt}(s) + \frac{c^2}{b}z_t(s)\right\|^2 + \frac{1}{2}\left\|v_{tt}(s) + \frac{c^2}{b}v_t(s)\right\|^2\right] \mathrm{d}s\\
\leq& \left(\alpha - \frac{c^2\tau}{b}\right)\int_0^t \left[2\|z_{tt}(s)\|^2 + \frac{1}{2}\left\|z_{tt}(s) + \frac{c^2}{b}z_t(s)\right\|^2\right] \mathrm{d}s\\
&+ \left(\alpha - \frac{c^2\tau}{b}\right)\int_0^t \left[3\|v_{tt}(s)\|^2 + \frac{c^4}{b^2}\|v_t(s)\|^2\right] \mathrm{d}s\\
\leq& \frac{5}{\tau}\left(\alpha - \frac{c^2\tau}{b}\right)\int_0^t \mathcal{E}_z(s)\mathrm{d}s\\
&+ \frac{3b^2 + \lambda_0 c^4}{b^2}\left(\alpha - \frac{c^2\tau}{b}\right)\int_0^t \left(\|v_{tt}(s)\|^2 + \left\|\mathcal{A}^{\frac{1}{2}} v_t(s)\right\|^2\right) \mathrm{d}s,
\end{aligned}$$

then we can finally get

$$\mathcal{E}_z(t) \leq M_1 \exp\left(\frac{5}{\tau}\left(\alpha - \frac{c^2\tau}{b}\right)T_1\right)\int_0^{T_1} \left(\|v_{tt}(s)\|^2 + \left\|\mathcal{A}^{\frac{1}{2}} v_t(s)\right\|^2\right) \mathrm{d}s, \tag{4.8}$$

where $M_1 = \frac{3b^2+\lambda_0 c^4}{b^2}\left(\alpha - \frac{c^2\tau}{b}\right)$. Combining (4.7) and (4.8) leads to the inequality

$$\mathcal{E}_v(0) \leq C_3 \int_0^{T_1} \left(\|v_{tt}(t)\|^2 + \left\|\mathcal{A}^{\frac{1}{2}} v_t(t)\right\|^2\right) \mathrm{d}t,$$

where C_3 is a positive constant that depends on T_1 and the physical parameters of our problem. This completes the proof. □

Declarations

Conflict of interest The authors declare no conflict of interest.

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